# On the Moving Mass versus Moving Load Problem

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ABSTRACT: In this contribution, a new method for a deflection shape determination of an infinite beam on a visco-elastic foundation traversed by uniformly moving mass is presented. The method invokes the dynamic stiffness matrix concept and for the sake of simplicity the results are shown on Euler-Bernoulli beams. The solution is presented in the context of a review of some methods for solution of uniformly moving mass and load problems on finite and infinite beams. Advantages and disadvantages of these methods are summarized.

KEY WORDS: Moving load; Moving mass; Eigenvalue expansion; Dynamic stiffness matrix.

## 1 INTRODUCTION

Dynamic analyses of beam structures under moving loads have attracted the engineering and scientific community from the middle of the 19th century, when railway construction began. Increasing demands on the railway network capacity leads to a necessity of better understanding of dynamic phenomena related to train-track-soil interactions and therefore questions regarding the moving load and moving mass problems are the still important subjects in nowadays investigations. New modelling approaches, as well as their solving methods, are needed to perform simulations that could reflect important features of dynamic systems. In this context analytical and semi-analytical solutions have the undoubted advantage of possibility of direct sensibility analysis on parameters involved in the problem.

Moving force problem is far simpler. It has a semi-analytical or analytical solution available for finite as well as infinite beams. Generalizations affecting the beam theory and foundation models, like extension from the Euler-Bernoulli theory to the Timoshenko-Rayleigh theory, or generalizations of Winkler foundation to Pasternak or other foundation models, introduction of foundations of finite depth or alterations from viscous to hysteretic damping models do not present substantial difficulty [1], except in cases when numerical solution of complex frequencies is necessary. In finite beams eigenvalue expansion techniques can be used and in infinite beams either Fourier transform or the concept of the dynamic stiffness matrix can be exploited [2, 3]. In the latter case two semi-infinite beams are connected by the continuity conditions at the load application point. Such a solution can easily be extended to the moving force with harmonic component [4] or non-uniform foundation [5, 6].

The inertial effects of both the beam and the moving vehicle were studied as early as in 1929 by Jeffcott [7] by the method of successive approximations. The moving mass problem does not have fully analytical solution. Analysing finite beams, it is seen that the governing equations in modal space remain coupled [3]. There is however a classical work [8], which is often taken as a bench-mark solution, but this solution does not consider all effects at the contact point as already depicted by others [9]. There are other papers repeating the same error [10], some of them corrected by Letters to the Editor [11].

If a steady-state solution exists for an infinite beam, then it exactly matches the solution for the moving force and the mass has no contribution as indicated in [2, 12]. If the solution is not steady, there is an oscillation around the steady-state deflection and the amplitude and frequency of this oscillation has to be determined. In this paper a new method for their determination is presented.

## 2 PROBLEM STATEMENT

Let a uniform motion of a constant vertical force and a mass along a horizontal beam on a linear visco-elastic foundation be assumed (Figure 1). The foundation is modelled as homogeneous distributed spring-and-dashpot sets. Simplifications for the analysis of vertical vibrations are outlined as follows:

(i) the beam obeys linear elastic Euler-Bernoulli theory;
(ii) the beam damping is proportional to the velocity of vibration;
(iii) the beam and mass are in continuous contact;
(iv) no other loading is added;
(v) the vertical displacement is measured from the equilibrium deflection position caused by the beam mass;
(vi) the velocity is maintained constant and no restriction is imposed on its magnitude.

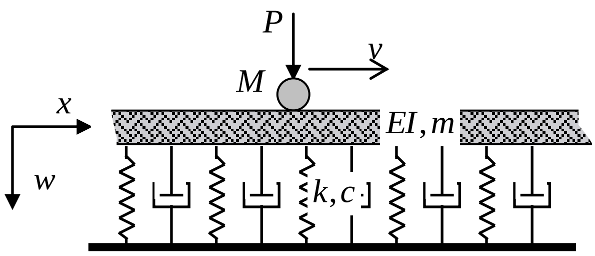


Figure 1. Structure under consideration.

The objective is to solve the time dependent deflection shape *w(x,t)*.

## 3 FINITE BEAMS

Equations If the previous model has a finite length designated as *L*, then several boundary conditions can be considered. Here we will show examples only for simply supported beam and left cantilever. Thus:

$$w(0,t)=0\,,\ \left.\frac{\partial w^2(x,t)}{\partial x^2}\right|_{x=0}=0\,,$$

$$w(L,t)=0\,,\ \left.\frac{\partial w^2(x,t)}{\partial x^2}\right|_{x=L}=0\quad \forall t \tag{1}$$

or

$$w(0,t)=0\,,\ \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0}=0\,,$$

$$\left.\frac{\partial w^2(x,t)}{\partial x^2}\right|_{x=L}=0\,,\ \left.\frac{\partial w^3(x,t)}{\partial x^3}\right|_{x=L}=0\quad \forall t \tag{2}$$

respectively, and, for the sake of simplicity initial conditions are assumed as homogeneous

$$w(x,0)=0\,,\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0}=0\quad \forall x \tag{3}$$

The equation of motion for the unknown field *w(x,t)* is written as:

$$EI\frac{\partial w^4}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}+kw=p(x,t) \tag{4}$$

For the constant mass and load

$$p(x,t)=\left(P-M\,\frac{\mathrm{d}^2 w_0(t)}{\mathrm{d}t^2}\right)\delta(x-vt)\,,\quad w_0(t)=w(vt,t) \tag{5}$$

which means that the loading term can be written in terms of the unknown field *w(x,t)* as:

$$\left(P-M\left(\frac{\partial^2 w(x,t)}{\partial t^2}+2v\frac{\partial^2 w(x,t)}{\partial x\partial t}+v^2\frac{\partial^2 w(x,t)}{\partial x^2}\right)\right)\delta(x-vt) \tag{6}$$

here *EI* and *m* represent the flexural rigidity and the mass per unit length of the beam, *c* and *k* are the damping coefficient and Winkler's constant of the foundation, *P* and *M* are the travelling force and mass. *w(x,t)* and $w_0(t)$ stand for the vertical displacement of the beam and of the point of load application (mass contact point), *v* is the constant velocity, *x* is the spatial coordinate, *t* is the time and $\delta$ is the Dirac function. *x* has its origin at the left extremity of the structure. Zero time corresponds to load position at *x*=0.

In Equation (6) the terms are as follows: the vertical loading force, the mass inertial force acting along the direction of deflection of the beam, the Coriolis force related to the rate of inclination of the beam; and the centrifugal force associated with the curvature of the beam. The last two terms are not used in [8, 10]. In some cases they can be neglected, but generally not.

Solution can be obtained by implementing the Fourier method of variable separation and assuming the existence of free harmonic vibrations:

$$w(x,t)=w(x)e^{i\omega t}\,,\ i=\sqrt{-1} \tag{7}$$

The frequency $\omega$ of these vibrations is named as the natural frequency and it is determined from the eigenvalue problem obtained from the homogeneous governing equation. Then the transient response in the time domain is expressed as infinite series of these modes, where each vibration mode (function of the spatial coordinate *x*) is multiplied by a generalized displacement (modal coordinate, amplitude function) that is a function of time.

$$w(x,t)=\sum_{j=1}^{\infty}q_j(t)w_j(x) \tag{8}$$

For the moving load problem, the modal analysis is facilitated by the uncoupling in the modal space. Nevertheless, even in self-adjoint systems, modal expansion is commonly governed by undamped vibration modes, because this allows their determination within the real domain and completeness of the eigenspace is guaranteed.

In moving mass problem the previous statements are not valid. Two methods can be used: (i) usually the expansion is performed over beam modes calculated without the moving mass contribution; (ii) the other possibility would be to include the moving mass in the beam mass. In the former case modal equations cannot be uncoupled; have the following form:

$$\mathbf{M}(t)\cdot\ddot{\mathbf{q}}(t)+\mathbf{C}(t)\cdot\dot{\mathbf{q}}(t)+\mathbf{K}(t)\cdot\mathbf{q}(t)=\tilde{\mathbf{q}}(t) \tag{9}$$

where matrices **M**, **C**, **K** are defined by introduction of vibration modes in their exact analytical form (without any discretization) as:

$$M_{ij}=\delta_{ij}+Mw_i(vt)w_j(vt) \tag{10}$$

$$C_{ij}=2Mvw_i(vt)w_j'(vt)+\delta_{ij}\frac{c}{m} \tag{11}$$

$$K_{ij}=Mv^2w_i(vt)w_j''(vt)+\delta_{ij}\omega_j^2 \tag{12}$$

Here $\delta_{ij}$ is the Kronecker delta and upper primes stand for the derivation with respect to the spatial variable. Standard techniques [2, 5, 6] can be used for wave numbers $\lambda_j/L$ determination, it is only recalled that

$$\omega_j=\sqrt{\left(\frac{\lambda_j}{L}\right)^4\frac{EI}{m}+\frac{k}{m}} \tag{13}$$

The modes in equations above are normalized by

$$N_j=\sqrt{\int_0^L mw_j^2(x)\,dx} \tag{14}$$

The system cannot be solved analytically, but numerically. Computational time increases exponentially with the number of modes involved. Precision of a solution obtained for a certain number of modes cannot be simply increased by including one more mode, but the whole system must be completely recalculated. If there is no elastic foundation, usually low number of modes is sufficient (around 10). With foundation included the number of modes must be much higher, depending on several factors it ranges around 100-200 or more [3]. Numerical solution can be obtained in commercial software, it is convenient to rewrite the system as a set of first order equations and use Matlab [12], but then the numerical precision is compromised, [2, 5, 6]. Double precision may not be sufficient for higher number of modes, especially when hyperbolic functions are involved in mode shapes.

As an example, solution of the moving mass and its corresponding weight on a cantilever is shown (Figure 2). This is the example presented in [8], [9]. It is seen that in this case the effect of Coriolis and centrifugal forces is significant. Numerical data are taken from [9] as: $L$=7.62m, $P$=25.79kN, $M$=2629kg, $EI$=9480.6kNm$^2$, $m$=46kg/m, $v$=50.8m/s. This is however not a very good example, since the deflection is quite large and the validity of the Euler-Bernoulli beam theory is compromised.

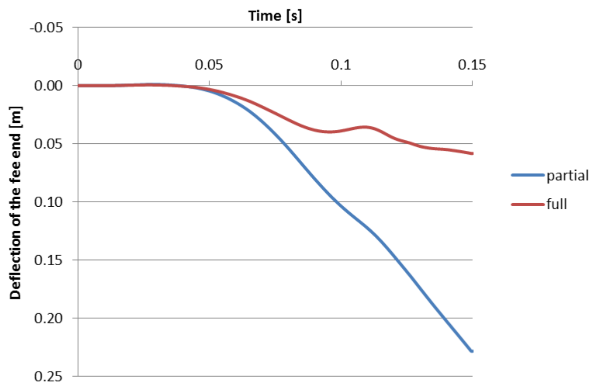


Figure 2. Deflection of the cantilever free end, “partial” means that some terms were omitted as in [8], “full” means that all terms are included.

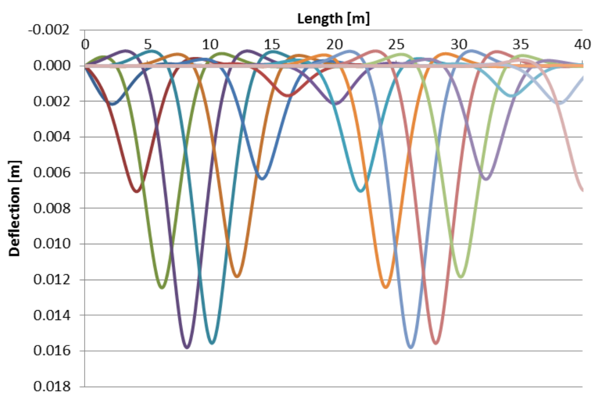


Figure 3. Deflection of the simply supported beam on an elastic foundation, initial 40m of the full length, defelections related to mass position at each 2m.

Another application is a simply supported beam on an elastic foundation (Figure 3). The input data are: $L$=100m, $P$=100kN, $M$=10ton, $EI$=6.4MNm$^2$, $m$=60kg/m, $k$=4MN/m$^2$, $v$=100m/s. The beam and foundation data are related to railway applications. The beam stands for one single rail. In this case 150 modes were necessary for a good accuracy of the solution, but for over 50 modes (even if in this case with purely sinusoidal shape) accumulated numerical errors caused unphysical excessive oscillations when the load approached the right support.

If the moving mass is added to the beam mass, then it is necessary to solve the vibration modes at each mass position. The modes are orthogonal at each such a position and can be determined following [3, 5, 6]. Due to adaptable numerical precision, Maple [13] is one of the most adequate software. Then the modes change their shape and frequency with the moving mass position and in fact only moving force problem should be solved [14]. It is necessary to introduce sufficiently small time discretization and it is possible to assume linear modes variation between discrete force positions. For uniform discretization $\Delta t=t_i-t_{i-1}$ is constant. An intermediate value reads as:

$$\hat{w}_{i,j}\left(v\tau\right)=w_j\left(vt_{i-1}\right)+\frac{w_j\left(vt_i\right)-w_j\left(vt_{i-1}\right)}{\Delta t}\tau \tag{15}$$

where $\tau$ is a local time starting at $t_{i-1}$ and $w_j(vt_i)$ designates $j$-th mode determined for mass position at $x_i=vt_i$. It holds:

$$\begin{aligned} q_j\left(t_i\right)=&\frac{1}{b_{i,j}}\int_0^{\Delta t}P\hat{w}_{i,j}\left(v\tau\right)e^{-\frac{c}{2m}\left(\Delta t-\tau\right)}\sin\left(b_{i,j}\left(\Delta t-\tau\right)\right)d\tau+\\ &q_j\left(t_{i-1}\right)e^{-\frac{c}{2m}\Delta t}\cos\left(b_{i,j}\Delta t\right)+\\ &\frac{1}{b_{i,j}}\left(\frac{c}{2m}q_j\left(t_{i-1}\right)+\frac{d}{dt}q_j\left(t_{i-1}\right)\right)e^{-\frac{c}{2m}\Delta t}\sin\left(b_{i,j}\Delta t\right)\end{aligned} \tag{16}$$

with

$$b_{i,j}=\sqrt{\omega_{i,j}^2-\left(\frac{c}{2m}\right)^2} \tag{17}$$

After some manipulations:

$$\begin{aligned} q_j\left(t_i\right)=&e^{-\frac{c}{2\mu}\Delta t}\left(D_i\cos\left(b_{i,j}\Delta t\right)+E_i\sin\left(b_{i,j}\Delta t\right)\right)+\\ &\frac{d_i}{\left(\frac{c}{2\mu}\right)^2+b_{i,j}^2}-\frac{ce_i}{\mu\left(\left(\frac{c}{2\mu}\right)^2+b_{i,j}^2\right)^2}+\frac{e_i\Delta t}{\left(\frac{c}{2\mu}\right)^2+b_{i,j}^2}\end{aligned} \tag{18}$$

$$\begin{aligned} \frac{d}{dt}q_j\left(t_i\right)=&-\frac{c}{2\mu}e^{-\frac{c}{2\mu}\Delta t}\left(D_i\cos\left(b_{i,j}\Delta t\right)+E_i\sin\left(b_{i,j}\Delta t\right)\right)+\\ &b_{i,j}e^{-\frac{c}{2\mu}\Delta t}\left(-D_i\sin\left(b_{i,j}\Delta t\right)+E_i\cos\left(b_{i,j}\Delta t\right)\right)+\frac{e_i}{\left(\frac{c}{2\mu}\right)^2+b_{i,j}^2}\end{aligned} \tag{19}$$

where

$$d_i = Pw_j\left(vt_{i-1}\right),\ e_i = P\frac{w_j\left(vt_i\right)-w_j\left(vt_{i-1}\right)}{\Delta t} \quad (20)$$

$$D_i = q_j\left(t_{i-1}\right) - \frac{d_i}{\left(\frac{c}{2\mu}\right)^2 + b_{i,j}^2} + \frac{ce_i}{\mu\left(\left(\frac{c}{2\mu}\right)^2 + b_{i,j}^2\right)^2} \quad (21)$$

$$E_i = \frac{1}{b_{i,j}}\left(q_j\left(t_{i-1}\right)\left(\frac{c}{2\mu}\right) + \frac{d}{dt}q_j\left(t_{i-1}\right)\right) + \frac{1}{b_{i,j}\left(\left(\frac{c}{2\mu}\right)^2 + b_{i,j}^2\right)^2} \left[-\left(\frac{c}{2\mu}\right)^3 d_i + \left(\frac{c}{2\mu}\right)^2 e_i - \frac{c}{2\mu}d_i b_{i,j}^2 - e_i b_{i,j}^2\right] \quad (22)$$

Expressions above are only approximations to the analytical solution, but their advantage is that they are very quick to evaluate. The other advantage is that the precision can simply be improved by adding more modes. There are some alterations that must be introduced. One of them is switching the modes in the way that the same order moves smoothly its shape and does not jump to the other side of the structure. Some results are shown in Figure 4, numerical data are related to the previously introduced cantilever.

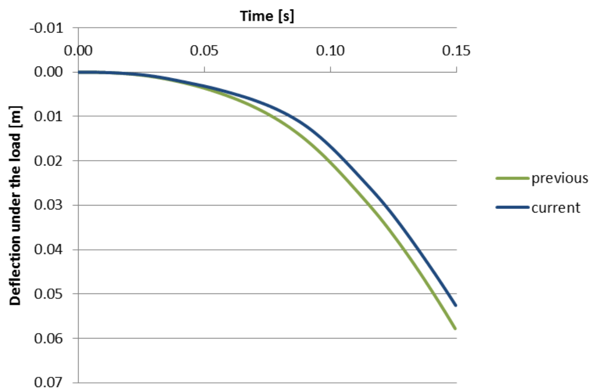


Figure 4. Deflection under the moving mass.

It is seen, however, that even for fine discretization, there is an error in the deflection under the load, which does not get better with finer discretization. These values are governed by the first vibration modes at each separate position.

## 4 INFINITE BEAMS

If an infinite beam is under consideration, a moving coordinate system can be introduced by $\tilde{x} = x - vt$, $t$=$t$. Then the left hand side of Equation (4) can be written as:

$$EI\frac{\partial\tilde{w}^4}{\partial\tilde{x}^4} + m\left(\frac{\partial^2\tilde{w}}{\partial t^2} - 2v\frac{\partial^2\tilde{w}}{\partial\tilde{x}\partial t} + v^2\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}\right) + c\left(\frac{\partial\tilde{w}}{\partial t} - v\frac{\partial\tilde{w}}{\partial\tilde{x}}\right) + k\tilde{w} \quad (23)$$

where $\tilde{w}\left(\tilde{x},t\right)$ is the unknown deflection field and the right hand side simplifies to:

$$\left(P - M\frac{\partial^2\tilde{w}}{\partial t^2}\right)\delta\left(\tilde{x}\right) \quad (24)$$

If $M$=0, then the problem can be solved following [2] or other known literature. Other possibility is to use the dynamic stiffness matrix. The definition is usually introduced in structures separated into $n$-subdomains. The local dynamic stiffness matrix of the $n$-th sub-domain can be calculated in the following way. The degrees of freedom are represented in Figure 5a). Excitation with unit amplitude and given circular frequency $\omega$ is assumed in the direction of one of the degrees of freedom, while the other degrees of freedom are kept fixed. Figure 5b) exemplifies implementation of the first degree of freedom and orientation of the corresponding terms of the stiffness matrix.

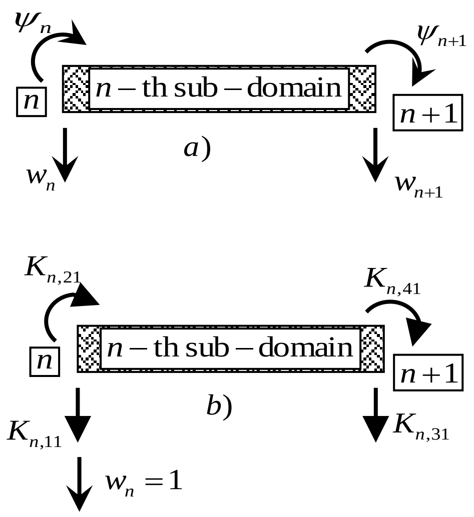


Figure 5. a) Degrees of freedom, b) construction of the local dynamic stiffness matrix of the $n$-th sub-domain.

For such an excitation, member-end generalized harmonic forces in the steady-state regime can be calculated. The procedure is repeated for the other degrees of freedom. More details can be found in [5]. If semi-infinite sub-domains are considered, then only two degrees of freedom have to be considered, as shown in Figure 6.

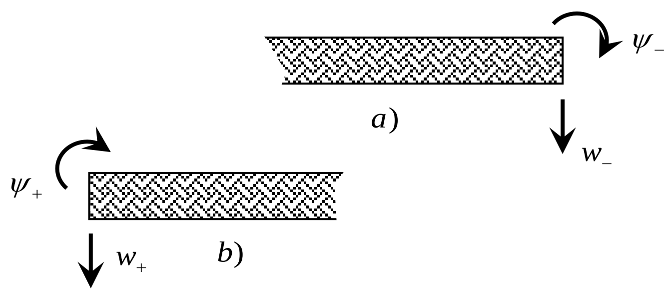


Figure 6. Degrees of freedom of semi-infinite sub-domains: a) negative, b) positive.

Wave equation is determined based on Equation (23). Four roots are separated according to the sign of the real part. Only the real negative-valued ones are used in the positive semi-infinite sub-domains to ensure vanishing of the displacements and rotations for *x* tending to positive infinity, and *vice versa*.

General form of all involved terms for moving harmonic force $Pe^{i\omega_f t}$ where $\omega_f$ is the excitation frequency is presented in [4]. Four constants defining deflection shapes of the two semi-infinite sub-domains are then determined from the four continuity equations, which are the continuity of displacements, rotations and bending moments and, in addition, internal shear force must be in equilibrium with the externally applied force. In the case of the harmonic force, the constants are time-dependent. For excitation directed by sine, imaginary part of the solution corresponds to the beam deflection.

The case with a moving mass can be solved in a similar way. According to [15], if the final solution is steady, then the mass effect is cancelled and the solution has the same form as if only moving force was introduced. When the solution is not steady, but stable, then periodic oscillation occur in the deflection shape around the steady state line. The frequency and amplitude of this additional movement has to be determined. The frequency can be solved by exploiting the fact that in the point of mass contact the vertical force exerted on the beam is equal to

$$-M \left.\frac{\partial^2 \tilde{w}}{\partial t^2}\right|_{\tilde{x}=0} = M\omega_M^2 \left.\tilde{w}\right|_{\tilde{x}=0} \tag{25}$$

where $\omega_M$ is the corresponding frequency of these oscillations. Therefore, the infinite beam can be solved for an externally applied harmonic force and tested that the force value is correct. One can just start with some frequency estimate, solve the wave numbers, establish the four continuity equations and compare the force with the expected result. Since the problem is linear, the actual estimated force value is not import, because for the same frequency the ratio of the introduced and obtained value will be the same. This allows defining a simple iteration procedure, where after the first iteration the forcing frequency is recalculated, and so on. This procedure has very quick convergence ratio and is also very stable. Frequencies do not have to be searched in a complex domain as suggested in [15].

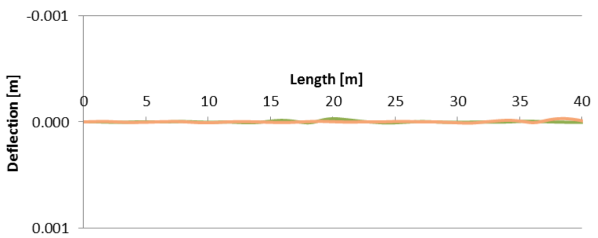


Figure 7. Deflection of the simply supported beam on an elastic foundation, initial 40m of the full length, deflections related to mass position at 18m and 36m.

Naturally, a good initial estimate will always help. In such a case one can use results on finite beams. In the case shown in this paper (Figure 3) it is seen that the lowest deflection shapes happen when the mass is at positions equal to 18m, 36m, etc. In this particular case it is seen that the deflection practically vanishes (Figure 7).This prediction indicates that the frequency should be around 34.91rad/s. In two steps this value can be corrected to 34.857rad/s, with difference between the two approximations less than 0.003%. This result also fits well the relation established [15]. Other cases were tested and same conclusions were taken. Then the solution procedure is finished following [15].

## 5 CONCLUSION

In this contribution several aspects related to the dynamic analysis of beam structures under moving loads were summarized. Differences in solution techniques and results were given for moving force and moving mass problems, as well as for finite and infinite beams. The concept of the dynamic stiffness matrix was posted as a general principle for finite, semi-infinite and infinite beams. This forms the base for the new solution technique for moving mass problem on infinite beams.

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