European Congress on Computational Methods
in Applied Sciences and Engineering (ECCOMAS 2012)
J. Eberhardsteiner et.al. (eds.)
Vienna, Austria, September 10-14, 2012

ENHANCED FORMULA FOR A CRITICAL VELOCITY OF A UNIFORMLY MOVING LOAD

Zuzana Dimitrovová

UNIC, Civil Engineering Department
Nova University of Lisbon, Caparica, Portugal
e-mail: zdim@fct.unl.pt

Keywords: transverse vibration, track modulus, dynamic stiffness, elastic foundation.

Abstract. *The response of rails to moving loads is of interest in the area of high-speed railway transport. For determination of critical velocity of the train a theoretical concept that is based on the assumption that the track structure acts as a continuously supported beam (the rail) resting on a uniform layer of springs is traditionally used. In this contribution dynamic equilibrium of the soil in the vertical direction is implemented to obtain two frequency dependent parameters that are capable of handling geometric damping and of accounting for the soil mass inertia activated by induced vibrations. The new approach is tested on finite beams and single moving force. It allows for determination of resonant velocities. Then the quasi-stationary deflection shape of an infinite beam can be determined from two semi-infinite beams and critical velocities can be obtained from the nullity condition of the determinant of the dynamic stiffness matrix of the structure.*

1 INTRODUCTION

The response of rails to moving loads is of interest in the area of high-speed railway transportation. If simple geometries of the track and subsoil are considered, a theoretical concept that is based on the assumption that the track structure acts as a continuously supported beam (the rail) resting on a uniform layer of springs can be introduced. This layer of springs represents the underlying remainder of the track structure. The stiffness of such spring layer along the length of the track is named as the track modulus and defines Winkler's model. The Winkler model is often referred to as a "one-parameter model". Such a simplified model is traditionally used to estimate the critical velocity of moving trains.

The first solution of steady-state dynamic response of an infinite beam on elastic foundation traversed by moving load was presented by Timoshenko [1]. In [2], the moving coordinate system is introduced to convert the governing equation to ordinary differential equation that can be solved by Fourier integral transformation. In [3], the concept of the dynamic stiffness matrix is implemented. Two semi-infinite beams are solved and connected by continuity equations. Then the critical velocity can be determined as the velocity that ensures the nullity of the determinant of the dynamic stiffness matrix.

The critical velocity of the load v_{cr} is defined as the phase velocity of the slowest free wave, which in this case is the one that in undamped case induces infinite displacements directed upward as well as downward. But in reality, this velocity should be compared to the Rayleigh-wave velocity of the ground [4], therefore it is strange that the mass of the foundation is not accounted for.

It can be proven that in the steady-state regime load exerts no inertial effects [2], which is probably the reason why also the mass inertia of the foundation was overlooked and the formula for the critical velocity was used for many years. This classical formula, however, predicts very high critical velocity, giving impression that is unreachable by high-speed trains and consequently no attention was paid to this fact during expansions of high-speed railway network. Unfortunately, practical experience showed that the realistic critical velocity is much lower and therefore the classical formula must be revised.

2 CRITICAL VELOCITY

The critical velocity of the load traversing an infinite Euler-Bernoulli beam on an elastic foundation is given by the classical formula:

$$v_{cr}^{E-B} = \sqrt[4]{\frac{4kEI}{\mu^2}} \tag{1}$$

where μ stands for the beam mass per unit length, EI for the beam bending stiffness and k for the Winkler constant of the foundation. This formula is closely related to a finite beam. Following [5, 6], the resonant velocity of a finite beam corresponds to the velocity for which the excitation frequency of the passing load equals to the corresponding beam natural frequency, thus:

$$v_{res} = \frac{L}{\lambda_j}\omega_j \tag{2}$$

where L is the beam length, λ_j / L is the wave number and ω_j is the corresponding natural frequency. Such a resonant velocity can be attributed to each vibration mode. The critical velocity is the lowest resonant velocity. For a beam without an elastic foundation $j_{cr} = 1$ is always verified. When an elastic foundation is included, then one can consider the previous equation as function of j and establish the extreme value. For an Euler-Bernoulli beam Equation (2) is simply

$$v_{j_r}^{E-B} = \frac{L}{j_r\pi}\sqrt{\left(\frac{j_r\pi}{L}\right)^4 \frac{EI}{\mu} + \frac{k}{\mu}} \tag{3}$$

and the minimum value is achieved for a non-integer j_{cr}:

$$j_{cr,1} = \frac{L}{\pi}\sqrt[4]{\frac{k}{EI}},\quad j_{cr,2} = -\frac{L}{\pi}\sqrt[4]{\frac{k}{EI}},\quad j_{cr,3} = i\frac{L}{\pi}\sqrt[4]{\frac{k}{EI}},\quad j_{cr,4} = -i\frac{L}{\pi}\sqrt[4]{\frac{k}{EI}} \tag{4}$$

where $i=\sqrt{-1}$ is the complex unit. Substituting j_{cr} back into Equation (3), Equation (1) is verified, as expected. Thus, the closest integer to j_{cr} indicates the critical velocity of the load passing over the finite Euler-Bernoulli beam. This value always overestimates the value related to the infinite beam.

3 GENERALIZATIONS

Several studies were performed over the years in order to generalize Equation (1).

3.1 Beam level

Generalization at the beam level are simple to derive, when [5] is followed. By nullity condition imposed on the determinant of the dynamic stiffness matrix, for instance the value for the Timoshenko-Rayleigh beam can be obtained as:

$$v_{cr}^{T-R} = \sqrt{\frac{1}{\mu\left(kr^2 - G\bar{A}\right)^2}\left(k\left(EI\left(kr^2 - G\bar{A}\right) - 2r^2G\bar{A}^2\right) + 2G\bar{A}\sqrt{kG\bar{A}}\sqrt{kr^4G\bar{A} - EI\left(kr^2 - G\bar{A}\right)}\right)} \tag{5}$$

where r stands for the radius of gyration of the beam cross-section and $G\bar{A}$ stands for the shear stiffness ($\bar{A}$ is the reduced cross sectional area by the Timoshenko shear coefficient).

3.2 Soil level

Improvements of Winkler's model were obtained by introduction of another parameter in so-called Filonenko–Borodich, Pasternak or Hetenyi models. This parameter can be explained as shear contribution and thus removes the disadvantage of Winkler's springs that do not interact between themselves and is especially important when extremity of finite beam is analyzed. It can equally be understood as distributed rotational springs. This representation is easier to implement when finite element confirmation of theoretical developments is required. The model is named as a "two-parameter model".

Because the concern is not on the wave propagation inside the soil, but merely on the deformation properties on the surface, *i.e.* at the contact with the beam structure, the two parameters can be simply determined. Nevertheless, in order to account for variation of the vertical soil displacement and the active depth of the soil, which is the part of the soil that is deformable down to a rigid base, other generalizations were developed.

It is assumed that the deflection w varies inside the soil according to a function $f(z)$ and $w(x,y,z,t) = w(x,y,t)f(z)$, where $w(x,y,t)$ equals the deflection of the beam/soil contact point, x,y,z are spatial coordinates and t is the time. Then $f(z)$ must verify $f(0)=1$ and $f(H)=0$, where H is the active depth. $f(z)$ can be expressed with the help of another parameter γ as:

$$f(z)=\frac{\sinh\left[\gamma\left(1-\frac{z}{H}\right)\right]}{\sinh\gamma} \tag{6}$$

This model introduced by Vlasov is usually referred to as a "three-parameter model". In the original development the parameter γ is arbitrary. One of possibilities of γ determination establishes a relation involving the vertical displacement w, still unknown, and therefore an interactive procedure must be introduced in the solution.

$$\left(\frac{\gamma}{H}\right)^2=\frac{1-2\nu}{2(1-\nu)}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\nabla w)^2\,dxdy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w^2\,dxdy} \tag{7}$$

This model is named as the modified Vlasov model [7]. In Equation (7) ν is the Poisson ration and ∇ is gradient operator. The soil mass is added by a term assuming a linear distribution of function $f(z)$ directly in the mass matrix of the structure. Analyzing relation (6) it can be concluded that the only viable shapes of the function $f(z)$ are contained within the region restricted by the linear distribution, as shown in Figure 1.

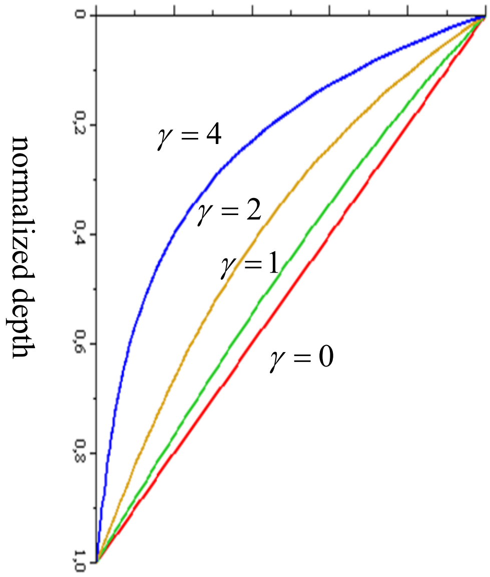

Figure 1: Dependence of function $f(z)$ on parameter γ.

The two soil parameters are given by:

$$k=\int_0^H E^{oed}\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 dz=\frac{E^{oed}}{H}\frac{\gamma\left(\sinh\gamma\cosh\gamma+\gamma\right)}{2\sinh^2\gamma} \tag{8}$$

$$k_p=\int_0^H Gf^2dz=GH\frac{\left(\sinh\gamma\cosh\gamma-\gamma\right)}{2\gamma\sinh^2\gamma} \tag{9}$$

where the oedometer modulus is $E^{oed}=E(1-\nu)/\left((1+\nu)(1-2\nu)\right)$ and E and G stand for the Young and shear modulus of the soil, respectively. For linear distribution of function $f(z)$, *i.e.* $\gamma=0$ previous parameters are given by

$$k = \frac{E^{oed}}{H}, \; k_p = \frac{GH}{3} \tag{10}$$

4 NEW APPROACH

4.1 Frequency dependent soil parameters

Generalizations on the soil level published in the literature do not account for the mass inertia influence that is activated in the underlying soil. In order to remove this drawback the two soil parameters of the Pasternak model should be considered as frequency dependent, $k(\omega)$ and $k_p(\omega)$. This will also remove the need of determination of additional parameter γ. If a harmonic motion inducing only transversal displacements is assumed, then the deflection w varies inside the soil according to a function $f(z)$ and same assumptions as before can be adopted $w(x,y,z,t) = w(x,y,t)\, f(z)$. In the static case the function $f(z)$ can be approximated by a linear function; improved values were shown in previous section. In the dynamic case, $f(z)$ shape is frequency dependent and more variability to its from should be given. Function $f(z)$ can be derived from the dynamic equilibrium of the soil in the vertical direction. Following [8]:

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2} \tag{11}$$

where σ and τ stand for normal and tangential stress components, respectively. The components of the deformation tensor are given by:

$$\varepsilon_z = w(x,y,t)\frac{\mathrm{d}f(z)}{\mathrm{d}z}, \; \gamma_{xz} = \frac{\partial w(x,y,t)}{\partial x} f(z), \; \gamma_{yz} = \frac{\partial w(x,y,t)}{\partial y} f(z) \tag{12}$$

where ε and γ stand for the extension and engineering distortion, respectively. Therefore, the stress components can be expressed as:

$$\sigma_z = E^{oed} w(x,y,t)\frac{\mathrm{d}f(z)}{\mathrm{d}z}, \; \tau_{xz} = G\frac{\partial w(x,y,t)}{\partial x} f(z), \; \tau_{yz} = G\frac{\partial w(x,y,t)}{\partial y} f(z) \tag{13}$$

Assuming harmonic vibrations and neglecting the shear stress derivatives, the differential equation for the function $f(z)$ reads as:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} f(z) + \lambda^2 f(z) = 0 \tag{14}$$

where the wave number λ is given by:

$$\lambda = \sqrt{\frac{\omega}{v_p}} = \sqrt[4]{\frac{\omega^2 \rho}{E^{oed}}} \tag{15}$$

and v_p is the velocity of the pressure waves. The solution of Equation (14) is:

$$f(z) = \cos\lambda z - \mathrm{cotg}\lambda H \sin\lambda z = \frac{\sin(\lambda(H-z))}{\sin\lambda H} \tag{16}$$

The total energy (both potential and kinetic) of the soil can be expressed as:

$$U = \frac{1}{2}\int_{\Omega}\left\{\int_0^H \left(E^{oed}\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 w^2 + Gf^2\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) - \omega^2 \rho f^2 w^2 \right)\mathrm{d}z\right\}\mathrm{d}\Omega$$
$$= \frac{1}{2}\int_{\Omega}\left(kw^2 + k_p\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right)\right)\mathrm{d}\Omega \tag{17}$$

If a sufficiently extensive area Ω is selected, the energy beyond this region can be neglected. In this formulation, the energy attributed to the Pasternak modulus in fact corresponds to the energy of distributed rotational springs. It follows:

$$k(\omega) = \int_0^H E^{oed}\left(\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 - (\lambda f)^2\right)dz = \frac{E^{oed}}{H}\lambda H\frac{\cos\lambda H}{\sin\lambda H} \tag{18}$$

$$k_p(\omega) = \int_0^H Gf^2 dz = \frac{1}{2}GH\left(\frac{\lambda H - \sin\lambda H\cos\lambda H}{\lambda H\sin^2\lambda H}\right) \tag{19}$$

and the vertical stress (the reaction pressure of the soil) at the contact is given by:

$$p_s(\omega) = k(\omega)w - k_p(\omega)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{20}$$

If λH tends to zero, static values of the Winkler and Pasternak parameters are verified (compare with Equation (10)).

In summary, the effect of the viscoelastic foundation can be represented by the soil pressure, which for beam structures takes the following form:

$$p_s(\omega) = k(\omega)w - k_p(\omega)\frac{\partial^2 w}{\partial x^2} \tag{21}$$

4.2 Natural frequencies of simply supported beam

In order to derive the new formula for the critical velocity, a finite beam on a frequency dependent foundation will be considered first. The governing equation of undamped free vibrations of the Euler-Bernoulli beam on a Pasternak foundation is given by:

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} - k_p(\omega)\frac{\partial^2 w_n}{\partial x^2} + k(\omega)w = 0 \tag{22}$$

By implementing the Fourier method of variable separation and assuming the existence of free harmonic vibrations in form of:

$$w(x,t) = w(x)e^{i\omega t} \tag{23}$$

one can derive:

$$EI\frac{d^4 w}{dx^4} - \omega^2\mu w - k_p(\omega)\frac{d^2 w_n}{dx^2} + k(\omega)w = 0 \tag{24}$$

Equation (24) is verified by e^{px}, thus:

$$EIp^4 - \omega^2\mu - k_p(\omega)p^2 + k(\omega) = 0 \tag{25}$$

Considering a practical example, let the following values as specified in Table 1 be adopted.

Property	Values
Beam bending stiffness EI (MNm^2)	6.4
Beam mass per unit length μ (kg/m)	60
Soil Young's modulus E (MPa)	200
Soil Poisson's ratio ν	0
Soil density ρ (kg/m^3)	2000
Beam length L (m)	200
Active depth H (m)	12

Table 1: Numerical data in a practical example.

Because of the simple supports, the only beam deflection shape that verifies the boundary conditions is given by:

$$w_j(x) = \sin\left(\frac{j\pi}{L}x\right) \tag{26}$$

thus $p = j\pi / L$. By substitution of this relation and Equations (19-20) in Equation (25), natural frequencies can be determined.

	sm1	sm2	sm3	sm4	sm5
bm1	41,29	123,87	206,45	289,04	371,62
bm2	41,29	123,87	206,45	289,04	371,62
bm3	41,29	123,87	206,45	289,04	371,62
bm4	41,29	123,87	206,45	289,04	371,62
bm5	41,29	123,87	206,45	289,04	371,62
bm6	41,29	123,87	206,45	289,04	371,62
bm7	41,29	123,87	206,45	289,04	371,62
bm8	41,29	123,87	206,45	289,04	371,62
bm9	41,29	123,87	206,46	289,04	371,62
bm10	41,29	123,87	206,46	289,04	371,62
bm11	41,30	123,87	206,46	289,04	371,62
bm12	41,30	123,88	206,46	289,04	371,62
bm13	41,30	123,88	206,46	289,04	371,62
bm14	41,31	123,88	206,46	289,04	371,62
bm15	41,31	123,88	206,46	289,04	371,62
bm16	41,32	123,88	206,46	289,04	371,62
bm17	41,32	123,88	206,46	289,04	371,62
bm18	41,33	123,89	206,46	289,04	371,62
bm19	41,34	123,89	206,46	289,04	371,62
bm20	41,35	123,89	206,47	289,05	371,63

Table 2: Natural frequencies of the Winkler beam.

Natural frequencies values for the numerical example are shown in Table 2. In the legend “bm” stands for beam modes and “sm” for soil modes. It is interesting to see that when the Pasternak contribution is omitted, there are infinite natural frequencies for a fixed j that have consecutive shapes of function $f(z)$. By analysis of Equation (25) it can be concluded that the second soil mode frequency related to the fundamental beam mode shape will never be lower than any frequency of the first soil mode.

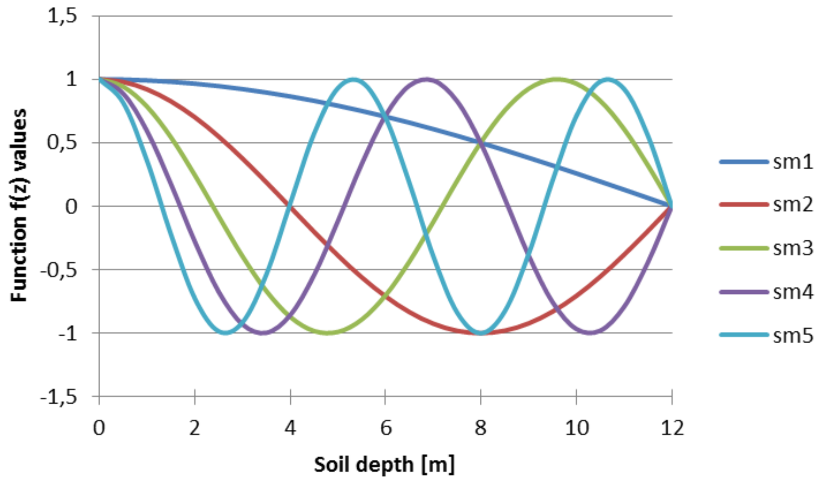

Figure 2: First five soil modes of the Winkler beam.

	sm1	sm2	sm3	sm4	sm5
bm1	41,14	82,86	123,82	165,61	206,42
bm2	40,70	83,08	123,67	165,73	206,34
bm3	39,95	83,45	123,43	165,91	206,19
bm4	38,91	83,95	123,08	166,17	205,98
bm5	37,56	84,58	122,63	166,50	205,71
bm6	35,88	85,33	122,08	166,90	205,38
bm7	33,87	86,20	121,43	167,37	204,99
bm8	31,46	87,18	120,67	167,91	204,54
bm9	28,60	88,27	119,79	168,52	204,03
bm10	25,13	89,49	118,79	169,20	203,45
bm11	20,80	90,83	117,64	169,95	202,80
bm12	14,90	92,34	116,31	170,77	202,08
bm13	94,07	114,75	171,68	201,28	252,51
bm14	96,12	112,84	172,67	200,39	253,18
bm15	98,84	110,25	173,77	199,39	253,90
bm16	174,99	198,27	254,69	283,40	335,92
bm17	176,37	196,99	255,55	282,61	336,55
bm18	177,97	195,48	256,49	281,73	337,23
bm19	179,95	193,59	257,52	280,76	337,97
bm20	182,80	190,82	258,67	279,67	338,77

Table 3: Natural frequencies of the Pasternak beam.

The first five soil modes related to the fundamental beam mode shape are represented in Figure 2.

If the Pasternak contribution is included, the regular sequence shown in Table 2 is interrupted. The first soil frequency decreases till the 12th beam frequency and then it jumps to a higher value. Some wave numbers of the first five soil modes yield very low value in function $\sin \lambda H$ and therefore some of the shapes are unrealistic, namely the 2nd and the 4th related to the fundamental beam mode shape. Valid modes are shown in Figure 3. Nevertheless, for the 13th beam frequency all soil modes seem realistic and are shown in Figure 4.

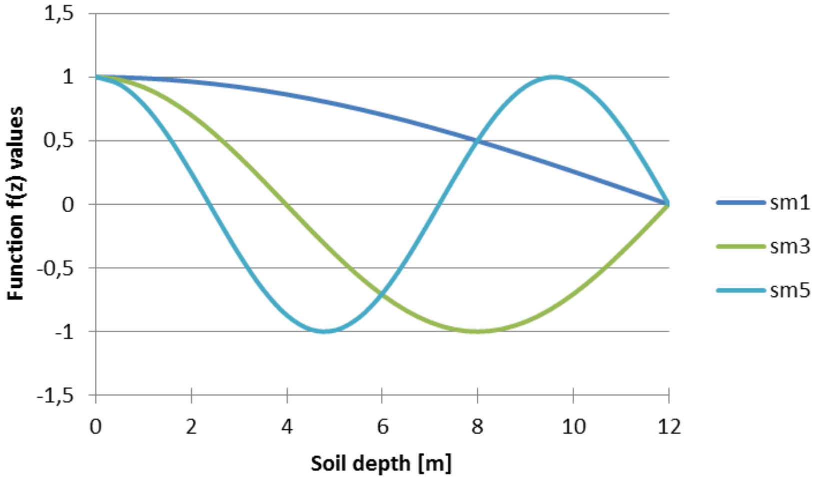

Figure 3: Valid soil modes of the Pasternak beam related to the fundamental beam mode shape.

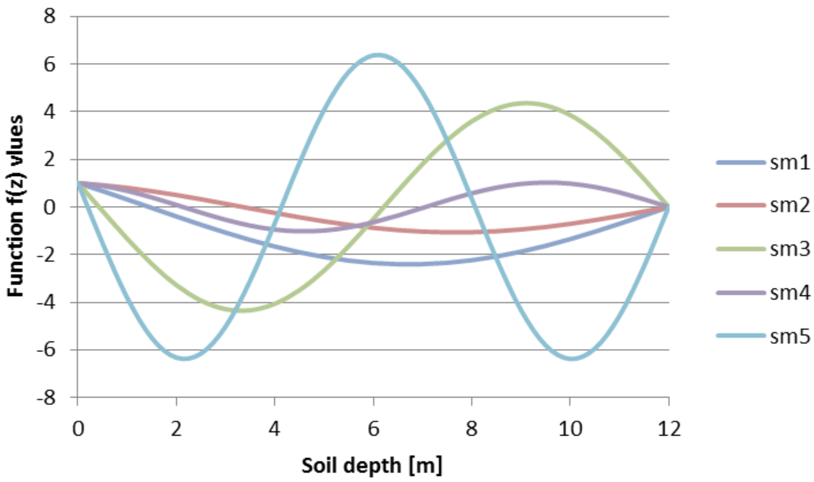

Figure 4: Soil modes of the Pasternak beam corresponding to the thirteenth beam frequency.

Globally, first frequencies are much lower than the frequencies of the classical Winkler case. This can be seen from the cut-off frequency of the Winkler beam that is

$$\omega_{cut-off} = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{E^{oed}}{H\mu}} = 527 rad/s \tag{27}$$

Having the natural frequencies and shapes it is possible to determine resonant velocities by standard procedures similar to [5, 6] and consequently the new value of the critical velocity.

5 CONCLUSIONS

In this contribution the disadvantages of the standard formula for critical velocity determination of a load moving on a beam with an elastic foundation were summarized. Generalizations of the theory already published were summarized. The new approach that can improve the formula was introduced and further direction that must be taken was established.

REFERENCES

[1] S.P. Timoshenko: Statical and dynamical stresses in rails, Proceedings of the 2nd International Congress for Applied Mechanics, Zürich (Switzerland), 12-17 September 1926, 407-418.

[2] L. Frýba: Vibration of solids and structures under moving loads, Research Institute of Transport, Prague (1972), 3rd edition, Thomas Telford, London, 1999.

[3] Y.H. Chen, Y.-H. Huang: Dynamic stiffness of infinite Timoshenko beam on viscoelastic foundation in moving co-ordinate. *International Journal for Numerical Methods in Engineering*, 48 (2000), 1-18.

[4] A.M. Kaynia, C. Madshus, P. Zackrisson: Ground vibration from high-speed trains: prediction and countermeasure. *Journal of Geotechnical and Geoenvironmental Engineering*, 126 (2000), 531-537.

[5] Y-H. Chen, Y-H. Huang: Dynamic characteristics of infinite and finite railways to moving loads. *ASCE Journal of Engineering Mechanics*, 129 (2003), 987-995.

[6] Z. Dimitrovová, A.S.F. Rodrigues: Critical velocity of a uniformly moving load. *Advances in Engineering Software*, 50 (2012), 44–56.

[7] C.V.G. Vallabhan, Y.C. Das: Modified Vlasov model for beams on elastic foundations. *Journal of Geotechnical Engineering*, 117 (1991), 956-966.

[8] J. Máca: Dynamic analysis of plates on elastic foundation, Proceedings of the 4th International Conference on Structural Dynamics, EURODYN2002, H. Grundmann and G.I. Schueller (eds.), Munich, Germany, 2-5 September 2002, 887-892.