New Methodology to Establish Bounds on Effective Properties of Cellular Media.

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ABSTRACT. A new methodology to establish bounds on effective constitutive constants of cellular media consisting of a network of beams is presented. The methodology is completely general, simple to use and besides of the bounds determines necessary and sufficient conditions on microstructure of optimal media, which can be sometimes written only in terms of geometrical parameters. With one exception, the bounds obtained in this paper are linear function of the relative density, thus they correspond to the linearization of general bounds. The new methodology is verified in two dimensions, where the bounds are already known.

KEYWORDS. Cellular media, effective properties, bounds, optimal media.

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1 Introduction.

A cellular medium (CM) is composed of an interconnected network of solid beam and shell parts, which can be assigned to cells, i.e. building blocks that are (perhaps with some modifications) repeated in the medium. CM are characterized by two essential features:

(i) dimensions of voids are very small compared to the size of the medium,

(ii) CM have high porosity (usually above 70%).

Thus when a CM is under consideration we are dealing with a highly heterogeneous composite with void and solid phases. We introduce the relative density, s, as the ratio of the density of the CM to the density of the solid phase and in the following the term basic material is used for the solid phase. Due to the low relative density it is obvious that at least one dimension of the basic material, thickness, at the cell level is small compared to the characteristic cell size. This condition motivates to the possibility of using structural theories in homogenization calculation of CM effective properties as we already discussed in detail in [5] and [6]. In the present paper we show that structural theories are a powerful tool in determination of bounds on CM effective properties as well.

CM have a wide range of applications. They can be used for absorption of the kinetic energy from impacts, as thermal isolation, they have useful electrical properties, etc. New technologies allow foaming of metals and ceramics, so there are many new materials in the class of CM and detailed description of their properties is required.

Determination of bounds on effective properties of highly heterogeneous composites has been the subject of considerable research for many years. In the present paper we make use of formulation of two-phase bounds of Hashin-Shtrikman for 2D effectively isotropic media ([9]-[11]) and of Allaire & Kohn from [1] for 2D media with effective square symmetry.

The main monograph on CM [8] has been published by Gibson and Ashby, but the problem of bounds on CM effective properties is not examined neither in [8] nor elsewhere. There are only available numerical optimization algorithms to determine microstructures consisting of trusses or beams, which correspond to prescribed properties [13]-[16].

Clearly, 2D-CM are composed only of beams while 3D media can be more general. The important restriction of the class of 3D-CM is thus to CM consisting of beams (CMB). This restriction allows determination of new CMB bounds that are strictly lower than the general ones since optimal media for 3D general bounds cannot be just from CMB class (the proof is in [2]). The main contribution of this paper is thus in the introduction of the new methodology to establish bounds on effective properties of CMB media. The methodology is in detail described and verified in 2D. The verification is made by comparison with the known 2D bounds. Consequently, the main conclusions obtained in this paper can be extended to 3D case. Determination of the new 3D-CMB bounds together with the extension of the methodology presented here we published in [7]. Moreover, using the same methodology, we also developed bounds on properties of CMB media with mainly bending and torsion response, see [4]. These ones we are going to publish soon.

2 General remarks about the new methodology.

2.1 SIMPLIFIED ASSUMPTIONS AND BASIC RELATIONS NEEDED FOR THE NEW METHODOLOGY.

When CMB are under consideration, there are two extreme possibilities for the structural model of the crossing points - either “hinges” or “rigid corners”. According to [15] we introduce the terms micro-frame media when rigid corners are assumed and micro-truss media when the beams are straight and connected only by hinges. The group of CM consisting of straight beams we denote by CMS. To each micro-frame medium from CMS group we can relate a micro-truss medium just by changing the behavior of the crossings. In true CMB the behavior of the crossings is somewhere between these two extreme cases. The corners are not really stiffened, only in some natural CMB the thickness is usually higher close to the crossing points. This influence is not very significant (see [17]), but it justifies the assumption of rigid corners. Fig. 2.1 illustrates the definition of the theoretical length and effective length. The former is the length between the theoretical crossing points, measured on the middle axes of the beams, and the latter is the theoretical length reduced by parts that belong to the crossings.

For the sake of simplicity we assume that the basic material is homogeneous isotropic linear elastic (E_{bm} and ν_{bm} stand for its Young's modulus and Poisson's ratio, respectively). Due to the homogeneity, the relative density coincides with the volume fraction of the basic material and thus it forms a counter part to the porosity. In a sequel, we adopt a maximum value for the relative density of CM as s=0.3, see [8].

Obviously, without loss of generality, we can treat only CMB with periodic microstructure and we can use some basic conclusions from homogenization techniques, thus a cell of a CMB medium can be conveniently rescaled and spatial variable **y** can be introduced there, [5]. Rescaling enables introduction of the unit related to some dimension of the cell. All other geometrical parameters are expressed proportionally to this unit.

As already mentioned, only 2D-CMB are considered in this paper and, in addition, assumption of the plane stress without specifying the perpendicular thickness is adopted. Furthermore, we restrict ourselves to media with effective isotropy or square symmetry and thus the dimensionless matrix of effective elastic stiffnesses **C*** is used in the following form:

$$\mathbf{C}^* = \begin{bmatrix} k^*+{}^1G^* & k^*-{}^1G^* & 0 \\ & k^*+{}^1G^* & 0 \\ \text{symm} & & {}^2G^* \end{bmatrix}. \qquad (2.1)$$

As usual k* stands for the plane stress bulk modulus, ${}^1G^*$ and ${}^2G^*$ are the shear moduli. The star * in notation means effective and dimensionless with respect to the basic material Young's modulus E_{bm}. A medium is effectively isotropic when ${}^1G^*={}^2G^*=G^*$. If we do not specify the stiffness constant, which we are referred to, we use the notation C*. Only bounds on constitutive constants mentioned above will be determined. Obviously, lower bounds of media with one void-phase are always zero, consequently only upper bounds will be examined in the following.

In derivation of each bound just basic knowledge from linear beam theory, some basic facts from linear algebra, Schwarz inequality and basic assumption for derivation of Voight bound (rule of mixtures) will be used. This assumption states uniform strain field in optimal media (see [12]), namely, when local strain field is uniform throughout the composite, then it is equal to the macroscopic one and the stiffness constant that corresponds to such macroload reaches a maximum possible value. Since in a micro-truss medium each bar has a uniform strain field with respect to its "bar" coordinates, to fulfil the previous assumption means that we have to find directions of bars, such that their strain field will coincide with the macroscopic one. Due to the small thickness of the bars, the influence of crossings can be neglected and the medium can be characterized only by its bars' middle axes. Their local displacements, **u**, must thus coincide with the uniform strain, **E**·**y**, where **E** is the macrostrain and **y** is the spatial variable. This requirement can be used to get necessary maximality condition for micro-truss media. The condition is not sufficient, since the requirement does not exclude "zero" bars from the optimal medium. Introduction of this maximality condition is inevitable in 3D case, but in 2D case can be obtained as a consequence of the estimations.

Base of the new methodology lies in well-known expressions for effective strain energy density W (Σ_M and $\boldsymbol{\Sigma}_D$ are volumetric and deviatoric parts of the effective stress tensor $\boldsymbol{\Sigma}$):

Medium with effective isotropy	Medium with effective square symmetry	
$W = \frac{1}{2}\left(\frac{\Sigma_M^2}{k^*} + \frac{\boldsymbol{\Sigma}_D : \boldsymbol{\Sigma}_D}{2G^*}\right)$	$W = \frac{1}{2}\left[\frac{\Sigma_M^2}{k^*} + \frac{\left(\Sigma_{D,11} - \Sigma_{D,22}\right)^2}{4\,{}^1G^*} + \frac{\Sigma_{D,12}^2}{{}^2G^*}\right]$	(2.2)

To be able to express from (2.2) just a specific C*, an appropriate test macroload must be applied to the medium. The ones that are needed in this paper are summarized in tab. 2.1. Due to the effective isotropy or square symmetry of assumed media, components of the macrostrain, **E**, developed in CMB under the macroloads, cannot be arbitrary. It must satisfy the same conditions as the macrostress (see column 3 of tab. 2.1).

Macroload	Constant to be determined	Specification
Σ^I	k*	$\Sigma_{11}=\Sigma_{22}\neq 0,\ \Sigma_{12}=0$
Σ^{II}	G*	$\Sigma_{11}=-\Sigma_{22}\neq 0,\ \Sigma_{12}\neq 0,\ \Sigma_{11}/\Sigma_{12}=\gamma$
Σ^{III}	${}^1G^*$	$\Sigma_{11}=-\Sigma_{22}\neq 0,\ \Sigma_{12}=0$
Σ^{IV}	${}^2G^*$	$\Sigma_{11}=\Sigma_{22}=0,\ \Sigma_{12}\neq 0$

Tab. 2.1 – Test macroloads and the corresponding specification of macrostress as well as macrostrain.

Since the effective stress and the strain energy density are defined by averaging [5], we apply averaging operator (see (3.1) later on) on the local stress and strain energy fields and express the results in terms of beam's internal forces and moments. For the W-expression we keep the following assumptions: each beam is assumed with its theoretical length, crossing points are assumed as rigid corners and the effect of shear deflection is neglected.

Then we express a fraction defining C* in terms of internal forces and moments and by step by step estimation using especially Schwartz inequality we finally obtain the form of the bound.

2.2 Some conclusions about the CMB optimal media

Besides of simplicity the main advantage of the present methodology is that from the step by step estimations immediately follow necessary and sufficient conditions on microstructure of optimal media, which can be in some cases written only in terms of geometrical parameters.

The most interesting partial results are:

(i) the optimal media are from CMS,

(ii) each beam of an optimal medium has a constant thickness,

(iii) except of G*-optimal media, only normal forces are developed under the corresponding macroload,

(iv) directions of the beams coincide with the principal directions of the macroload.

To the point (i) it should be add, that there exists, to the authors' knowledge, only one 2D-medium, that is Σ^{I}-optimal and cannot be determined by our methodology. It is the fictitious Hashin's medium – the composite cylinders. This medium has random microstructure of special kind and in the bulk test it is in the state of uniform local strains that corresponds to extension in radial symmetry. A cell of the Hashin's medium, in fact, contains infinite number of composite cylinders, thus it can be hardly included to CM. In addition, we start our estimations with the assumption that a cell of a CMB medium is composed by finite number of beams (see Section 3.1), which is the other reason why the Hashin's medium was not discovered by our methodology. The proof that the straight beams are preferable in optimal media we included in [7], thus here we restrict directly to CMS media.

The point (ii) follows simply from the Schwarz inequality and will be justified later on, in the first proof and then it will be introduced as the assumption.

In addition to the point (iii) we remark that (except of G*-bound) each bound is a linear function of the relative density. Obviously, this linear function is the tangent to the general bound (compare with [6]). In the case of G*-bound the CMB bound lies between the general bound and the linearized bound. The linearized one is the tangent to both and corresponds to G*-optimal medium taken as a micro-truss. We can show, see [7], that the nonlinear effect in G*-CMB bound is negligible.

In this context it is also useful to remark, that the validity of the CMB-bounds is related to the validity of the linear beam theory, that is used in their derivation. Specifically, the linear beam theory is reliable only if the ratio of the theoretical length of each beam to its thickness remains inside the interval [4,R], where R is related to the stability limit. In a sequel, other nonlinearities should be included in bounds' form for higher relative densities, when some ratio is close to the lower limit 4, but this effect cannot be large. Also the range of the validity of each bound in terms of the relative density should not be simply s=0.3, but should be adjusted to the existence of at least one optimal medium, for which the beam theory is still acceptable.

Finally, the point (iv) is related to the theory of Michell trusses, [2]. Directions of the beams are not exactly specified in $\mathbf{\Sigma}^{I}$ and $\mathbf{\Sigma}^{II}$ –optimal media, since in the former case each direction is principal and in the latter case the macroload is not uniquely determined (see tab. 2.1) and consequently each direction can be assumed as principal. We also remark, that in 3D, on contrary to the 2D case, beams' directions of ^{2}G*-optimal media could hardly be determined from the principal directions of the macroload (see [7]).

2.3 GENERAL TWO-PHASE BOUNDS AND CMB BOUNDS.

Following tab. 2.2 summarizes the general bounds (specified to media with one void-phase) together with their linearized forms with respect to the relative density. Obviously, in linearized forms does not appear the basic material Poisson's ratio ν_{bm}. For the sake of completeness also CMB bounds and the state of conditions that is possible to obtain on optimal media is included in tab. 2.2, δ represents small coefficient in front of the bending contribution to G*-CMB bound. We define G*-optimal media as media, which taken as micro-trusses saturate the linearized bound. The coefficient δ is different for different G*-optimal media.

Bound	k^*_+	${}^1G^*_+ = {}^2G^*_+$	G^*_+
General form	$\frac{s}{2[2-s(1+\nu_{bm})]}$	$\frac{s}{2[2-s(1-\nu_{bm})]}$	$\frac{s}{2[4-s(3-\nu_{bm})]}$
Linearized form	s/4	s/4	s/8
CMB bound	s/4	s/4	$s/8+\delta s^3$
Optimal medium	necessary and sufficient conditions	fully specified by geometrical relations	necessary and sufficient conditions

Tab. 2.2 – General bounds, CMB bounds and characterization of the CMB optimal media.

3 2D-CMB media.

3.1 BASIC FORMULAS OBTAINED BY APPLICATION OF THE AVERAGING OPERATOR.

As explained above, we can restrict ourselves directly to CMS media. Let us take a cell of a CMS medium and assume that the cell, V, contains finite number of beams. Now we express macrostress **Σ** and macroscopic strain energy density W as (see [5]):

$$\boldsymbol{\Sigma} = \frac{1}{|V|}\int_{V^*} \boldsymbol{\sigma} d\mathbf{y} = \frac{1}{|V|}\sum_i \int_{V^*_i} \boldsymbol{\sigma}^i d\mathbf{y} = \sum_i \left\langle \boldsymbol{\sigma}^i \right\rangle,$$
$$\Sigma_M = \sum_i \left\langle \sigma^i_M \right\rangle,\ \boldsymbol{\Sigma}_D = \sum_i \left\langle \boldsymbol{\sigma}^i_D \right\rangle,\ W = \sum_i \left\langle w^i \right\rangle, \tag{3.1}$$

where $\boldsymbol{\sigma}^i$ is the local stress field, its deviatoric and volumetric parts are denoted by $\boldsymbol{\sigma}^i_D$ and σ^i_M, respectively, and w^i is the local strain energy density corresponding to one single i-beam. The area of the full cell is $|V|$ while the one corresponding to the i-beam is $|V^*_i|$. $|V^*_i|$ must include the area related to the effective length and to the connected parts of crossings in the way that $\sum_i |V^*_i| = |V^*|$ and $\dim(V^*_i \cap V^*_j) \leq 1$ $\forall i \neq j$, where V* is the material part of the cell. Due to the periodicity we do not have to treat separately the case, when the i-beam is cut but the boundary of the cell.

Let us take <u>one single i-beam</u> and compute its <u>contributions</u> $\langle \boldsymbol{\sigma}^i \rangle$ and $\langle w^i \rangle$. Detailed analysis is needed for $\langle \boldsymbol{\sigma}^i \rangle$, since influence of crossing points is important. The i-beam has the theoretical length l_i, the effective length p_i and arbitrarily oriented middle axis by the angle $\alpha_i = \langle 0,2\pi)$ with respect to the cell coordinate y_1 (see fig. 3.1). Let us denote by $h_{i\,l}$ and $h_{i\,r}$ the parts of the theoretical length l_i, which correspond to the left and right end of the beam and are included in the crossing points. Thus $p_i = l_i - h_{i\,l} - h_{i\,r}$. Generally thickness is variable within the beam's effective length, where local coordinate ${}^i z_1$ is introduced with the origin in the middle of the theoretical length, thus $t_i = t_i({}^i z_1)$.

We introduce the internal forces and moments N_i, Q_i, M_i in a way that is specified in fig. 3.1. Distribution of bending moments corresponding to M_i is <u>constant</u> and corresponding to $Q_i p_i / 2$ is <u>linear with "zero area"</u>. First we compute the stress contribution (using Green's formula) with respect to the beam's local coordinate system and in its simplified (as explained below) form $\langle \tilde{\boldsymbol{\sigma}}^i \rangle$. Obviously $\langle \tilde{\boldsymbol{\sigma}}^i \rangle = \langle \tilde{\boldsymbol{\sigma}}^i_{bm} \rangle + \langle \tilde{\boldsymbol{\sigma}}^i_{cpl} \rangle + \langle \tilde{\boldsymbol{\sigma}}^i_{cpr} \rangle$, where contribution with subscript "bm" relates to the beam with effective length, with "cpl" and "cpr" to the left and right crossing faces, respectively, see fig. 3.1. It holds:

$$|V|\langle\tilde{\boldsymbol{\sigma}}^{i}_{bm}\rangle = p_i\begin{bmatrix} N_i & Q_i \\ Q_i & 0 \end{bmatrix}$$

and

$$|V|\langle\tilde{\boldsymbol{\sigma}}^{i}_{cpl}\rangle = \begin{bmatrix} N_i h_{il} & Q_i h_{il} \\ -Q_i p_i/2 + M_i & 0 \end{bmatrix};\ |V|\langle\tilde{\boldsymbol{\sigma}}^{i}_{cpr}\rangle = \begin{bmatrix} N_i h_{ir} & Q_i h_{ir} \\ -Q_i p_i/2 - M_i & 0 \end{bmatrix}. \tag{3.2}$$

Relations in (3.2) do not include the contributions from the integration over internal faces of crossings, since they would cancel after summing inside the cell. This is the simplification mentioned above. Finally:

$$\langle\tilde{\boldsymbol{\sigma}}^{i}\rangle = \frac{l_i}{|V|}\begin{bmatrix} N_i & Q_i \\ 0 & 0 \end{bmatrix}. \tag{3.3}$$

We can see that (3.3) does not include constant bending moments M_i and that i-beam contribution is non-symmetric, due to the missing contributions. Transformation to the cell coordinates yields in:

$$\langle\boldsymbol{\sigma}^{i}\rangle = \frac{l_i}{|V|}\begin{bmatrix} N_i\cos^2\alpha_i - Q_i\sin\alpha_i\cos\alpha_i & N_i\sin\alpha_i\cos\alpha_i + Q_i\cos^2\alpha_i \\ N_i\sin\alpha_i\cos\alpha_i - Q_i\sin^2\alpha_i & N_i\sin^2\alpha_i + Q_i\sin\alpha_i\cos\alpha_i \end{bmatrix}.$$

Since the tractions on the boundary of the cell are self-equilibrated, $\boldsymbol{\Sigma}$ must be symmetric ([5]) and consequently we can take from each contribution just its symmetric part, since then $\boldsymbol{\Sigma} = \sum_i\langle\boldsymbol{\sigma}^{i}\rangle = \sum_i\langle\boldsymbol{\sigma}^{i}_{sym}\rangle$. Finally:

$$\boldsymbol{\Sigma} = \sum_i\langle\boldsymbol{\sigma}^{i}\rangle = \frac{1}{|V|}\sum_i l_i\begin{bmatrix} N_i\cos^2\alpha_i - \frac{Q_i\sin 2\alpha_i}{2} & \frac{Q_i\cos 2\alpha_i}{2} + \frac{N_i\sin 2\alpha_i}{2} \\ \text{symm} & N_i\sin^2\alpha_i + \frac{Q_i\sin 2\alpha_i}{2} \end{bmatrix}, \tag{3.4}$$

$\Sigma_M = \frac{1}{2|V|}\sum_i N_i l_i$ and expression for $\boldsymbol{\Sigma}_D$ is obvious.

Since (3.4) does not include constant moments M_i, we can simply conclude, that if in optimal medium <u>bending moments</u> develop, they <u>have zero area</u>. Consequently the contribution $\langle w^i\rangle$ in optimal media is (under the assumptions already taken in Section 2.1):

$$\left\langle w^i \right\rangle = \frac{1}{2|V|E_{bm}} \left[N_i^2 \int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1 + 12 Q_i^2 \int_{-l_i/2}^{l_i/2} \frac{{}^i z_1^2}{t_i^3\left({}^i z_1\right)} d^i z_1 \right]$$

or when the thickness is constant within i-beam:

$$\left\langle w^i \right\rangle = \frac{1}{2|V|E_{bm}} \left[N_i^2 \frac{l_i}{t_i} + Q_i^2 \left(\frac{l_i}{t_i} \right)^3 \right].$$

Now the orientation of the coordinate ${}^i z_1$ is unimportant and we can only assume that $\alpha_i \in \langle 0, \pi)$.

3.2 DERIVATION OF 2D CMB BOUNDS.

Two ways in the bound's proofs can be established, one for general micro-frames (proof A) and the other specified just for micro-truss media with constant thickness within each truss (proof B). We define the following vectors to facilitate expressions in proof B:

$$\mathbf{N} = \left\{ N_1 \sqrt{\frac{l_1}{t_1}}, N_2 \sqrt{\frac{l_2}{t_2}}, ..., N_n \sqrt{\frac{l_n}{t_n}} \right\},$$

$${}^1\mathbf{R} = \left\{ \sqrt{l_1 t_1} \cos^2 \alpha_1, \sqrt{l_2 t_2} \cos^2 \alpha_2, ..., \sqrt{l_n t_n} \cos^2 \alpha_n \right\},$$

$${}^2\mathbf{R} = \left\{ \sqrt{l_1 t_1} \sin^2 \alpha_1, \sqrt{l_2 t_2} \sin^2 \alpha_2, ..., \sqrt{l_n t_n} \sin^2 \alpha_n \right\},$$

$$\mathbf{P} = \left\{ \sqrt{l_1 t_1} \cos(2\alpha_1), \sqrt{l_2 t_2} \cos(2\alpha_2), ..., \sqrt{l_n t_n} \cos(2\alpha_n) \right\},$$

$$\mathbf{Q} = \left\{ \sqrt{l_1 t_1} \sin(2\alpha_1), \sqrt{l_2 t_2} \sin(2\alpha_2), ..., \sqrt{l_n t_n} \sin(2\alpha_n) \right\},$$

$$\mathbf{L} = \left\{ \sqrt{l_1 t_1}, \sqrt{l_2 t_2}, ..., \sqrt{l_n t_n} \right\}.$$

Obviously:

$$\mathbf{P} = {}^1\mathbf{R} - {}^2\mathbf{R}, \quad {}^1\mathbf{R} + {}^2\mathbf{R} = \mathbf{L} \quad \text{and} \quad \|\mathbf{P}\|^2 + \|\mathbf{Q}\|^2 = \|\mathbf{L}\|^2,$$

where $\| \; \|$ denotes the Euclidean norm and n is the number of trusses in the cell. The relative density can be expressed as $s = \|\mathbf{L}\|^2 / |V|$ and relation (3.4) now takes the form:

$$\boldsymbol{\Sigma}^T = \frac{1}{|V|}\mathbf{S}\cdot\mathbf{N}^T = \frac{1}{|V|}\begin{Bmatrix} {}^1\mathbf{R} \\ {}^2\mathbf{R} \\ \mathbf{Q}/2 \end{Bmatrix}\cdot\mathbf{N}^T\,, \tag{3.5}$$

where we used the usual vector representation $\boldsymbol{\Sigma} = \{\Sigma_{11}, \Sigma_{22}, \Sigma_{12}\}$ and **S** is a modified statical matrix. If the local strains are uniform, we can write compatibility conditions in the form:

$$\mathbf{S}^T \cdot \mathbf{E}^T = \mathbf{N}^T / E_{bm}\,, \tag{3.6}$$

where analogously $\mathbf{E} = \{E_{11}, E_{22}, 2E_{12}\}$. Specifications of components of $\boldsymbol{\Sigma}$ from tab. 2.1 introduced into (3.5) results in <u>additional conditions</u> on possible normal forces, while similar specifications of **E** together with (3.6) yield in the necessary <u>maximality conditions</u> on normal forces.

In the proofs we first <u>establish the bound</u> together with <u>maximality and additional conditions</u> on possible internal forces. Then we summarize the <u>necessary and sufficient conditions</u> on optimal media and try to express them, as much as possible, as <u>geometrical conditions</u>.

3.2.1 BULK MODULUS.

<u>Σ^I-macroload.</u>

<u>Proof A</u>. It can be written directly:

$$k^* = \frac{1}{|V|}\cdot\frac{\left(\sum_i \frac{1}{2}N_i l_i\right)^2}{\sum_i\left[N_i^2\int_{-l_i/2}^{l_i/2}\frac{1}{t_i({}^iz_1)}d\,{}^iz_1 + 12Q_i^2\int_{-l_i/2}^{l_i/2}\frac{{}^iz_1^2}{t_i^3({}^iz_1)}d\,{}^iz_1\right]} \le \frac{1}{4|V|}\cdot\frac{\left(\sum_i N_i l_i\right)^2}{\sum_i\left[N_i^2\int_{-l_i/2}^{l_i/2}\frac{1}{t_i({}^iz_1)}d\,{}^iz_1\right]}. \tag{3.7}$$

In addition, we can use Schwarz inequality in the form:

$$\left(\sum_i N_i l_i\right)^2 = \left\{\sum_i \left[N_i \sqrt{\int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1}\right] \cdot \left[\frac{l_i}{\sqrt{\int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1}}\right]\right\}^2 \leq$$

$$\left[\sum_i N_i^2 \int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1\right] \cdot \left[\sum_i \frac{l_i^2}{\int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1}\right]. \tag{3.8}$$

Substituting (3.8) into (3.7) results in:

$$k^* \leq \frac{1}{4|V|} \sum_i \frac{l_i^2}{\int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1} \leq \frac{1}{4|V|} \sum_i \int_{-l_i/2}^{l_i/2} t_i\left({}^i z_1\right) d^i z_1 = \frac{1}{4} s\,, \tag{3.9}$$

where the Schwarz inequality was used again, now in the following integral form:

$$l_i^2 = \left[\int_{-l_i/2}^{l_i/2} \sqrt{t_i\left({}^i z_1\right)} \cdot \sqrt{\frac{1}{t_i\left({}^i z_1\right)}} d^i z_1\right]^2 \leq \left[\int_{-l_i/2}^{l_i/2} t_i\left({}^i z_1\right) d^i z_1\right] \cdot \left[\int_{-l_i/2}^{l_i/2} \frac{1}{t_i\left({}^i z_1\right)} d^i z_1\right]. \tag{3.10}$$

Relation (3.10) states the important conclusion of constant thickness within each beam of an optimal medium, since only in this case the equality in (3.10) is obtained and consequently the equality holds also in the second inequality of (3.9). Obviously, relations analogous to (3.9) are obtained in estimations related to the others stiffness constants, thus (3.10) can be also applied there. In the sequel, from now on the assumption of constant thickness within each beam is adopted.

Maximality conditions on possible internal forces are the ones that ensure equalities in the above relations (3.7-9), i.e. only normal forces can be developed and N_i/t_i must be constant within the cell. Thus optimal media can be taken as micro-trusses and the maximality condition is written as:

$$\mathbf{N} // \mathbf{L}\,. \tag{3.11}$$

Additional conditions, with (3.11) substituted, can be written as geometrical ones:

$${}^{1}\mathbf{R}\cdot\mathbf{L}^{T} = {}^{2}\mathbf{R}\cdot\mathbf{L}^{T} \ \&\ \mathbf{Q}\perp\mathbf{L} \ \Leftrightarrow \ \left\|{}^{1}\mathbf{R}\right\| = \left\|{}^{2}\mathbf{R}\right\| \ \&\ \mathbf{Q}\perp\mathbf{L}\,. \tag{3.12}$$

(3.11-12) are necessary and sufficient conditions on optimal media for $\mathbf{\Sigma}^{I}$-macroload. Unfortunately (3.11) cannot be expressed only by geometrical parameters. Simple examples of optimal media with structural hexagonal symmetry are in fig. 3.2 and with structural square symmetry in fig. 3.3. Some interesting conclusions about optimal media with structural hexagonal symmetry are in [4].

Proof B:

$$k^{*} = \frac{1}{4|V|}\cdot\frac{\left(\mathbf{N}\cdot\mathbf{L}^{T}\right)^{2}}{\|\mathbf{N}\|^{2}} \le \frac{s}{4}\,,$$

consequently the maximality condition is (3.11) and the rest of the discussion is the same as before.

The other possibility in proof B is to start with (3.6), taking into account tab. 2.1. Then (3.11) is derived directly and we see, that in this case (3.6) ensure not only necessary but also sufficient maximality conditions as it is obvious from (3.11) that now “zero” trusses are excluded.

3.2.2 SHEAR MODULI.

$\mathbf{\Sigma}^{II}$-macroload.

Proof A. A useful condition in this case is(: stands for the tensorial multiplication):

$$\left\langle \boldsymbol{\sigma}_{D}^{i} \right\rangle : \left\langle \boldsymbol{\sigma}_{D}^{i} \right\rangle = \frac{l_{i}^{2}}{2|V|^{2}}\left(N_{i}^{2} + Q_{i}^{2}\right),$$

but following the similar estimations as for $\mathbf{\Sigma}^{I}$-macroload, we can obtain only a bound equal to s/4, with maximality conditions that cannot be satisfied by any medium. This happens due to the nonlinearity of the bound.

Σ^{III}-macroload.

Proof A. Let us express first:

$$\left(\Sigma_{D,11}-\Sigma_{D,22}\right)^2=\left[\sum_i\left(\left\langle\sigma^i_{D,11}\right\rangle-\left\langle\sigma^i_{D,22}\right\rangle\right)\right]^2=\left[\sum_i\left(\sqrt{l_i t_i}\right)\cdot\left(\frac{\left\langle\sigma^i_{D,11}\right\rangle-\left\langle\sigma^i_{D,22}\right\rangle}{\sqrt{l_i t_i}}\right)\right]^2\le$$

$$\left(\sum_i l_i t_i\right)\cdot\left[\sum_i\frac{\left(\left\langle\sigma^i_{D,11}\right\rangle-\left\langle\sigma^i_{D,22}\right\rangle\right)^2}{l_i t_i}\right]=\left(\sum_i l_i t_i\right)\cdot\left\{\frac{1}{|V|^2}\sum_i\left[N_i\cos(2\alpha_i)+Q_i\sin(2\alpha_i)\right]^2\frac{l_i}{t_i}\right\}.$$

Step by step estimations yield in:

$$^1G^*\le\frac{1}{4|V|}\cdot\frac{\left(\sum_i l_i t_i\right)\cdot\left\{\sum_i\left[N_i\cos(2\alpha_i)+Q_i\sin(2\alpha_i)\right]^2\frac{l_i}{t_i}\right\}}{\sum_i\left[N_i^2\frac{l_i}{t_i}+Q_i^2\left(\frac{l_i}{t_i}\right)^3\right]}\le$$

$$\frac{1}{4|V|}\cdot\frac{\left(\sum_i l_i t_i\right)\cdot\left[\sum_i\left(N_i^2+Q_i^2\right)\frac{l_i}{t_i}\right]}{\sum_i\left[N_i^2\frac{l_i}{t_i}+Q_i^2\left(\frac{l_i}{t_i}\right)^3\right]}\le\frac{1}{4|V|}\cdot\left(\sum_i l_i t_i\right)=\frac{s}{4}.$$

Thus maximality conditions show again that we can assume only micro-truss media, consequently we will not specify any other condition now, since the direct determination from proof B is easier.

Proof B:

$$^1G^*=\frac{1}{4|V|}\cdot\frac{\left(\mathbf{N}\cdot\mathbf{P}^T\right)^2}{\|\mathbf{N}\|^2}=\frac{1}{4|V|}\cdot\|\mathbf{P}\|^2\cos^2\left(\mathbf{N},\mathbf{P}\right)\cdot\frac{\|\mathbf{L}\|^2}{\|\mathbf{L}\|^2}=\frac{s}{4}\cdot\frac{\|\mathbf{P}\|^2}{\|\mathbf{P}\|^2+\|\mathbf{Q}\|^2}\cos^2\left(\mathbf{N},\mathbf{P}\right)\le\frac{s}{4},\qquad(3.13)$$

where $\left(\mathbf{N},\mathbf{P}\right)$ denotes the angle between the vectors **N** and **P**. Maximality conditions from (3.13) result in $\mathbf{N}//\mathbf{P}$ and $\mathbf{Q}=\mathbf{0}$; additional ones (from (3.5)) are $\mathbf{N}\perp\mathbf{L}$ and $\mathbf{N}\perp\mathbf{Q}$. In this case the optimal medium can be fully geometrically specified. From $\mathbf{Q}=\mathbf{0}$ we directly obtain that only horizontal and vertical trusses can be in the cell, thus it is convenient to assume that the cell is rectangular, with in-line arrangement (fig. 3.4). Since we have to exclude zero trusses, equilibrium in crossings results in continuous bars passing through the cell. Additionally, $\mathbf{P}\perp\mathbf{L}$ yield in:

$$\sum_{\text{horiz}} l_i t_i = \sum_{\text{vert}} l_j t_j \,.$$

Thus the total vertical trusses' area must be the same as the area of the horizontal trusses. Using the maximality condition $\mathbf{N}//\mathbf{P}$, e.g. forces in horizontal bars are positive and in vertical ones negative proportions of the corresponding thickness. Again due to equilibrium in crossings, forces must be equal in each passing bar, which enforce <u>the thickness</u> to be <u>constant within the bar</u>, which completes the full geometrical specification. We call the optimal medium the <u>uniform perpendicular lattice</u> (UPL). Let us denote the dimensions of the cell L and H. Thus UPL group is characterized in the following way:

- the cell consists only of horizontal and vertical continuous bars (beams),
- condition $L\sum_{i=1}^{n} t_{L,i} = H\sum_{j=1}^{m} t_{H,j}$ is satisfied, where n and m is the number of horizontal and vertical bars, respectively. The other notation is marked in fig. 3.4.

No other medium can be optimal for $\boldsymbol{\Sigma}^{III}$-macroload. The simplest example from UPL group is the regular square lattice (from fig. 3.3).

Starting from (3.6) we would get the maximality condition $\mathbf{N}//\mathbf{P}$, but not $\mathbf{Q}=\mathbf{0}$. Thus in this case (3.6) yield only in the necessary but not sufficient maximality conditions.

<u>$\boldsymbol{\Sigma}^{IV}$-macroload.</u>

Since by rotation of global coordinates by 45 degrees the two shear moduli exchange its position in the matrix of effective elastic stiffnesses (2.1), the situation now is exactly the same as for $\boldsymbol{\Sigma}^{III}$-macroload, just rotated. The bound is the same, the optimal media are the UPL media rotated by 45 degrees and we name them the <u>uniform diagonal lattices</u> (UDL), since the bars' directions are in directions of diagonals of the unit square. The simplest example from UDL group is the regular diagonal lattice from fig. 3.3.

Σ^{II}-macroload.

Going back to Σ^{II}-macroload we restrict ourselves to micro-trusses to obtain at least the linearized bound. Suppose that we have already an optimal medium for Σ^{II}-macroload. By application of the Σ^{III} and Σ^{IV}-macroloads separately, we get the best estimates as (since vectors $\|\mathbf{Q}\|$ or $\|\mathbf{P}\|$ cannot be zero):

$$G^* \leq \frac{s}{4}\cdot\frac{\|\mathbf{P}\|^2}{\|\mathbf{P}\|^2+\|\mathbf{Q}\|^2} \ \&\ G^* \leq \frac{s}{4}\cdot\frac{\|\mathbf{Q}\|^2}{\|\mathbf{P}\|^2+\|\mathbf{Q}\|^2} \Rightarrow G^*_+=s/8. \tag{3.14}$$

Maximality conditions are the following:

$$^{III}\mathbf{N}\,//\,\mathbf{P}\ \&\ {}^{IV}\mathbf{N}\,//\,\mathbf{Q} \Rightarrow \mathbf{N}=\lambda_1\mathbf{P}+\lambda_2\mathbf{Q}, \tag{3.15}$$

where $^{III}\mathbf{N}$ and $^{IV}\mathbf{N}$ correspond to the normal forces developed under the Σ^{III} and Σ^{IV}-macroloads, respectively, and λ_1, λ_2 are coefficients. Additional conditions are thus:

$$\begin{gathered} ^{III}\mathbf{N}\perp\mathbf{L},\ {}^{III}\mathbf{N}\perp\mathbf{Q},\ {}^{IV}\mathbf{N}\perp{}^{1}\mathbf{R},\ {}^{IV}\mathbf{N}\perp{}^{2}\mathbf{R} \Rightarrow \\ \|{}^{1}\mathbf{R}\| = \|{}^{2}\mathbf{R}\|\ \&\ \mathbf{Q}\perp{}^{1}\mathbf{R}\ \&\ \mathbf{Q}\perp{}^{2}\mathbf{R}, \end{gathered} \tag{3.16}$$

where the geometrical ones on the second line of (3.16) ensure the opposite implication in (3.15). Thus:

$$^{III}\mathbf{N}\,//\,\mathbf{P}\ \&\ {}^{IV}\mathbf{N}\,//\,\mathbf{Q} \Leftrightarrow \mathbf{N}=\lambda_1\mathbf{P}+\lambda_2\mathbf{Q}.$$

It remains to prove, that at least one optimal medium exists. We could do it by showing the known optimal media, e.g. the ones with structural hexagonal symmetry are the regular triangle lattice and the triangle & honeycomb lattice from fig. 3.2, but we also present another interesting proof in the following section.

Conditions (3.15-16) state again necessary and sufficient settings on optimal micro-truss media, but their full geometrical specification is difficult. All G*-optimal media taken as micro-frames exhibit bending contribution that puts the G*-value slightly above the linearized bound, but there are no G*-optimal media, which taken as micro-trusses would not be characterized by necessary and sufficient conditions (3.15-16). As we proved in [7], the bending contribution is negligible.

3.3 SUPERPOSITION OF THE RESULTS.

Let us take two $\mathbf{\Sigma}^k$-optimal micro-truss media, k=I, II, III or IV, with the same cell shape and size and combine them together in the sense, that if the trusses cross, we assume a crossing point there. Let us adjust the value of the macroloads s.t. ${}^1\mathbf{\Sigma}^k$ and ${}^2\mathbf{\Sigma}^k$ applied on the original first and second cells, respectively, produce the same macrostrain i.e. the same local uniform strain. Now let us introduce the superposition of the macroloads ${}^1\mathbf{\Sigma}^k + {}^2\mathbf{\Sigma}^k$ to the combined cell and suppose, that the internal normal forces are $\mathbf{N} = {}^1\mathbf{N} + {}^2\mathbf{N}$, where ${}^i\mathbf{N}$ are normal forces developed in i-cell under ${}^i\mathbf{\Sigma}^k$, i=1,2. Equilibrium in the combined cell is obviously satisfied. Consequently for the stiffness constant, that corresponds to the applied macroload hold:

$$C^* \leq {}^1C^* + {}^2C^*, \tag{3.17}$$

where the superscripts correspond to the first and to the second medium. Since the local strain was the same for both cells, it is the same in combined cell as well, thus compatibility is satisfied and consequently equality holds in (3.17).

As an example let us take <u>the square lattice with two oblique bars</u> from fig. 3.3, that was obtained by combination of the regular square lattice (thickness t_1) and the regular diagonal lattice (thickness t_2), both in fig. 3.3. It holds (${}^1s$ and ${}^2s$ are the original relative densities of the two cells):

$$k^* = {}^1k^* + {}^2k^* = \frac{{}^1s}{4} + \frac{{}^2s}{4} = \frac{s}{4}, \text{ since } s = {}^1s + {}^2s$$

More generally, when the original media are neither micro-trusses nor optimal, the strain developed in the cells is not necessarily uniform. If after combination of the cells, compatibility is obtained for $\mathbf{N}={}^{1}\mathbf{N}+{}^{2}\mathbf{N}$, the equality in (3.17) holds as well. Thus, for the same example taken as micro-frame:

$$ {}^{1}G^{*}={}^{11}G^{*}+{}^{21}G^{*}=\frac{{}^{1}s}{4}+\frac{{}^{2}s^{3}}{16}=\frac{1}{2}t_{1}+\sqrt{2}t_{2}^{3}, $$

$$ {}^{2}G^{*}={}^{12}G^{*}+{}^{22}G^{*}=\frac{{}^{1}s^{3}}{16}+\frac{{}^{2}s}{4}=\frac{1}{2}t_{1}^{3}+\frac{\sqrt{2}}{2}t_{2}. $$

As a consequence, the relation to ensure effective isotropy of the square lattice with two oblique beams is $t_1=\sqrt{2}t_2$, as obviously follows from $G^{*}={}^{1}G^{*}={}^{2}G^{*}$ (see fig. 3.5). Then:

$$ {}^{1}s={}^{2}s=\frac{s}{2} \text{ and } G^{*}=\frac{1}{8}s+\frac{1}{128}s^{3}. $$

Going back to (3.14) one can see, that we found a G*-optimal medium.

4 Conclusion.

The presented new methodology for determination of bounds on effective constitutive constants of CM has many advantages and is fully general. It can be thus used CMB with more general effective behavior, than the ones presented in this paper.

In addition, we summarize some interesting facts. First, the maximality and additional conditions are in tab. 4.1.

Macroload	Maximality conditions	Additional conditions
$\boldsymbol{\Sigma}^{\mathrm{I}}$	$\mathbf{N}\,//\,\mathbf{L}$	$\left\Vert {}^{1}\mathbf{R}\right\Vert=\left\Vert {}^{2}\mathbf{R}\right\Vert$ & $\mathbf{Q}\perp\mathbf{L}$
$\boldsymbol{\Sigma}^{\mathrm{II}}$	$\mathbf{N}=\lambda_1\mathbf{P}+\lambda_2\mathbf{Q}$	$\left\Vert {}^{1}\mathbf{R}\right\Vert=\left\Vert {}^{2}\mathbf{R}\right\Vert$ & $\mathbf{Q}\perp{}^{1}\mathbf{R}$ & $\mathbf{Q}\perp{}^{2}\mathbf{R}$
$\boldsymbol{\Sigma}^{\mathrm{III}}$	$\mathbf{N}\,//\,\mathbf{P}$ & $\mathbf{Q}=\mathbf{0}$	not included due to the full geometrical specification
$\boldsymbol{\Sigma}^{\mathrm{IV}}$	$\mathbf{N}\,//\,\mathbf{Q}$ & $\mathbf{P}=\mathbf{0}$	not included due to the full geometrical specification

Tab. 4.1 – Maximality and additional conditions.

From additional conditions, we can conclude, that there may exist Σ^{I}-optimal media, which are not Σ^{II}-optimal. It is, e.g., the regular honeycomb lattice from fig. 3.2. For the discussion of the opposite statement, i.e. of the existence of media with “large” G* and “small” k* see [6].

Second, some other comparisons are in tab. 4.2, where we also included the column with the tension test (because the conclusions there are interesting) and we did not include a column with Σ^{II}-macroload.

Macroload	Optimal media	Under applied macroload the medium is …			
		Σ^{I}	Σ^{III}	Σ^{IV}	Tension test
Σ^{I}	UPL, UDL, others without full geom. spec.	optimal	optimal	optimal	not unique cathegory
Σ^{II}	without full geom. spec.	see [6]	optimal	optimal	see [6]
Σ^{III}	UPL	optimal	optimal	kinematical mechanism	optimal
Σ^{IV}	UDL	optimal	kinematical mechanism	optimal	kinematical mechanism

Tab. 4.1 – Final comparisons.

Other interesting facts are: media from UPL group have zero effective Poisson’s ratio and Σ^{I} & Σ^{II}-optimal media have the effective Poisson’s ratio equal to 1/3. This statement is already published in [3].

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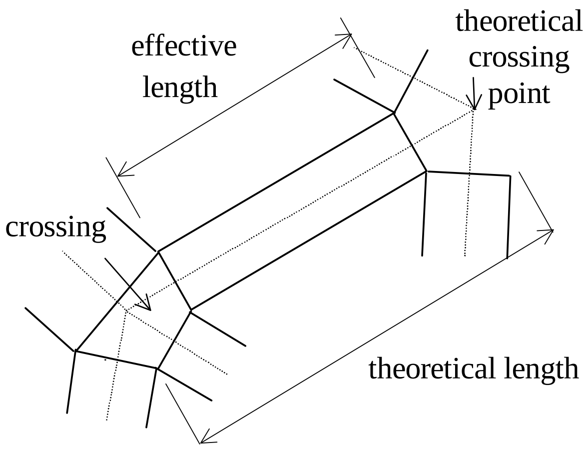

Fig. 2.1 – Introduction of theoretical and effective lengths.

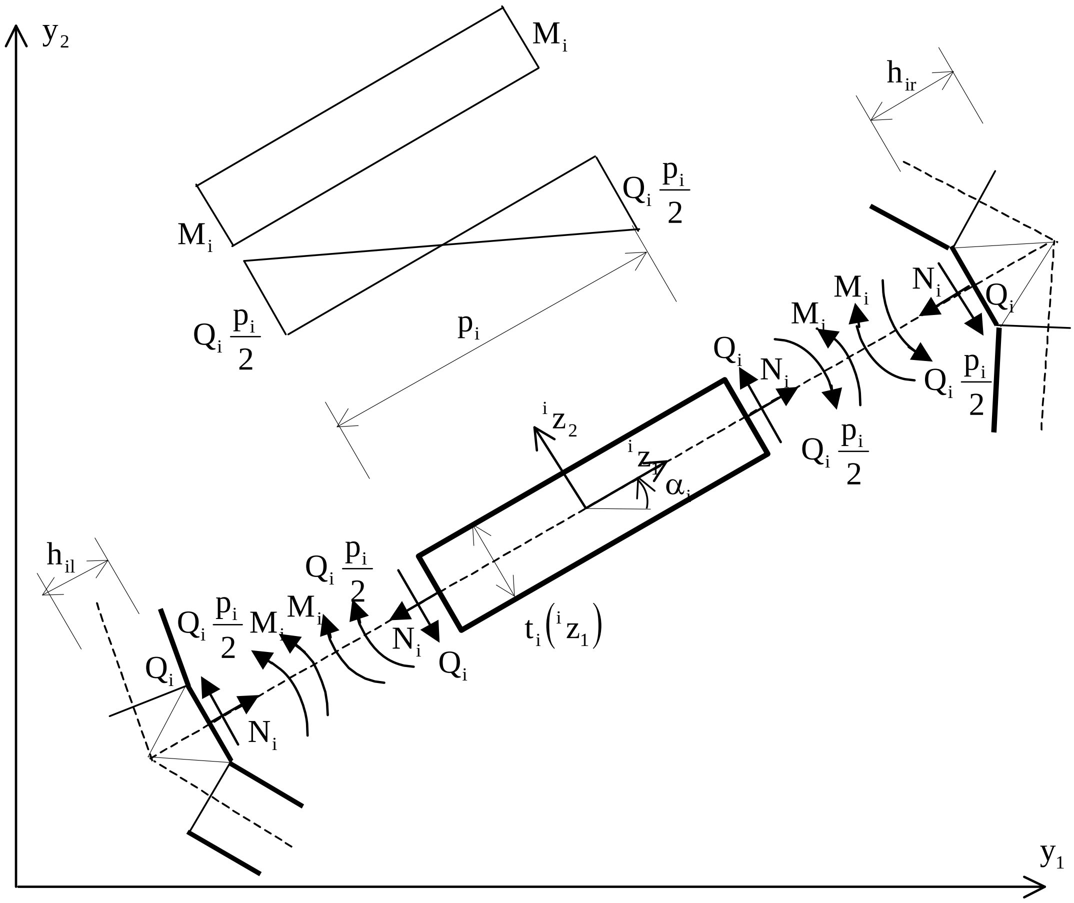

Fig. 3.1 – i-beam's location in the cell and its internal forces and moments.

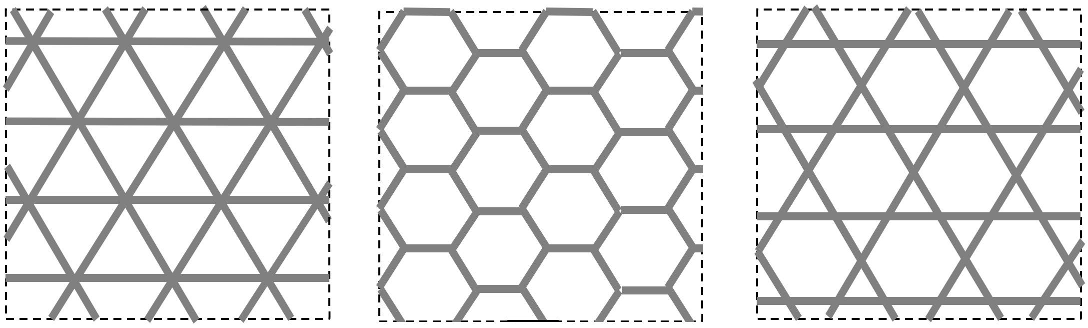

Fig. 3.2 – Σ^{I}-optimal media with structural hexagonal symmetry.

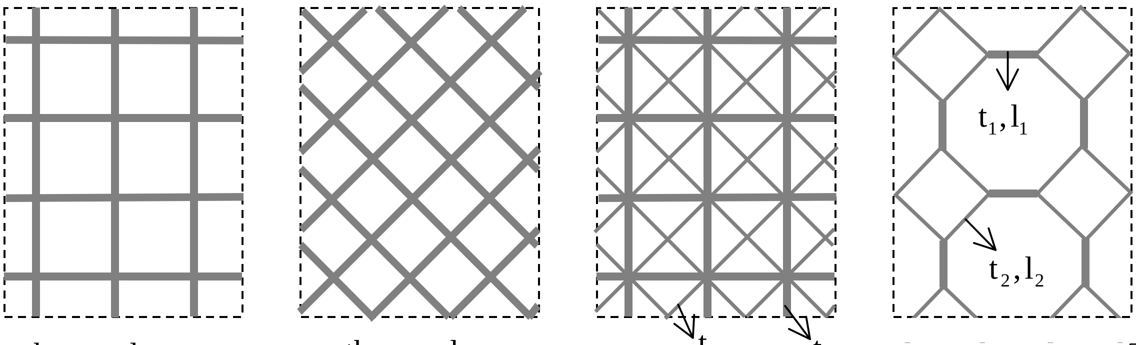

Fig. 3.3 – Σ^{I}-optimal media with structural square symmetry.

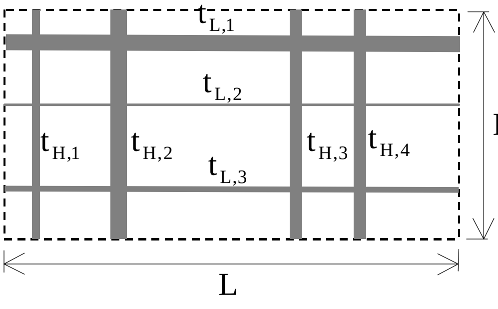

Fig. 3.4 – Σ^{III}-optimal media – UPL group.

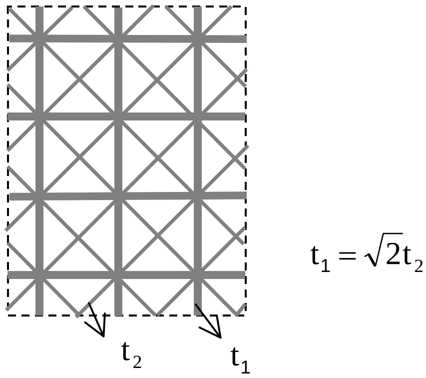

Fig. 3.5 - Σ^{II}-optimal medium with structural square symmetry.

LIST OF FIGURES.