International Conference on Engineering Vibration
Ljubljana, Slovenia, 7-10 September 2015

UNIFORMLY MOVING LOAD ON A BEAM SUPPORTED BY A FOUNDATION WITH FINITE DEPTH

Zuzana Dimitrovová*[1]

[1]Departamento de Engenharia Civil, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa and IDMEC, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
zdim@fct.unl.pt

Keywords: Transverse Vibration, Critical Velocity, Active Soil Depth, Visco-elastic Foundation, Moving Load, Analytical Solution.

Abstract. *In this paper some new analytical solutions for moving load problems are presented. In the first part, a new analytical formula for the critical velocity of a uniformly moving force on a beam supported by a foundation of finite depth is given. This formula just indicates a more realistic estimate of the critical velocity because it considers only the part of the foundation that is dynamically activated. The formula accounts for the effect of the normal force acting on the beam and the shear influence of the foundation. The critical velocity is expressed as a function of the mass ratio that relates the foundation mass with the beam mass. The new formula approaches the classical formula for the low mass ratio and the velocity of propagation of shear waves for the high mass ratio. In the second part, a new analytical formula is presented for the deflection shape of an infinite beam that is traversed by a moving mass and supported by a visco-elastic foundation. In such a case the deflection shape resembles the one associated with the moving force with an additional oscillation around it. The frequency of this oscillation is induced by the foundation characteristics and the amplitude can by derived analytically. It is also shown that if the force associated with the mass has a harmonic component, then its frequency is superposed with the one induced by the foundation.*

1 INTRODUCTION

The investigations on moving load problems were initiated since the early 20th century, when increased importance was attributed to the railway lines performance. Since then, numerous studies have been published on this subject. At first, simple models have been undertaken by analytical and semi-analytical approaches, because of the lack of advanced methods capable to deal with more complex formulations. But these solutions did not lose their utility, because they possess several advantages: they cover only the relevant data, making them easier to analyse; the parameter dependence of the results is preserved, permitting direct sensitivity analysis and providing physical insight into the problem; the numerical evaluation of the results can be carried out for the places of interest only (both in space and time), and more importantly, solving realistic and complicated three-dimensional finite element models still presents some difficulties in form of the high computational cost; necessity to solve the problem over the whole time domain; large number of results to analyse; need for special boundary conditions; several uncertainties in the input data and in the level of discretization, etc.

Regardless the excessive number of published works, there are still some unsolved issues. The aim of this paper is therefore to review some analytical approaches and provide some missing solutions. In Section 2 a simplified model of a beam supported by a foundation with finite depth and subjected to a moving force is introduced. A new analytical formula for the critical velocity of the uniformly moving force is presented. The formula accounts for the effect of the normal force acting on the beam and the shear influence of the foundation. In Section 3 the moving mass problem on a finite beam is reviewed. Then a new analytical formula is presented for the deflection shape of an infinite beam that is traversed by a moving mass and supported by a visco-elastic foundation. In such a case the deflection shape resembles the one associated with the moving force and an additional oscillation around it is induced by the mass. The frequency of this oscillation depends on the foundation characteristics and the amplitude can be derived analytically. It is also proven that if the force associated with the mass has a harmonic component, then its frequency is superposed on the one induced by the foundation. The paper is concluded in Section 4.

2 MOVING FORCE

The problem of a uniformly moving force on a beam supported by a visco-elastic foundation of the Winkler or Pasternak type has an analytical solution for finite as well as infinite beams. Fully analytical closed form solution is only available for simply supported finite beam, thus other cases must be accompanied by a numerical solution of an additional equation, but otherwise the deflection shape of the beam can be presented in a closed explicit form. The most severe simplifications that contradict the railway lines applications are: (i) missing inertial effects of the foundation; and (ii) linearity of the springs representing the foundation, in the sense that the foundation is not tensionless, as it should be.

In this paper one of the possible extensions of the simple beam model is presented, namely, a foundation of a finite depth with inertia and shear effects is introduced. Such a finite depth corresponds to an effective depth of the foundation that is dynamically activated. A uniform motion of a constant vertical force P along a horizontal infinite beam posted on an elastic foundation of finite depth H is assumed (Figure 1). Simplifications for the analysis of beam vertical vibrations are outlined as follows:

(i) the beam obeys linear elastic Euler-Bernoulli theory;

(ii) the beam vertical displacement is measured from the equilibrium deflection position caused by the beam weight;

(iii) the force velocity is maintained constant and no restriction is imposed on its magnitude;

(iv) the foundation is represented by a finite strip of width *b* under plane strain condition.

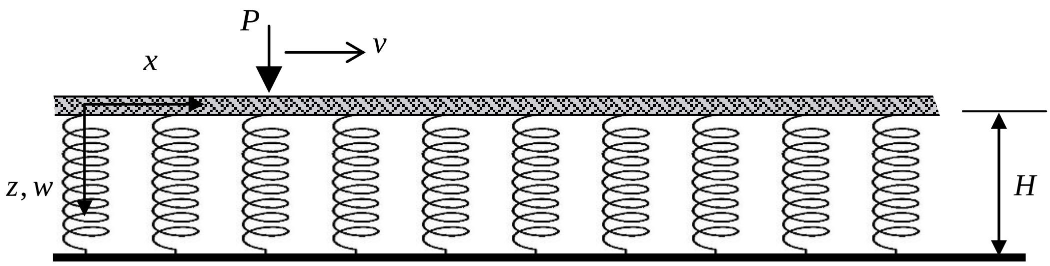

Figure 1: Infinite beam on an elastic foundation of finite depth represented by frequency dependent Winkler's springs, subjected to a moving force.

The governing equation for transverse vibrations $w(x,t)$ can be written as:

$$EIw_{,xxxx} + Nw_{,xx} + mw_{,tt} + c_b w_{,t} + p_s = P\delta(x - vt) \tag{1}$$

where *EI*, *m*, c_b and *N* stand for the bending stiffness, mass per unit length and coefficient of viscous damping of the beam and for the axial force acting on the beam, considered positive when inducing compression. δ is the delta Dirac function, *v* is the force velocity and p_s is the foundation pressure that will be substituted later. Spatial variables are *x* and *z* and *t* is the time. Derivatives are designated by the respective variable in subscript position, preceded by a comma. Initial conditions are assumed as homogeneous.

Some preliminary results are presented in [1]. Further extensions of the analytical solutions presented in [1] are still under review [2], therefore only main features will be given here. The vertical soil displacement is introduced as $u(x,z,t)$, and the necessary boundary conditions are:

$$u(x,0,t) = w(x,t) \text{ and } u(x,H,t) = 0 \tag{2}$$

Then the soil dynamic equilibrium in the vertical direction with the shear influence in a simplified form can be written as:

$$\bar{\rho} u_{,tt} + c_f u_{r,t} = \bar{k}_{st} H u_{,zz} + \bar{G} u_{,xx} \tag{3}$$

where $\bar{\rho}$ and $\bar{G}$ are the density and shear modulus of the foundation soil, where the upper bar means that these values are related to the foundation strip, i.e. multiplied by *b*. c_f is the viscous damping coefficient of the foundation, which can only be introduced correctly if the relative vertical displacement u_r is used. $\bar{k}_{st}$ is the static value of the foundation modulus, $k_{st} = E^{oed} / H$, where E^{oed} is the oedometer modulus of the soil and the upper bar has the same meaning as before. Eqs. (1, 3) can be simplified by introduction of the moving coordinate $s = x - vt$ and further restricted to the steady state conditions:

$$EIw_{,ssss} + \left(N + mv^2\right) w_{,ss} - vc_b w_{,s} + p_s = P\delta(s) \tag{4}$$

$$\bar{\rho} v^2 u_{,ss} - c_f v u_{r,s} = \bar{k}_{st} H u_{,zz} + \bar{G} u_{,ss} \tag{5}$$

Further alterations will involve dimensionless parameters. Displacement components can be divided by the static displacement w_{st} to get dimensionless $\hat{u}_r$ and $\hat{w}$. In addition:

$$\chi = \sqrt[4]{\frac{\bar{k}_{st}}{4EI}}\,,\; w_{st} = \frac{P\chi}{2\bar{k}_{st}}\,,\; \eta_b = \frac{c_b}{2\sqrt{m\bar{k}_{st}}}\,,\; \eta_f = \frac{c_f H}{\sqrt{\bar{k}_{st} m}}\,,\; \vartheta_s = \frac{v_s}{v_{cr}}\,,\; \alpha = \frac{v}{v_{cr}} \tag{6}$$

and also

$$v_{cr} = \sqrt[4]{\frac{4\bar{k}_{st}EI}{m^2}} = \frac{1}{\chi}\sqrt{\frac{\bar{k}_{st}}{m}}\,,\; \xi = \chi s\,,\; z = \zeta H\,,\; \hat{u}_r = \sum_{j=1}^{\infty} U_j \sin(j\pi\zeta)\,,\; \eta_N = \frac{N}{N_{cr}} = \frac{N}{2\sqrt{k_{st}EI}} \tag{7}$$

will be used. The soil equilibrium:

$$\begin{aligned} &\left(1-\left(\frac{\vartheta_s}{\alpha}\right)^2\right)\sum_{j=1}^{\infty} U_{j,\xi\xi}\sin(j\pi\zeta) - \frac{\eta_f}{\alpha\mu^2}\sum_{j=1}^{\infty} U_{j,\xi}\sin(j\pi\zeta) + \left(\frac{j\pi}{\alpha\mu}\right)^2 \sum_{j=1}^{\infty} U_j \sin(j\pi\zeta) \\ &= -\left(1-\left(\frac{\vartheta_s}{\alpha}\right)^2\right)(1-\zeta)\,\hat{w}_{,\xi\xi} \end{aligned} \tag{8}$$

can be solved by the Fourier transform:

$$U_j{*} = \frac{\omega^2 \dfrac{2}{j\pi}\left(1-\left(\dfrac{\vartheta_s}{\alpha}\right)^2\right)}{-\omega^2\left(1-\left(\dfrac{\vartheta_s}{\alpha}\right)^2\right) - i\omega\dfrac{\eta_f}{\alpha\mu^2} + \left(\dfrac{j\pi}{\alpha\mu}\right)^2} W{*} \tag{9}$$

and then the foundation pressure p_s can be obtained from the contact condition:

$$p_s = -(1+i\eta_h)\bar{k}_{st}\left(\sum_{j=1}^{\infty} j\pi u_j - w\right) \tag{10}$$

where η_h stands for the hysteretic damping of the foundation. Finally:

$$\hat{w}_{,\xi\xi\xi\xi} + 4\left(\alpha^2+\eta_N\right)\hat{w}_{,\xi\xi} - 8\eta_b\alpha\hat{w}_{,\xi} + 4(1+i\eta_h)\left(\hat{w} - \sum_{j=1}^{\infty} j\pi U_j\right) = 8\delta(\xi) \tag{11}$$

which can also be solved by the Fourier transform. At the end, the inverse transform to the time domain has to be done numerically, because of the trigonometric functions that will appear in the Fourier image. By examinations of the deflection shapes, the improved formula for the critical velocity $v_{sh,N}$ is obtained as:

$$v_{sh,N} = v_{cr}\left[\left(\sqrt{1-\eta_N} - \vartheta_s\right)\sqrt{\frac{2}{2+\mu^{2+\sqrt{\vartheta_s}}}} + \vartheta_s\right] \tag{12}$$

where the mass ratio is defined as $\mu = \sqrt{\bar{\rho}H/m}$.

The new formula was checked numerically and some results are presented in Figure 2. In this figure $\eta_N = 0.5$ and the mass ratio is varied between 0 and 10. Three different shear ratios were tested. It is seen that for the low mass ratio the critical velocity approaches the classical formula and for the high mass ratio it reaches the velocity of propagation of the shear waves in the foundation soil.

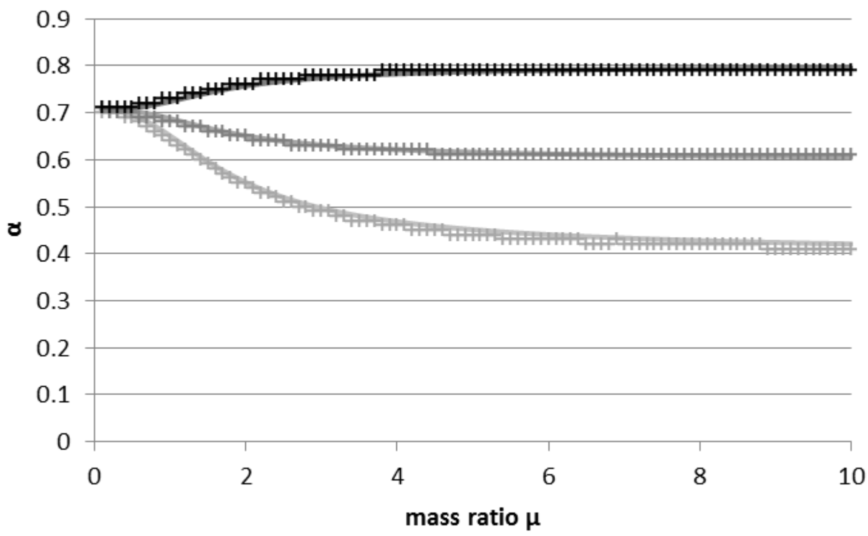

Figure 2: Critical velocity as a function of the mass ratio: full lines represent the prediction according to formula (12) for $\vartheta_s = 0.4$ (light grey), $\vartheta_s = 0.6$ (medium grey) and $\vartheta_s = 0.8$ (dark grey), respectively, crosses represent the numerical values.

3 MOVING MASS

Regarding the finite beam, the solution of the problem of the moving mass can be expressed by the eigenvalue expansion method. The first solution of this kind was presented in [3]. Solution presented in [4] is a semi-analytical one and can be easily extended to Timoshenko beams or account for the effect of the normal force, etc. The disadvantage is that the equations in the modal space are coupled, thus even if no discretization is involved and mode shapes are introduced in their analytical form, the generalized coordinate must be solved numerically.

The equation of motion for the unknown vertical deflection field $w(x,t)$ can be written as:

$$EIw_{,xxxx} + mw_{,tt} + c_b w_{,t} + kw = p(x,t) \tag{13}$$

where k is the Winkler's constant of the foundation. For a constant mass M and an associated constant force P with harmonic component, the loading term $p(x,t)$ can be written as:

$$p(x,t) = \left(P + P_0 \sin\left(\omega_f t + \varphi_f\right) - Mw_{0,tt}(t)\right)\delta(x - vt) \tag{14}$$

Function sine is used to keep this term in the real domain, because then, the whole solution can be solved in the real domain. The mass displacement $w_0(t) = w(vt,t)$, i.e. the mass is always in contact with the beam and its horizontal position is determined by the velocity. Initial conditions are considered as homogeneous. x has its origin at the left extremity of the beam and zero time corresponds to load position at x=0. For the solution it is necessary to remove the additional unknown $w_0(t)$ and express it in terms of the unknown field $w(x,t)$ as:

$$\left(P + P_0 \sin\left(\omega_f t + \varphi_f\right) - M\left(w_{,tt}(x,t) + 2vw_{,xt}(x,t) + v^2 w_{,xx}(x,t)\right)\right)\delta(x - vt) \tag{15}$$

Thus

$$EIw_{,xxxx}+mw_{,tt}+c_bw_{,t}+kw+M\delta\left(x-vt\right)\left(w_{,tt}\left(x,t\right)+2vw_{,xt}\left(x,t\right)+v^2w_{,xx}\left(x,t\right)\right)$$
$$=\left(P+P_0\sin\left(\omega_ft+\varphi_f\right)\right)\delta\left(x-vt\right) \qquad (16)$$

Several boundary conditions can be considered.

$$w\left(0,t\right)=0\,,\ w_{,xx}\left(x,t\right)\Big|_{x=0}=0\,,\ w\left(L,t\right)=0\,,\ w_{,xx}\left(x,t\right)\Big|_{x=L}=0\,, \qquad (17)$$

$$w\left(0,t\right)=0\,,\ w_{,x}\left(x,t\right)\Big|_{x=0}=0\,,\ w_{,xx}\left(x,t\right)\Big|_{x=L}=0\,,\ w_{,xxx}\left(x,t\right)\Big|_{x=L}=0\,, \qquad (18)$$

Above these conditions are written for simply supported beam and left cantilever, respectively, and the beam length is designated as L. Solution of the problem can be obtained by implementing the Fourier method of variable separation and assuming the existence of free harmonic vibrations:

$$w\left(x,t\right)=w\left(x\right)e^{i\omega t}\,,\ i=\sqrt{-1} \qquad (19)$$

The frequency ω of these vibrations is named as the natural frequency and it is determined from the eigenvalue problem obtained from the homogeneous governing equation. Then the transient response in the time domain is expressed as infinite series of these modes, where each vibration mode (function of the spatial coordinate x) is multiplied by a generalized displacement (modal coordinate, amplitude function) that is a function of time.

$$w\left(x,t\right)=\sum_{j=1}^{\infty}q_j\left(t\right)w_j\left(x\right) \qquad (20)$$

The same designation "w" can be used for the deflection field as well as for the vibration modes, because the vibration modes are distinguished by the corresponding subscript. As usual, the modes are normalized by mass, therefore:

$$\delta_{jk}=\int_0^L mw_j\left(x\right)w_k\left(x\right)dx \qquad (21)$$

where δ_{jk} is the Kronecker delta. Modal expansion is commonly governed by undamped vibration modes, because this allows their determination within the real domain and completeness of the eigenspace is guaranteed. Unfortunately, equations in modal space are coupled as shown below:

$$\mathbf{M}\left(t\right)\cdot\ddot{\mathbf{q}}\left(t\right)+\mathbf{C}\left(t\right)\cdot\dot{\mathbf{q}}\left(t\right)+\mathbf{K}\left(t\right)\cdot\mathbf{q}\left(t\right)=\tilde{\mathbf{q}}\left(t\right) \qquad (22)$$

In the equation above matrices **M**, **C**, **K** are defined by introduction of vibration modes in their exact analytical form (without any discretization) as:

$$M_{ij}=\delta_{ij}+Mw_i\left(vt\right)w_j\left(vt\right) \qquad (23)$$

$$C_{ij}=2Mvw_i\left(vt\right)w_{j,x}\left(vt\right)+\delta_{ij}\frac{c_b}{m} \qquad (24)$$

$$K_{ij}=Mv^2w_i\left(vt\right)w_{j,xx}\left(vt\right)+\delta_{ij}\omega_j^2 \qquad (25)$$

$$\tilde{q}_j=\left(P+P_0\sin\left(\omega_ft+\varphi_f\right)\right)w_j\left(vt\right) \qquad (26)$$

Standard techniques can be used for wave numbers λ_j / L determination, it is only recalled that

$$\omega_j = \sqrt{\left(\frac{\lambda_j}{L}\right)^4 \frac{EI}{m} + \frac{k}{m}} \tag{27}$$

The system (22) cannot be solved analytically, but numerically. For numerical solution in Matlab code, the system should be written in the state space form as:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \tilde{\mathbf{q}} \end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\tilde{\mathbf{q}} \end{Bmatrix} \tag{29}$$

Computational time increases exponentially with the number of modes involved. Precision of a solution obtained for a certain number of modes cannot be simply increased by including one more mode, but the whole system must be recalculated again. If there is no elastic foundation, usually low number of modes is sufficient (around 5-10). With the foundation included, the number of modes must be much higher, depending on several factors and it ranges around 100-200, or more. As an example, the solution of the moving mass and its corresponding weight on a cantilever is shown in Figure 3. This is one of the examples that are presented in [3]. Numerical data from [3] are slightly adapted to: L=7.62m, P=25.79kN, M=2629kg, EI=9480.6kNm^2, m=46kg/m, v=50.8m/s. It is seen that in this case the effect of the Coriolis and centrifugal forces is significant. This is, however, not a very good example, since the deflection is quite large and the validity of the Euler-Bernoulli beam theory is compromised.

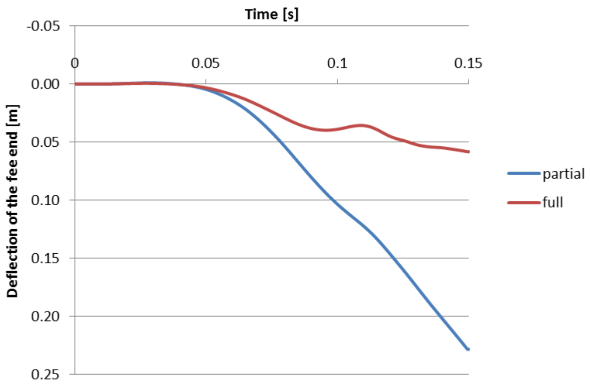

Figure 3: Deflection of the cantilever free end, “partial” means that some terms were omitted as in [3], “full” means that all terms are included.

Another application is a simply supported beam on an elastic foundation (Figure 4). The input data are: L=100m, P=100kN, M=10ton, EI=6.4MNm^2, m=60kg/m, k=4MN/m^2, v=100m/s. The beam and foundation data are related to railway applications. The beam stands for one single rail. In this case 150 modes were necessary for a good accuracy of the solution, but for over 50 modes (even if in this case with purely sinusoidal shape) accumulated numerical errors caused unphysical excessive oscillations when the load approached the right support. The solution of this case is already similar to the one on an infinite beam, thus the new analytical solution that will be presented next, can be checked on it.

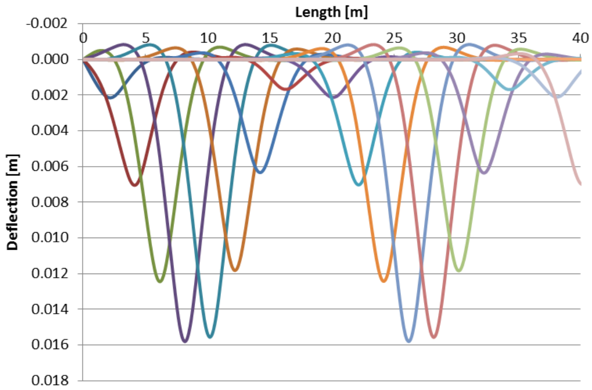

Figure 4: Deflection of the simply supported beam on an elastic foundation, initial 40m of the full length, deflections related to mass position at each 2m.

The governing equation of the vibrations of an infinite beam can be written as:

$$EIw_{,xxxx} + \left(N - k_p\right) w_{,xx} + m w_{,tt} + c_b w_{,t} + kw = -\left(P + P_0 e^{i\omega_f t} + M w_{0,tt}\right)\delta\left(x - vt\right) \tag{30}$$

As compared to Eq. (13), other effects like Pasternak coefficient k_p and the normal force N can be easily introduced. In this formulation there is no problem to express the harmonic component of the force with complex numbers as $P_0 e^{i\omega_f t}$. The unknown deflection w is assumed positive when acting upward. Similarly as before the additional unknown $w_0\left(t\right)$ must be expressed in terms of the unknown field $w\left(x,t\right)$. The moving coordinate $s = x - vt$ and dimensionless parameters can be introduced in accordance with Section 2 to obtain:

$$\begin{aligned} &\widehat{w}_{,\xi\xi\xi\xi} + 4\left(\eta_N - \eta_S + \alpha^2\right)\widehat{w}_{,\xi\xi} + 4\widehat{w}_{,\tau\tau} - 8\alpha\widehat{w}_{,\xi\tau} + 8\left(\eta_b \widehat{w}_{,\tau} - \eta_b \alpha \widehat{w}_{,\xi}\right) + 4\widehat{w} \\ &= -4\left(2 + 2\eta_P e^{i\widehat{\omega}_f \tau} + \eta_M \widehat{w}_{,\tau\tau}\right)\delta\left(\xi\right) \end{aligned} \tag{31}$$

where $\bar{k}_{st}$ was substituted by k, and moreover

$$\widehat{\omega}_f = \frac{\omega_f}{\chi v_{cr}},\ \eta_S = \frac{k_p}{2\sqrt{kEI}},\ \eta_M = \frac{M\chi}{m},\ \eta_P = \frac{P_0}{P} \tag{32}$$

were used.

It can be shown that by application of the Fourier transform in the form of:

$$F\left(p,q\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\xi,\tau\right) e^{-i\left(p\xi + q\tau\right)} d\xi d\tau \tag{33}$$

The governing equation gets the form:

$$W\left(p,q\right) D\left(p,q\right) = -16\pi\delta\left(q\right) - 16\pi\eta_P \delta\left(q - \widehat{\omega}_f\right) + 4\eta_M q^2 \tilde{W}\left(0,q\right) \tag{34}$$

where

$$D\left(p,q\right) = p^4 - 4p^2\left(\eta_N - \eta_S + \alpha^2\right) - 4q^2 + 8\alpha pq - 8iq\eta_c + 8ip\alpha\eta_c + 4 \tag{35}$$

and the final solution for the deflection under the load is:

$$w(0,\tau) = -\int_{-\infty}^{\infty} \frac{8i\left[\delta(q)+\eta_P\delta\left(q-\hat{\omega}_f\right)\right]\left(D_{,p}\left(p_1,q\right)+D_{,p}\left(p_2,q\right)\right)e^{iq\tau}dq}{\left(D_{,p}\left(p_1,q\right)D_{,p}\left(p_2,q\right)-4\eta_M q^2 i\left(D_{,p}\left(p_1,q\right)+D_{,p}\left(p_2,q\right)\right)\right)} \tag{36}$$

where p_1 and p_2 are roots of Eq. (35) according to the integration around upper and lower semi-circle.

From Eq. (36) it is clear that if $\eta_P = 0$, then the value of η_M has no influence on the final result and the deflection under the load is constant, i.e. steady, which is not correct. This proves that first Laplace transform must be implemented

$$\tilde{F}\left(\xi,\bar{q}\right) = \int_0^{\infty} f\left(\xi,\tau\right)e^{-\bar{q}\tau}d\tau \tag{37}$$

with the connection to the previous variables $\bar{q} = iq$, and only after that the Fourier one. Then the main difference is already seen in Eq. (34) which has now the form as:

$$W\left(p,\bar{q}\right)\bar{D}\left(p,\bar{q}\right) = -\frac{8}{\bar{q}} - \frac{8\eta_P}{\bar{q}-i\hat{\omega}_f} - 4\eta_M\bar{q}^2\tilde{W}\left(0,\bar{q}\right) \tag{38}$$

i.e.

$$W\left(p,q\right)D\left(p,q\right) = +\frac{8i}{q} + \frac{8i\eta_P}{q-\hat{\omega}_f} + 4\eta_M q^2\tilde{W}\left(0,q\right) \tag{39}$$

This procedure is described in [5], however, no results of the deflection shape are presented there. It is necessary to solve numerically for the frequency that is induced by the foundation. This can be done by simple iterative algorithm which convergence is secured. Once this is known, the deflection under the load can be expressed analytically by the inverse transform. Its form is quite complicated and thus its presentation is omitted here due to the limited number of pages. Then it is straightforward to join to this result the rest of the full deflection shape. If the applied force is constant, then the mass oscillates around the stationary position, as demonstrated in Figure 4. If the force has a harmonic component, than both harmonic movements are superposed around the stationary position.

4 CONCLUSIONS

In this paper some new analytical solutions for moving load problems were presented. In the first part, a new analytical formula for the critical velocity of a uniformly moving force on a beam supported by a foundation of finite depth was given. The formula accounts for the effect of the normal force acting on the beam and the shear influence of the foundation. The critical velocity is expressed as a function of the mass ratio that relates the foundation mass with the beam mass. The new formula approaches the classical formula for the low mass ratio and the velocity of propagation of shear waves for the high mass ratio.

In the second part, firstly, the analytical solution of the moving mass problem on a finite beam is presented. The importance of the Coriolis and centrifugal forces is highlighted. The solution with elastic foundation indicates the form of the deflection shape on infinite beams. Secondly, a new analytical solution is presented for the deflection shape of an infinite beam that is traversed by a moving mass and supported by a visco-elastic foundation. In such a case the deflection shape resembles the one associated with the moving force with an additional oscillation around it. The frequency of this oscillation is induced by the foundation characteristics and the amplitude can by derived analytically. When the force associated with the mass has a harmonic component, then its frequency is superposed with the one induced by the foundation.

ACKNOWLEDGEMENTS

This work was supported by the Fundação para a Ciência e a Tecnologia (FCT), Portugal, both directly in the form of a project PTDC/EME-PME/67658/2006 and through IDMEC, under LAETA, project UID/EMS/50022/2013.

REFERENCES

[1] Z. Dimitrovová, Enhanced formula for a critical velocity of a uniformly moving load, *European Congress on Computational Methods in Applied Sciences and Engineering* (ECCOMAS 2012), 11-14 September, 2012, Vienna, Austria, Vienna Technical University, ISBN 978-3-9502481-8-0, proceedings ISBN 978-3-9502481-9-7.

[2] Z. Dimitrovová, Critical velocity of a uniformly moving load on a beam supported by a finite depth foundation, *Journal of Sound and Vibration*, under review, 2015.

[3] J.E. Akin, M. Mofid, Numerical solution for response of beams with moving mass, *ASCE Journal of Structural Engineering*, **115**, 120–31, 1989.

[4] Z. Dimitrovová, On the moving mass versus moving load problem, *IX International Conference on Structural Dynamics* (EURODYN2014), 30 June - 2 July, 2014, Porto, Portugal.

[5] A.V. Metrikine, H.A. Dieterman, Instability of vibrations of a mass moving uniformly along an axially compressed beam on a visco-elastic foundation, *Journal of Sound and Vibration,* **201**, 567–576, 1997.