Moving Loads on a Visco-Elastically Supported Beam with Localized Disturbances

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Abstract

In this paper a finite Euler-Bernoulli beam on viscoelastic foundation subjected to moving loads is analysed. The beam and the viscoelastic foundation have uniform properties, however, it is considered that concentrated masses, springs and dampers are also parts of the structure. Masses can contribute by transverse as well as rotational inertia, springs and dampers can be linear or rotational. They will be named as localized disturbances. Ways of analytical solution based on modal expansion are reviewed. Advantages and disadvantages are discussed, regarding the computational difficulty and efficiency.

Keywords: distributed dynamic system, localized disturbances, eigenvalue expansion, modal space, complex frequencies, computational efficiency.

1 Introduction

The transient behaviour of general one-dimensional distributed dynamic systems, like vibrations of beam structures induced by moving loads, are often studied by implementing the Fourier method of variable separation and assuming the existence of free harmonic vibrations. Then the transient response in the time domain is expressed as an infinite series, where each vibration mode (function of the spatial coordinate x) is multiplied by a generalized displacement (modal coordinate, amplitude function) that is a function of time. Vibration modes and their frequencies are determined by the corresponding eigenvalue problem.

In this contribution a finite Euler-Bernoulli beam on viscoelastic foundation subjected to moving loads is analyzed. The beam and the viscoelastic foundation have uniform properties. It is considered moreover that this beam can contain concentrated masses, springs and dampers. Masses can contribute by transverse as well as rotational inertia, springs and dampers can be linear or rotational. They will be named as localized disturbances.

When localized disturbances are considered, one has generally two ways to follow. Either the disturbances will be inherent parts of vibration modes and frequencies or they will only be included in the modal space. We will name the former approach as “the expansion over full vibration modes”, and the latter one as “the expansion over simplified vibration modes”. Advantages of the former approach are evident, because orthogonality conditions can be defined and consequently modal equations will be uncoupled. The difficulty is attributed to the vibration modes and frequencies determination. The latter approach does not require complicated determination of vibration modes, the difficulty arises from the fact that the governing equations in the modal space are coupled.

There is another fundamental difference. If there is no damping, either distributed or concentrated dampers, full vibration modes can be determined within fully real-valued analysis. Uncoupled modal equations can be written in a standard form. Then in case of no localized dampers the viscoelastic contribution can be considered in an approximate way via the damping ratio and critical damping. Modal coordinates can be then determined directly by the Duhamel integral.

When concentrated dampers are included, full vibration modes must take into account also the viscous part of the foundation in order to ensure orthogonality conditions and uncoupling in the modal space. Sparse works are available about modal expansion in such cases. The difficulty arises from the fact that the frequencies as well as the vibration modes are complex [1, 2]. Roots search is not a simple task, because it involves calculation of complex roots of complex-valued function. It can only be performed within a software with adaptable number of digits precision, like Maple, or similar [3]. Analytical results in such cases are very important, because finite element codes usually fail to determine such a behaviour with acceptable accuracy.

Uncoupling modal equations does not seem to be so important for beams without viscoelastic foundation, because then (i) the contribution of the first mode is dominant and usually (ii) few vibration modes (around 10) contribution is necessary for achieving acceptable accuracy, at least in terms of displacements. The coupled equations can be solved directly in Matlab [4], for instance, without any necessity of discretization. This is not true for beams on viscoelastic foundation. In such a case more modes are necessary for sufficient accuracy and it is not the first mode that has the dominant contribution. The mode number with dominant contribution depends on the actual load velocity. It is also possible to define a resonant velocity to each vibration mode and consequently the critical velocity of the structure.

The new contribution of this work is a detailed comparison of computation difficulty and efficiency of the approaches named above. Such a comparison was not published yet. The computational efficiency is evaluated by the time necessary for calculation and the number of vibration modes needed for sufficient accuracy. The number of modes is determined by convergence studies, by L2-norm evaluation of the difference function defined as subtraction of the function to be evaluated from the target function. Comparison with finite element calculation is performed by transient analysis in general purpose finite element software ANSYS [5]. Convergence properties are studied against the number of elements. Cases where

finite element method fails to predict correct results are identified. The work presented is performed under the SMARTRACK project.

The paper is organized in the following way: in Section 2 governing equations are summarized. In Section 3 solution methods are reviewed, particular details are given for expansion over full and simplified modes. Numerical comparisons are shown in Section 4 and the paper is concluded in Section 5.

2 Governing equations

Let a uniform motion of a constant load along a uniform finite horizontal Euler-Bernoulli beam be assumed. The load inertia is neglected. Let a finite number of concentrated masses, springs and dampers is added. Masses can contribute by transverse as well as rotational inertia, springs and dampers can be linear or rotational. They will be called as localized disturbances.

In Figure 1, a finite simply supported beam without viscoelastic foundation, with one concentrated mass, m, linear spring of stiffness k, and linear viscous damper of damping coefficient c is shown. P stands for the moving force, v is its constant velocity, x and w are the spatial coordinate and the vertical deflection. The deflection is assumed positive when oriented downward and is measured from the equilibrium position, when the beam is only loaded with its own weight. At zero time (t=0), the load is located at the origin of the spatial coordinate x.

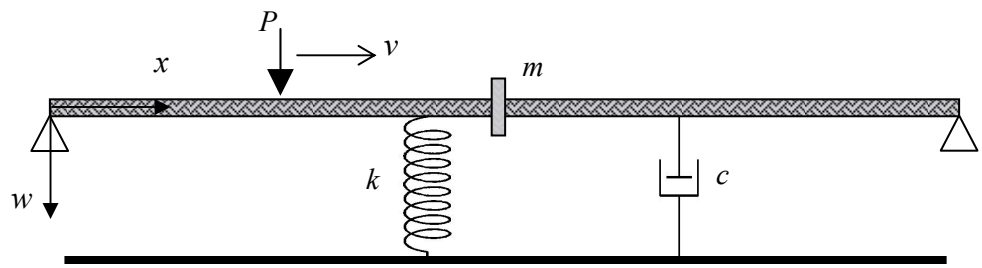

Figure 1: A finite beam considered in this study.

The governing equation can be written as:

$$EI\frac{\partial^4 w}{\partial x^4}+\mu\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}+kw+\sum_{n=1}^{b}\delta(x-x_n)\left(m_n\frac{\partial^2 w}{\partial t^2}+c_{L,n}\frac{\partial w}{\partial t}+k_{L,n}w\right)$$
$$+\sum_{n=1}^{b}\frac{\mathrm{d}\delta}{\mathrm{d}x}(x-x_n)\left(J_n\frac{\partial^3 w}{\partial x\partial t^2}+c_{R,n}\frac{\partial^2 w}{\partial x\partial t}+k_{R,n}\frac{\partial w}{\partial x}\right)=P\delta(x-vt) \tag{1}$$

where EI and μ stand for the bending stiffness and the mass per unit length, $k_{L,n}$, $k_{R,n}$, $c_{L,n}$, $c_{R,n}$, are the stiffness and damping coefficients of the concentrated linear and rotational springs and dampers, m_n and J_n are the mass and the mass moment of inertia of the concentrated masses and δ is the Dirac delta function. b is the number of positions of localized disturbances, including supports. Naturally, in each position

x_n there are not necessarily all kinds of the disturbances, therefore Equation (1) accepts zero characteristics of $k_{L,n}$, $k_{R,n}$, $c_{L,n}$, $c_{R,n}$, m_n or J_n. For simply supported beam of length L from Figure 1 the boundary conditions are:

$$w(0,t)=0\,,\ w(L,t)=0\,,\ \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0}=0\,,\ \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=L}=0 \quad \forall t \tag{2}$$

and the initial conditions can be assumed homogeneous, for the sake of simplicity.

$$w(x,0)=0\,,\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0}=0 \quad \forall x \tag{3}$$

For other supports than the ones represented in Figure 1, the corresponding boundary conditions can easily be written.

3 Solution methods

In order to express the solution of Equations (1-3) analytically, it is usual to adopt the Fourier method of variable separation and to accept an existence of free harmonic vibrations. This means that a solution of the homogeneous problem can be written as:

$$w(x,t)=w(x)e^{i\omega t} \tag{4}$$

where $i=\sqrt{-1}$. The frequency ω of these vibrations is named as the natural frequency and it is determined from the corresponding eigenvalue problem. Note that it is usual to use same designations for $w(x,t)$ and $w(x)$. Then the transient response in the time domain is expressed as an infinite series of these modes, where each vibration mode is multiplied by a modal coordinate that is a function of time [6]:

$$w(x,t)=\sum_{j=1}^{\infty} q_j(t)w_j(x)\,, \tag{5}$$

This solution method is called the eigenvalues expansion. The method looks straightforward; however, for the structure considered, several alternatives are possible that will be described in the following.

Essential conditions for modal coordinates determination is:

$$\int_{x=0}^{L}\frac{d^4 w(x,t)}{dx^4}w_j\mathrm{d}x=\int_{x=0}^{L}w(x,t)\frac{\mathrm{d}^4 w_j}{\mathrm{d}x^4}\mathrm{d}x \quad \forall j \tag{6}$$

This can be verified by carrying out integration by parts:

$$\int_{x=0}^{L} \frac{d^4 w(x,t)}{dx^4} w_j \mathrm{d}x = \left[\frac{\mathrm{d}^3 w(x,t)}{\mathrm{d}x^3} w_j \right]_0^L - \left[\frac{\mathrm{d}^2 w(x,t)}{\mathrm{d}x^2} \frac{\mathrm{d}w_j}{\mathrm{d}x} \right]_0^L + \left[\frac{\mathrm{d}w(x,t)}{\mathrm{d}x} \frac{\mathrm{d}^2 w_j}{\mathrm{d}x^2} \right]_0^L$$
$$- \left[w(x,t) \frac{\mathrm{d}^3 w_j}{\mathrm{d}x^3} \right]_0^L + \int_{x=0}^{L} w(x,t) \frac{\mathrm{d}^4 w_j}{\mathrm{d}x^4} \mathrm{d}x = \int_{x=0}^{L} w(x,t) \frac{\mathrm{d}^4 w_j}{\mathrm{d}x^4} \mathrm{d}x \tag{7}$$

If fact limits should be separated by x_n points. Nevertheless in places of localized disturbances continuity of deflection and rotation is also verified. The trick is to image that the boundary disturbances act infinitely close to the boundary, but still inside the beam.

3.1 Case with no damping, full vibration modes

When no damping is considered, the solution procedure can be kept within the real-valued domain. Two variants are possible: either full or simplified vibration modes can be used. The full vibration modes include all localized disturbances directly. By substituting Equation (4) into homogeneous Equation (1) and by omitting the damping term, one obtains:

$$\left(EI \frac{\mathrm{d}^4}{\mathrm{d}x^4} + k + \sum_{n=1}^{b} \delta(x - x_n) k_{L,n} \right) w + \sum_{n=1}^{b} \frac{\mathrm{d}\delta}{\mathrm{d}x} (x - x_n) k_{R,n} \frac{\mathrm{d}w}{\mathrm{d}x}$$
$$- \omega^2 \left(\mu + \sum_{n=1}^{b} \delta(x - x_n) m_n \right) w - \omega^2 \sum_{n=1}^{b} \frac{\mathrm{d}\delta}{\mathrm{d}x} (x - x_n) J_n \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \tag{8}$$

In order to solve Equation (6) the beam is separated into uniform parts, called sub-domains. The separation points are defined by positions of the disturbances, and the global dynamic stiffness matrix is assembled by the direct stiffness method [7]. The local dynamic stiffness matrix of the n-th sub-domain, following x_n separation point, can be calculated in the following way. The degrees of freedom are represented in Figure 2a).

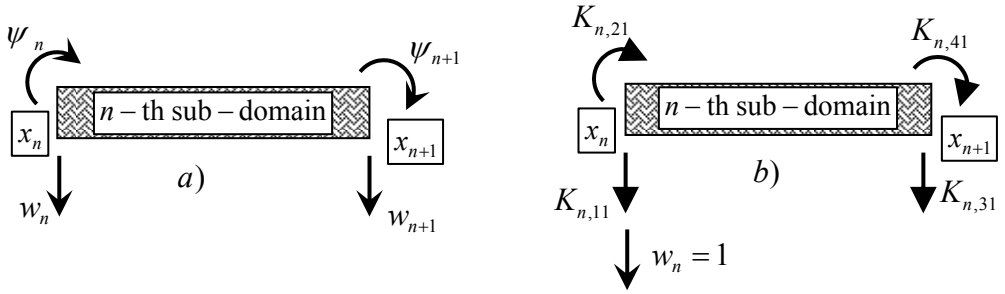

Figure 2: a) Degrees of freedom, b) construction of the local dynamic stiffness matrix of the n-th sub-domain.

Excitation with unit amplitude and given circular frequency ω is assumed in the direction of one of the degrees of freedom, while the other degrees of freedom are kept fixed. Figure 2b) exemplifies implementation of the first degree of freedom and orientation of the corresponding terms of the stiffness matrix. Within each sub-domain the mode shape is given by the general equation:

$$\begin{aligned} w_n(x) &= C_{n,1}\cos\left(\frac{\lambda_n}{L_n}x\right) + C_{n,2}\sin\left(\frac{\lambda_n}{L_n}x\right) \\ &+ C_{n,3}\cosh\left(\frac{\lambda_n}{L_n}x\right) + C_{n,4}\sinh\left(\frac{\lambda_n}{L_n}x\right), \quad n = 1,\ldots,b-1 \end{aligned} \tag{9}$$

where L_n is the length of the sub-domain and the wave number λ_n/L_n is given by:

$$\frac{\lambda_n}{L_n} = \sqrt[4]{\frac{\omega^2\mu - k}{EI}} \tag{10}$$

The member-end generalized harmonic forces in the steady-state regime can be calculated exploiting Equation (7) and

$$M_n(x) = -EI\frac{\mathrm{d}^2 w_n(x)}{\mathrm{d}x^2}, \; Q_n(x) = -EI\frac{\mathrm{d}^3 w_n(x)}{\mathrm{d}x^3} \tag{11}$$

where M and Q designate the bending moment and the shear force, respectively. Obviously:

$$K_{n,1s} = -Q_{n,s}(x_n), \; K_{n,2s} = M_{n,s}(x_n), \tag{12a}$$

$$K_{n,3s} = Q_{n,s}(x_{n+1}), \; K_{n,4s} = -M_{n,s}(x_{n+1}), \tag{12b}$$

where s designates the order of the degree of freedom that was used for excitation. The local dynamic stiffness matrix is a symmetric 4x4 matrix composed by harmonic functions with amplitudes shown below in Equation (13).

$$\begin{Bmatrix} -Q_n(x_n) \\ M_n(x_n) \\ Q_n(x_{n+1}) \\ -M_n(x_{n+1}) \end{Bmatrix} = EI \begin{bmatrix} F_{n,6}/L_n^3 & -F_{n,4}/L_n^2 & F_{n,5}/L_n^3 & F_{n,3}/L_n^2 \\ & F_{n,2}/L_n & -F_{n,3}/L_n^2 & F_{n,1}/L_n \\ & & F_{n,6}/L_n^3 & F_{n,4}/L_n^2 \\ symm. & & & F_{n,2}/L_n \end{bmatrix} \cdot \begin{Bmatrix} w_n \\ \psi_n \\ w_{n+1} \\ \psi_{n+1} \end{Bmatrix}, \tag{13}$$

The terms in Equation (11) make use of the following Koloušek functions [8]:

$$F_{1,n} = -\lambda_n \frac{\sinh\lambda_n - \sin\lambda_n}{\cosh\lambda_n\cos\lambda_n - 1}, \; F_{2,n} = -\lambda_n \frac{\cosh\lambda_n\sin\lambda_n - \sinh\lambda_n\cos\lambda_n}{\cosh\lambda_n\cos\lambda_n - 1}, \tag{14a}$$

$$F_{3,n} = -\lambda_n^2 \frac{\cosh\lambda_n - \cos\lambda_n}{\cosh\lambda_n \cos\lambda_n - 1}, \; F_{4,n} = \lambda_n^2 \frac{\sinh\lambda_n \sin\lambda_n}{\cosh\lambda_n \cos\lambda_n - 1}, \tag{14b}$$

$$F_{5,n} = \lambda_n^3 \frac{\sinh\lambda_n + \sin\lambda_n}{\cosh\lambda_n \cos\lambda_n - 1}, \; F_{6,n} = -\lambda_n^3 \frac{\cosh\lambda_n \sin\lambda_n + \sinh\lambda_n \cos\lambda_n}{\cosh\lambda_n \cos\lambda_n - 1}. \tag{14c}$$

Localized disturbances are added directly into diagonal terms of the global matrix, namely $k_{L,n}$ appears in position [2(n-1)+1, 2(n-1)+1] with a positive sign, $k_{R,n}$ appears in position [2n, 2n] with a positive sign, m_n will add a term $-m_n\omega^2$ in position [2(n-1)+1, 2(n-1)+1] and J_n will add a term $-J_n\omega^2$ in position [2n, 2n].

Frequency in each sub-domain must be the same, therefore the wave numbers are also the same and Equation (10) can be substituted. Then the determinant of the global dynamics stiffness matrix is set to zero. The roots of this equation are the natural frequencies. Except for very simple cases, the determinant contains a quite complicated combination of trigonometric and hyperbolic functions, requiring numerical search of the roots, for instance by the bisection method in software allowing for adaptable digits precision like Maple or similar. The determinant is real-valued and the roots are real as well. Actually it is simpler to solve for ω^2 than for ω. In addition, the determinant has many singularities coincident with all natural frequencies of each sub-domain considered separately. In order to avoid special treatment around the singularities, it is possible to solve the roots in the numerator. More details can be found in [7].

It is worthwhile to mention that the local dynamic stiffness matrix in Equation (13) corresponds to an interior sub-domain with both ends fixed. A boundary sub-domain can be simplified by methods of degrees of freedom condensation, depending on the actual boundary conditions.

Having calculated the natural frequencies, it is possible to substitute back into the global dynamic stiffness matrix, determine displacements and rotations of separation points, and consequently establish the vibration modes. After that Equation (5) can be substituted into Equation (1), which can be further multiplied by one of the modes and subject to integration over the full domain L. Exploiting orthogonality uncoupled equations in the modal space can finally be obtained.

The following two orthogonality conditions are fulfilled:

$$\int_0^L \mu w_j w_l \mathrm{d}x + \sum_{n=1}^{b} m_n w_j(x_n) w_l(x_n) + \sum_{n=1}^{b} J_n \frac{\mathrm{d}}{\mathrm{d}x} w_j(x_n) \frac{\mathrm{d}}{\mathrm{d}x} w_l(x_n) = 0, \quad \forall j \neq l \tag{15}$$

therefore the modal mass is given by:

$$M_j = \int_0^L \mu w_j^2 \mathrm{d}x + \sum_{n=1}^{b} m_n w_j^2(x_n) + \sum_{n=1}^{b} J_n \left(\frac{\mathrm{d}}{\mathrm{d}x} w_j(x_n) \right)^2 \tag{16}$$

and the modal equation by:

$$M_j \frac{\mathrm{d}^2 q_j}{\mathrm{d}t^2} + \omega_j^2 M_j q_j = \tilde{q}_j \tag{17}$$

where

$$\tilde{q}_j = \int_{x=0}^{L} P\delta(x - vt) w_j(x) dx = P w_j(vt)(H(L - vt) - H(-vt)) \tag{18}$$

The Heaviside function H in the brackets states that this term is only valid when the force is on the beam.

If damping is included in the foundation, then this term prevents the uncoupling because the term

$$C_{lj} = \int_{x=0}^{L} c\left(\sum \frac{\mathrm{d}q}{\mathrm{d}t} w_l\right) w_j \mathrm{d}x \tag{19}$$

cannot be simplified. When damping is low, it can be further assumed that $C_{lj} \cong 0 \; \forall l \neq j$ and $C_j = \xi c_{cr} = 2\xi M_j \omega_j$, where ξ is called the damping ratio and $c_{cr} = 2M_j \omega_j$ is the critical damping. In summary:

$$\frac{\mathrm{d}^2 q_j}{\mathrm{d}t^2} + 2\xi\omega_j \frac{\mathrm{d}q_j}{\mathrm{d}t} + \omega_j^2 q_j = \frac{1}{M_j} \tilde{q}_j \,. \tag{20}$$

The solution of Equation (20) can be expressed by the Duhamel integral:

$$q_j(t) = \frac{1}{M_j \omega_{d,j}} \int_{\tilde{t}=0}^{t} \tilde{q}_j(\tilde{t}) \mathrm{e}^{-\xi\omega_j(t-\tilde{t})} \sin(\omega_{d,j}(t - \tilde{t})) \mathrm{d}\tilde{t} \tag{21}$$

where

$$\omega_{d,j} = \omega_j \sqrt{1 - \xi^2} \tag{22}$$

3.2 Simplified vibration modes

When expansion over simplified vibration modes is considered, that means that localized disturbances are omitted in vibration modes calculation. Then the frequencies and the vibration modes are known in advance and do not have to be calculated any time the structure changes its localized disturbances. Viscous term in foundation can be again neglected, thus undamped vibration modes are determined from:

$$\left(EI \frac{\mathrm{d}^4}{\mathrm{d}x^4} + k\right) w - \omega^2 \mu w = 0 \tag{23}$$

The simplest case is the simply supported beam, because then the natural frequencies, the modes and the modal mass are given by fully analytical expressions:

$$\omega_j = \sqrt{\left(\frac{j\pi}{L}\right)^4 \frac{EI}{\mu} + \frac{k}{\mu}} \tag{24}$$

$$w_j(x) = \sin\left(\frac{j\pi}{L}x\right),\ M_j = \frac{1}{2}\mu L \tag{25}$$

The difficulty now lies in the solution of modal equations that are coupled and given by:

$$\mathbf{M}\cdot\ddot{\mathbf{q}}(t) + \mathbf{C}\cdot\dot{\mathbf{q}}(t) + \mathbf{K}\cdot\mathbf{q}(t) = \tilde{\mathbf{q}}(t), \tag{26}$$

where

$$M_{lj} = \delta_{lj}M_j + \sum_{n=1}^{b}\left[m_n w_l(x_n)w_j(x_n) + J_n \frac{\mathrm{d}w_l}{\mathrm{d}x}(x_n)\frac{\mathrm{d}w_j}{\mathrm{d}x}(x_n)\right] \tag{27}$$

$$\mathrm{C}_{lj} = \delta_{lj}\frac{c}{\mu}M_j + \sum_{n=1}^{b}\left[c_{L,n}w_l(x_n)w_j(x_n) + c_{R,n}\frac{\mathrm{d}w_l}{\mathrm{d}x}(x_n)\frac{\mathrm{d}w_j}{\mathrm{d}x}(x_n)\right] \tag{28}$$

$$K_{lj} = \delta_{lj}M_j\omega_j^2 + \sum_{n=1}^{b}\left[k_{L,n}w_l(x_n)w_j(x_n) + k_{R,n}\frac{\mathrm{d}w_l}{\mathrm{d}x}(x_n)\frac{\mathrm{d}w_j}{\mathrm{d}x}(x_n)\right] \tag{29}$$

For other supports frequencies can be quickly calculated, as they depend only on the boundary conditions considered. Coupling terms in modal space can be expressed using analytical vibration modes, therefore no discretization is necessary. Nevertheless final equations must be solved numerically.

It is worthwhile to point out another disadvantage: due to the coupling effect modal coordinates depend on the number of modes considered. When this number is increased, it is not sufficient, like for full vibration modes, simply add the additional contribution. It is necessary to repeat the full solution of the coupled Equation (26).

3.3 Case with localized dampers, full vibration modes

In this case it is necessary for vibration modes determination to include all terms entering the governing Equation (1), except for the loading term. This implies that frequencies are complex. One can proceed exactly in the same way as in Section 3.1, $c_{L,n}$ will add a term $ic_{L,n}\omega$ in position [2(n-1)+1, 2(n-1)+1] and $c_{R,n}$ will add a term $ic_{R,n}\omega$ in position [2n, 2n]. The substitution to introduce into the global dynamic stiffness matrix is:

$$\frac{\lambda_n}{L_n} = \sqrt[4]{\frac{\omega^2\mu - i\omega c - k}{EI}} \tag{30}$$

There are pairs of natural frequencies, which when multiplied by i will have complex conjugate values. The corresponding pairs of wave numbers and vibration mode shapes have also complex conjugate values. The orthogonality condition is specified as:

$$\begin{aligned}&\left(i\mu\left(\omega_j+\omega_l\right)+c\right)\int_0^L w_j w_l \mathrm{d}x+\sum_{n=1}^{b}\left(im_n\left(\omega_j+\omega_l\right)+c_L\right)w_j\left(x_n\right)w_l\left(x_n\right)\\&+\sum_{n=1}^{b}\left(iJ_n\left(\omega_j+\omega_l\right)+c_R\right)\frac{\mathrm{d}}{\mathrm{d}x}w_j\left(x_n\right)\frac{\mathrm{d}}{\mathrm{d}x}w_l\left(x_n\right)=0,\quad \forall j\neq l\end{aligned} \tag{31}$$

and the modal mass is given by:

$$M_j=\left(2i\omega_j\mu+c\right)\int_0^L w_j^2\mathrm{d}x+\sum_{n=1}^{b}\left(2i\omega_j m_n+c_L\right)w_j^2\left(x_n\right)+\sum_{n=1}^{b}\left(2i\omega_j J_n+c_R\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}w_j\left(x_n\right)\right)^2 \tag{32}$$

Modal coordinates are determined by:

$$q_j(t)=\frac{1}{M_j\omega_j}\int_{\tilde{t}=0}^{t}\tilde{q}_j\left(\tilde{t}\right)\mathrm{e}^{i\omega_j\left(t-\tilde{t}\right)}\mathrm{d}\tilde{t} \tag{33}$$

The sum specified in Equation (5) must always involve the counter part of complex conjugate, because the sum of each pairs of terms is real-valued.

4 Examples

Two case studies are considered. In the first one simply supported unit beam without viscoelastic foundation is tested over localized disturbances in form of springs and masses. In the second one similar beam, but in form of a cantilever, with localized dampers is analysed. Input data in both cases are considered dimensionless.

4.1 First case study

A simply supported beam with unit properties (EI=1, μ=1) and unit length (L=1) is assumed. A unit moving force (P=1) is traversing the beam by unit velocity v=1. It is worthwhile to mention that in this case the critical velocity is equal to π.

Four different cases of localized disturbances are considered: (i) linear spring of rigidity 10 at position 0.5, (ii) rotational spring of rigidity 100 at position 0.25, (iii) mass of value 1 at position 0.5 and (iv) mass rotational inertia of value 1 at position 0.25. Values introduced were chosen in the way to cause significant difference in displacement field (over 50%) with respect to the fully homogeneous beam without any localized disturbances. Two methods, expansion over full and simplified modes are compared. The former one is programmed in Maple due to the required high precision for frequencies evaluation, the latter one is programmed in Matlab.

In case (i) sufficient accuracy is obtained with 10 vibration modes. The normalized vibration modes in both methods are exactly the same. Modal coordinates are different. When simplified modes are considered, they depend on the number of modes considered. Nevertheless coupling is insignificant. It can be shown that off-diagonal terms in Equation (24) could be neglected. This is shown in graph of Figure 3. In the legend "map" stands for the Maple calculation, *i.e.* full vibration modes and "mat" for the Matlab calculation, *i.e.* simplified vibration modes. Last number designates the first modal coordinate and the other numbers specify how many modes were included into calculation. This number is relevant only for the simplified modes. It is seen that no significant difference has been found between all curves.

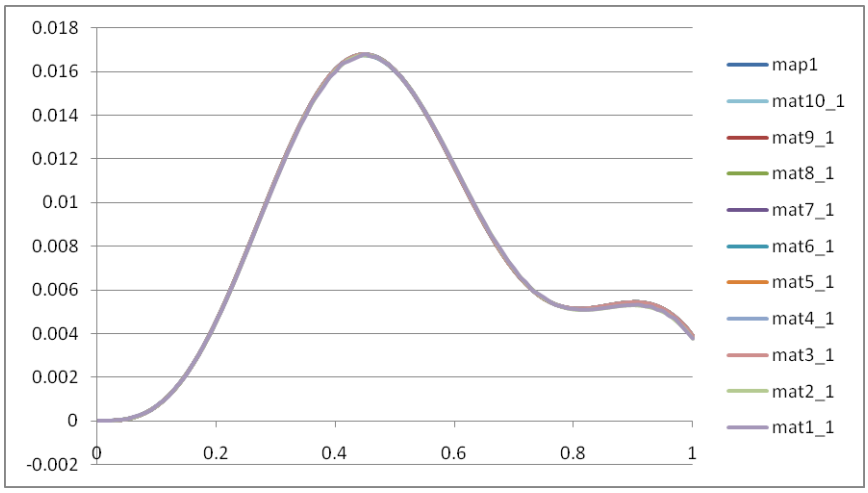

Figure 3: First modal coordinate comparison, case (i).

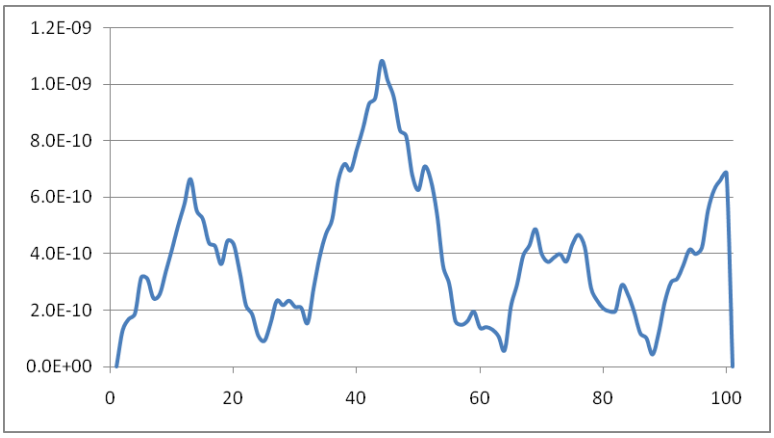

Figure 4: L2-norm comparison, case (i).

Next Figure 4 show L2-norm comparison for 101 time steps along the structure of the difference function composed of full modes and simplified modes calculation with 10 vibration modes. It is seen that the solution can be considered identical.

Same conclusions cannot be taken in case (ii). 10 vibration modes are sufficient only in full vibration modes method. Normalized vibration modes as well as the first modal coordinate exhibit significant differences. The same comparison as in Figure 3 is shown in Figure 5.

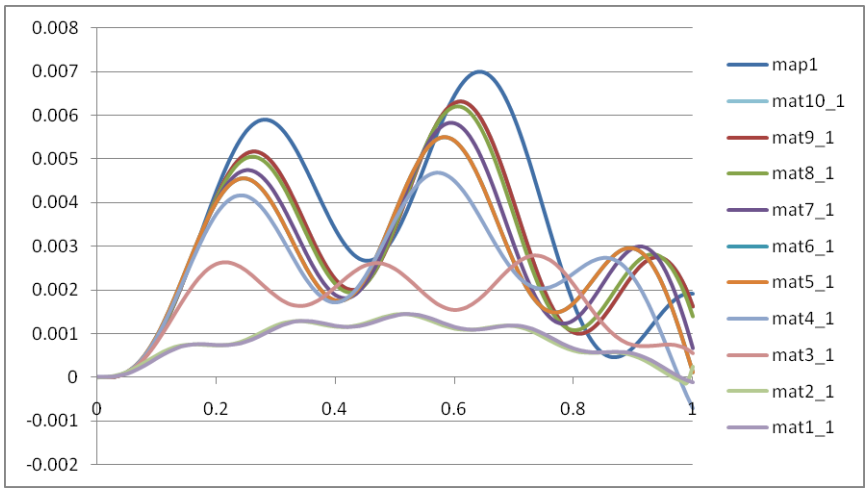

Figure 5: First modal coordinate comparison, case (ii).

Regarding L2-norm comparison, 10 modes of full vibration modes represent already an accurate solution. But it is seen in Figure 6 that even 30 modes of simplified modes method still exhibit a significant error. This is exemplified also in Figure 7, where deflection shape for load position at 0.8 is compared.

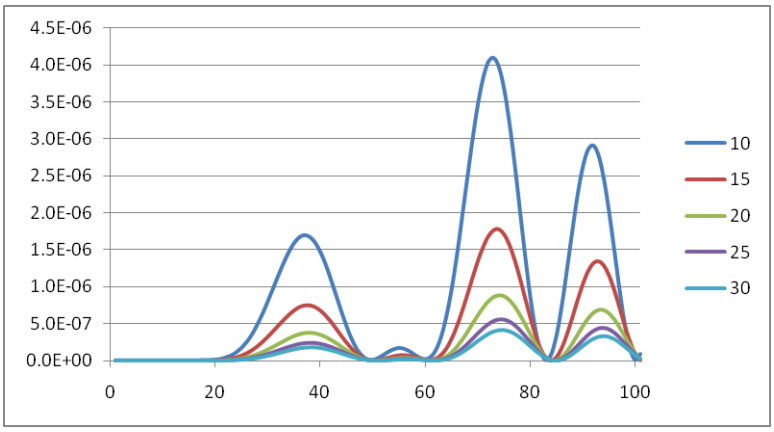

Figure 6: L2-norm comparison, case (ii).

It is necessary to point out that in this case the evaluation takes already a significant time (more than 1 hour, as it grows exponentially with the number of modes included) and moreover it is affected by reduced precision (only double precision) of Matlab. Nevertheless, the same coupled calculation would be inexecutable in Maple software, therefore the simplified modes method looses all its advantages, unless discretization is implemented.

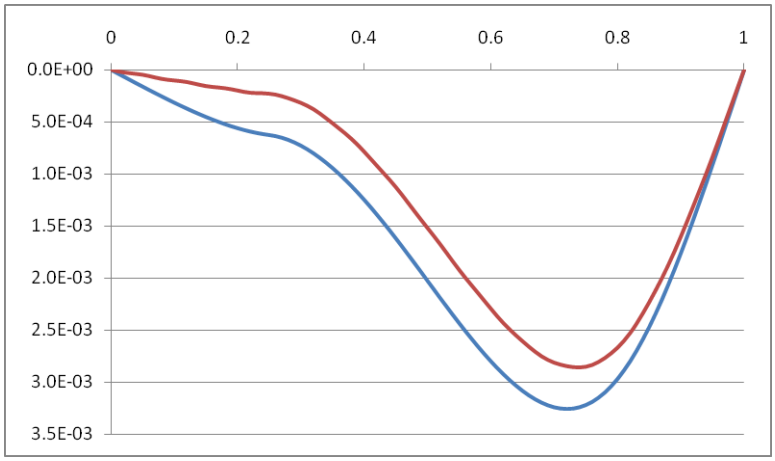

Figure 7: Load at position 0.8, case (ii), 10 full modes – blue, 30 simplified modes - red.

Regarding case (iii), similar conclusion can be taken as in case (i), except for vibration modes. Odd normalized vibration modes in the two methods are significantly different, while even modes are exactly the same as expected. Coupling is insignificant, but odd modal coordinates are also significantly different. Regarding L2-norm comparison, solutions can be considered identical for 10 vibration modes, as it is shown in Figure 8.

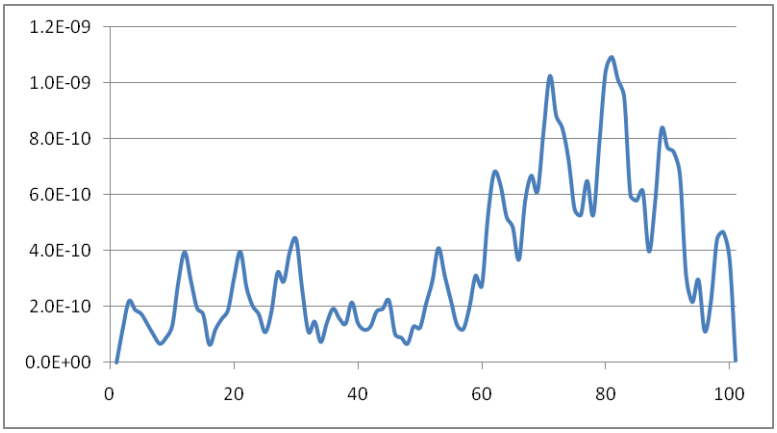

Figure 8: L2-norm comparison, case (iii).

Also in case (iv) 10 vibration modes are sufficient in expansion over full vibration modes. Vibration modes are very different between the methods. First two full modes have only one wave over the structures, therefore then the third mode is similar to the second simplified, *etc*. Acceptable accuracy is obtained for 20 simplified modes in term of the L2-norm comparison as it is shown in Figure 9, but it takes 15 minutes to obtain these results, and there is still a significant error as it is shown in Figure 10.

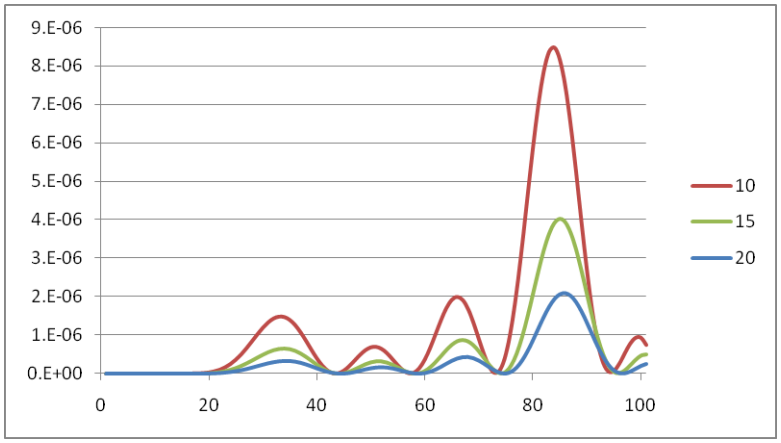

Figure 9: L2-norm comparison, case (iv), 10 full vibration modes versus simplified.

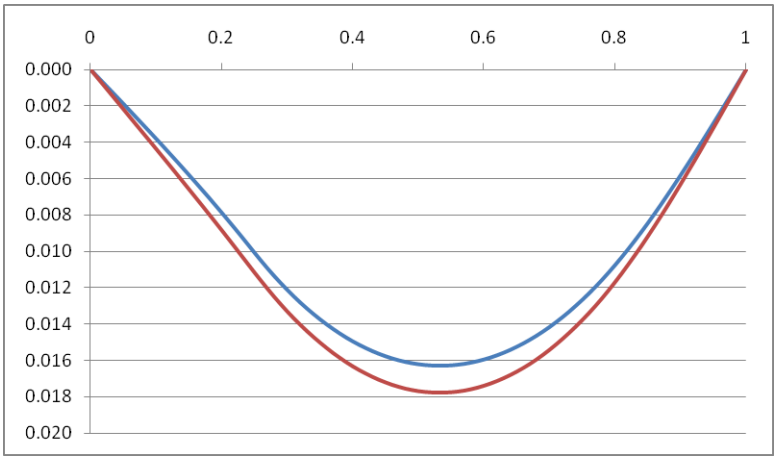

Figure 10: Load at position 0.8, case (iv), 10 full modes – blue, 20 simplified modes - red.

4.2 Second case study

Second case study contain localized damper. For the sake of better comparison, same structure as in [2] is selected. The structure is represented in Figure 11. Numerical data are the following: EI=1, μ=16, L=1, c=1.6, k_L=8, c_L=4, m=4. As in the first case study unit force P=1 is traversing the structure at constant velocity v=1. Implementation of full modes expansion is programmed in Maple. Complex frequencies are calculated by incorporated Maple solution procedure, specifying an appropriate interval. This is quite simple, because the imaginary part is small and decreasing, while the real part increases. High number of digits precisions must be invoked. It is possible to verify visually, that the real part of the vibration modes follow the usual predictions, in order to be sure that none of the frequencies is missing.

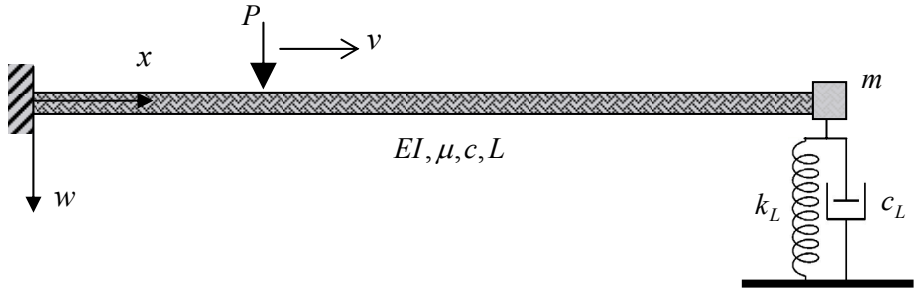

Figure 11: Second case study.

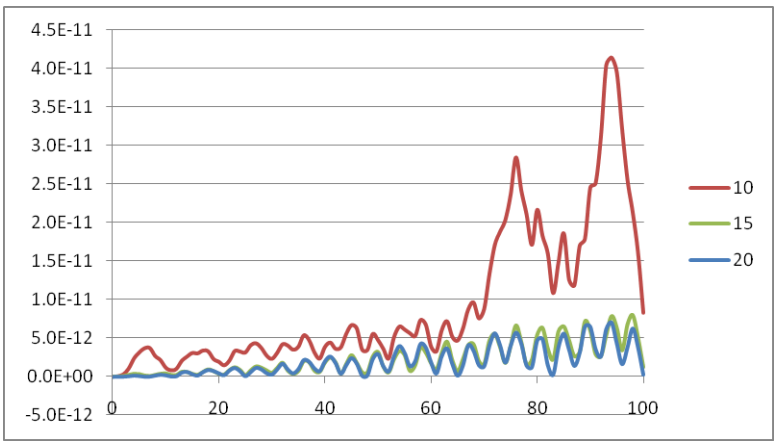

Figure 12: L2-norm comparison for solution exploiting simplified modes.

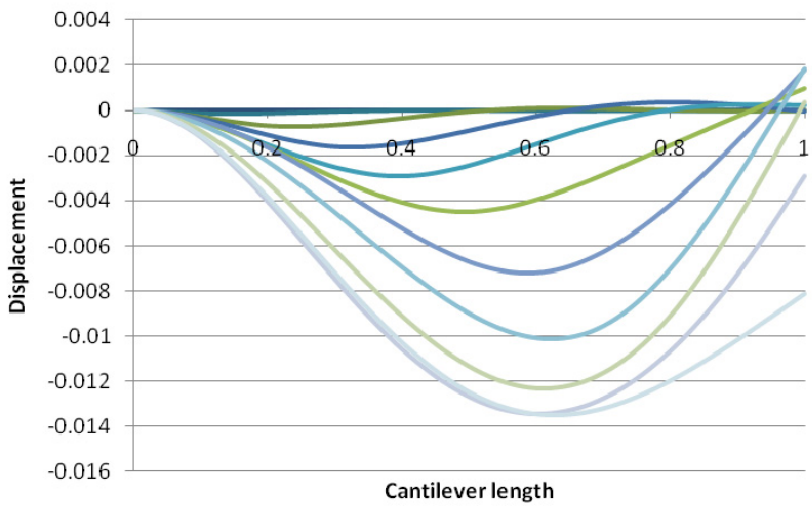

Figure 13: Deflection shapes recorded for force position at 0, 0.1, 0.2, 0.3, … 1.

Simplified modes lead to quick solution with reasonable precision already for 10 modes, as can be seen from L2-norm comparison. In this figure the target function is taken as the solution obtained with 30 simplified modes. Deflection shapes are depicted in Figure 13.

5 Conclusions

In this paper a finite Euler-Bernoulli beam on viscoelastic foundation subjected to moving loads is analysed. The beam and the viscoelastic foundation have uniform properties, however, it is considered that concentrated masses, springs and dampers are also parts of the structure. Masses can contribute by transverse as well as rotational inertia, springs and dampers can be linear or rotational. Ways of analytical solution based on modal expansion are reviewed. Two approaches named as "the expansion over full vibration modes", and "the expansion over simplified vibration modes" were introduced. Full analytical solutions in both methods were presented. Advantages and disadvantages were discussed, based on the computational difficulty and efficiency. Generally, when damping is omitted the full vibration modes expansion performs better, on the other hand for structure with damping this method is less efficient.

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