Semi-analytical Approach to Vibrations Induced by Oscillator Moving on a Beam Supported by a Finite Depth Foundation

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Abstract Structures subject to moving loads have several in rail, road and bridge engineering. When the velocity of the moving system approaches the critical velocity, then the induced vibrations are significantly augmented and safety and stability of the structure as well as of the moving system are compromised. The classical models predict the critical velocity much higher than the one observed in reality, because the wave propagation is restricted to the beam structure. But if the beam is supported by elastic continuum, then the waves can be dominant in the foundation and the interaction with the beam cannot be overlooked. This contribution analyses the critical velocity of an oscillator moving on a beam supported by a foundation of finite depth by semi-analytical methods.

1 Introduction

The objective of this contribution is to fill the gap in semi-analytical solutions related to wave propagation induced by moving loads. An analysis of the critical velocity of an oscillator moving on a beam supported by a foundation of finite depth is presented. Such analysis is important for transport engineering, as structures subjected to moving loads have several applications in rail, road and bridge engineering. The classical model where the beam structure is supported by massless linear springs is still used by several companies, due to its simplicity. Nevertheless, in this model the wave propagation is restricted to the beam structure and no dynamic interaction with the foundation can be accessed. Models considering beams placed on elastic continua of finite depth provide better access to the interaction mechanism. Nevertheless, as the inertial effects are important in the

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supporting structure, they are also important is the moving system. This means that the moving system should not be approximated by moving forces, as it is commonly done, but at least by a moving oscillator.

There are already several works dedicated to analytical and semi-analytical solutions, mainly concerned with stability issues of the moving oscillator, [1, 2]. In this paper different approach is presented directly linked to the issue of the critical velocity. At first, the problem of a moving force travelling over a beam supported by a finite depth elastic continuum in two-dimensions will be reviewed. Then, the semi-analytical solution of the mass moving on a beam supported by elastic springs will be extended to moving oscillator and validated by software LS-DYNA. Finally, the foundation will be substituted by frequency dependent springs, which is an acceptable approximation of the elastic finite depth continuum, and conclusions on the critical velocity will be drawn.

2 Finite Depth Elastic Continuum

It is assumed that the load P is traversing an infinite beam supported by a foundation of finite depth H, by a constant velocity v, as depicted on Fig. 1. It is further assumed that: (i) the beam obeys linear elastic Euler–Bernoulli theory; (ii) the foundation is represented by a linear elastic homogeneous continuum of finite width b under plane strain condition; (iii) the beam may be subjected to a normal force N acting on its axis (considered positive when inducing compression); (iv) gravitational effects on the beam and on the foundation are neglected.

It was derived in [3] that if horizontal displacements are omitted, but shear deformations are accounted for in the vertical dynamic equilibrium of the foundation, then the critical velocity ratio α_{cr} defined as $\alpha_{cr} = V_{cr}/v_{cr}$, where V_{cr} is the new value of the critical velocity and v_{cr} is the classical value of the critical velocity of the load passing a beam on Winkler's foundation, is governed by an approximate formula

$$\alpha_{cr} = \left(\sqrt{1-\eta_N} - \vartheta_s\right)\sqrt{\frac{2}{2+M^{2+\sqrt{\vartheta_s}}}} + \vartheta_s \tag{1}$$

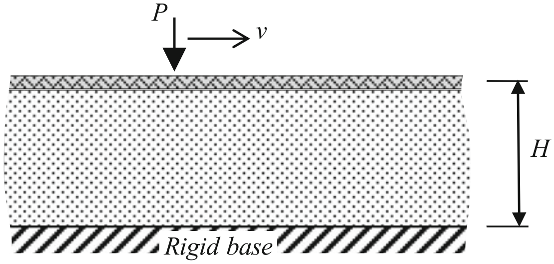


Fig. 1 Infinite beam on an elastic foundation of finite depth subjected to a moving load

where, ϑ_s is the shear ratio defined as $\vartheta_s = v_s/v_{cr}$ with v_s being the shear-wave velocity, and M is the mass ratio defined as the square root of the foundation mass to the beam mass $M = \sqrt{\rho b H/m}$ with ρ being the soil density and m the mass per length of the beam. Thus, for a lower mass ratio, the critical velocity approaches the classical value $v_{cr,N}$ with the effect of the normal force specified by the ratio η_N; and for a higher mass ratio, it approaches the velocity of propagation of shear waves in the foundation, which is the lowest wave-velocity of propagation related to the model adopted, because the Rayleigh waves cannot be developed if horizontal displacements are not properly introduced.

In the full two-dimensional model, both dynamic equilibrium equations are considered and thus horizontal displacements are not omitted. The system of governing equations read

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = \rho\mathbf{u}_{,tt} \tag{2}$$

$$EIw_{,xxxx} + Nw_{,xx} + c_b w_{,t} + mw_{,tt} + p_f = P\delta(x - vt) \tag{3}$$

where $\mathbf{u} = (u_x, u_z)$ is the vector of the displacement field in the foundation, λ and μ are Lame's constants of the elastic continuum, EI and c_b stand for the bending stiffness and the coefficient of viscous damping of the beam, p_f is the foundation pressure and t is the time. In this paper, derivatives will be designated by the corresponding variable symbol in the subscript position, preceded by a comma. Moreover, ∇ is the gradient and Δ is the Laplace operators applied on spatial variables x, z. The unknown beam deflection $w(x, t)$, displacement u_z, spatial coordinate z and force P are assumed positive when acting downward. Spatial coordinate x is positive to the right, the load travels from the left to the right and finally, δ is the Dirac delta function.

The solution method exploits displacement potentials and moving coordinate. Then, for the steady-state solution time dependent terms can be neglected and after the Fourier transform, analytical solution can be obtained in the Fourier space. The additional interface condition is ambiguous and can be written in form of zero horizontal displacement (ZHD), zero shear stress (ZSS) or some combination of these two in form of horizontal interface spring. The critical velocity can be identified in a semi-analytical way by identification of double poles on the real axis of the Fourier variable of the beam displacement image. Approximate formula for the critical velocity ratio is given as a sum of two parts, one is an addition to the previous Eq. 1 and the other one is an adaption Eq. 1

$$\alpha_{cr,add} = a_1\vartheta_j\left(\sqrt{\frac{2}{2 + M^{2+\sqrt{\vartheta_j}}}} - \sqrt{\frac{2}{2 + a_2(\vartheta_j + a_N)M^{2+\sqrt{\vartheta_j - a_N}}}}\right) \tag{4}$$

$$\alpha_{cr,prev} = \left(\sqrt{1 - \eta_N} - a_3\vartheta_j\right)\sqrt{\frac{2}{2 + M^{2+\sqrt{\vartheta_j}}}} + a_3\vartheta_j \tag{5}$$

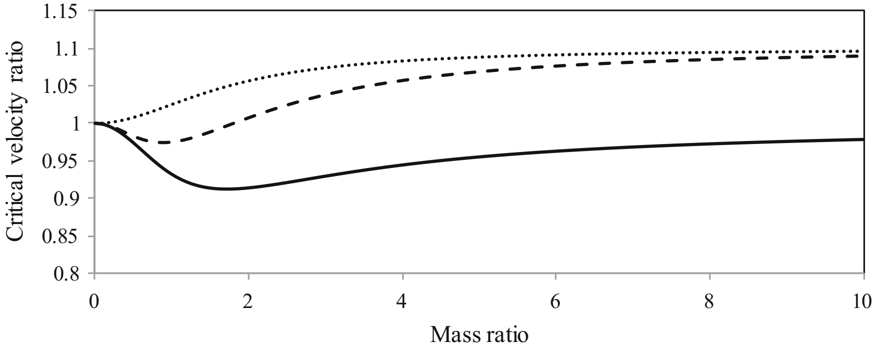


Fig. 2 Critical velocity ratio: simplified model (dotted), ZHD condition (dashed), ZSS condition (full), $\vartheta_s = 1.1$

where $a_N = 2a\eta_N(\vartheta_j - \eta_N)^2$ with $a = 1$ or 10 when $\vartheta_j < \eta_N$. In addition the subscript j is related to the velocity of propagation of shear waves, thus $j = s$, $a_1 = 0.3$, $a_2 = 0.4$, $a_3 = 0.99$ for ZHD; and for ZSS it is related to the velocity of propagation of Rayleigh waves, thus $j = R$, $a_1 = 0.5$, $a_2 = 0.4$ and $a_3 = 0.98$, so $\vartheta_R = v_R/v_{cr}$, where v_R is the velocity of propagation of Rayleigh waves. Results of the critical velocity are slightly affected by this additional interface condition, as can be seen in Fig. 2.

In Fig. 3 the influence of the normal force is shown for $\eta_N = 0.5$, simplified approach and ZSS condition are placed together in one graph for $\vartheta_s \in \langle 0.5; 1.5 \rangle$.

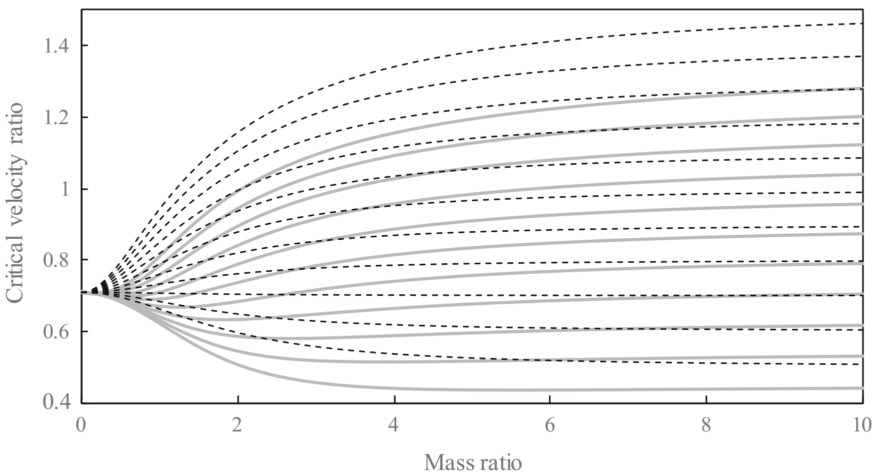


Fig. 3 Critical velocity ratio: simplified model (dashed), ZSS condition (full grey), $\vartheta_s \in \langle 0.5; 1.5 \rangle$, for each case the curves are starting from the bottom

It is seen, that there are some differences between the results of the critical velocity ratio related to ZHD and ZSS reflected especially in the asymptotic value of the critical velocity, which is v_s and v_R for the two conditions, respectively. On the other hand the asymptotic value of the simplified model and ZHD is approximately the same.

3 Massless Springs

In order to extend the previous analysis to the moving oscillator, finite beams on elastic springs foundation will be considered first. The reason is that, as shown in [4], the results on long finite beams provide very good approximation to the results related to infinite ones, so long as the mass is added to the moving force. Then the semi-analytical technique for solution of the moving mass problem by eigenvalues expansion can be extended to the moving oscillator and finally to the frequency dependent foundation. Last step would be to consider the complete two-dimensional continuum.

Thus, let a uniform motion of a constant mass and a vertical force with harmonic component along a horizontal infinite beam posted on a two-parameter visco-elastic foundation be assumed. Besides the previous simplifying assumptions, it is assumed that the mass is always in contact with the beam and its horizontal position is determined by its velocity. The problem at hand is depicted in Fig. 4.

The equation of motion for the unknown vertical displacement reads

$$EIw_{,xxxx} + \left(N - k_p\right) w_{,xx} + mw_{,tt} + c_b w_{,t} + kw = p(x,t) \tag{6}$$

with the loading term being

$$p(x,t) = \left(P + P_0 \sin\left(\omega_f t + \varphi_f\right) - Mw_{0,tt}(t)\right)\delta(x - vt) \tag{7}$$

Because this analysis will be used only for representation of the infinite situation, boundary conditions can be selected in the most convenient way. Assuming that

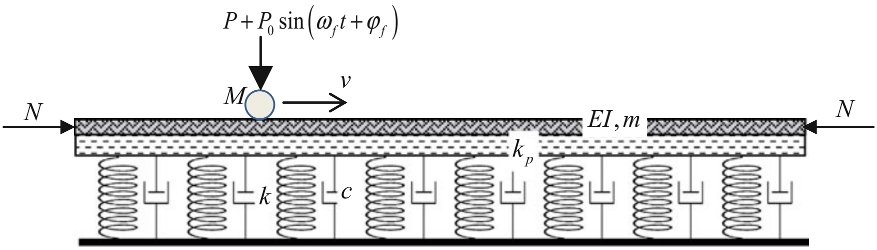


Fig. 4 Finite beam on a visco-elastic two-parameter foundation subjected to a moving load and a normal force

only n modes will be used, a compact matrix form for the generalized coordinate calculation can be presented as

$$\mathbf{M}(t) \cdot \mathbf{q}_{,tt}(t) + \mathbf{C}(t) \cdot \mathbf{q}_{,t}(t) + \mathbf{K}(t) \cdot \mathbf{q}(t) = \mathbf{p}(t) \tag{8}$$

where square $n \times n$ matrices **M**, **C**, **K** are not approximations resulting from some discretization of the problem, but are defined by vibration modes in their exact analytical form

$$M_{jk} = \delta_{jk} + Mw_j(vt)w_k(vt) \tag{9}$$

$$C_{jk} = \delta_{jk}\frac{c}{m} + 2Mvw_j(vt)w_{k,x}(vt) \tag{10}$$

$$K_{jk} = \delta_{jk}\omega_j^2 + Mv^2w_j(vt)w_{k,xx}(vt) \tag{11}$$

$$p_j = \left(P + P_0\sin\left(\omega_f t + \varphi_f\right)\right)w_j(vt) \tag{12}$$

These terms can be reordered, keeping the diagonal part of the matrices and introducing an additional variable λ that will join the necessary terms

$$\boldsymbol{M}(t) = \begin{bmatrix} M_{jk}^D & -w_j(\upsilon t) \\ Mw_k(\upsilon t) & 1 \end{bmatrix}, \boldsymbol{C}(t) = \begin{bmatrix} C_{jk}^D & 0 \\ 2Mvw_{k,x}(\upsilon t) & 0 \end{bmatrix} \tag{13}$$

$$\boldsymbol{K}(t) = \begin{bmatrix} K_{jk}^D & 0 \\ Mv^2w_{k,xx}(\upsilon t) & 0 \end{bmatrix} \tag{14}$$

Thus there is only one coupled equation and the second time derivative of the additional variable equals the contact force. If the oscillator is added, then the loading term from Eq. 7 must be completed by

$$(-k_{os}w_0(t) + k_{os}w_{os}(t))\delta(x - vt) \tag{15}$$

and additional equation read

$$M_{os}w_{os,tt}(t) + k_{os}w_0(t) - k_{os}w_{os}(t) = P_{os} \tag{16}$$

where the subscript "os" designates the quantities related to the oscillator. For typical values related to railways applications and very soft foundation, it is seen that mass induced frequency is superposed with the oscillator frequency (Fig. 5).

Therefore, the method for the critical velocity determination follows the identification of the resonant case. The mass induced vibration can be determined by semi-analytical methods presented in [4]. The typical graph of the real part of the induced frequencies is shown in Fig. 6.

Then according to the natural frequency of the oscillator, the velocity of the moving system is determined.

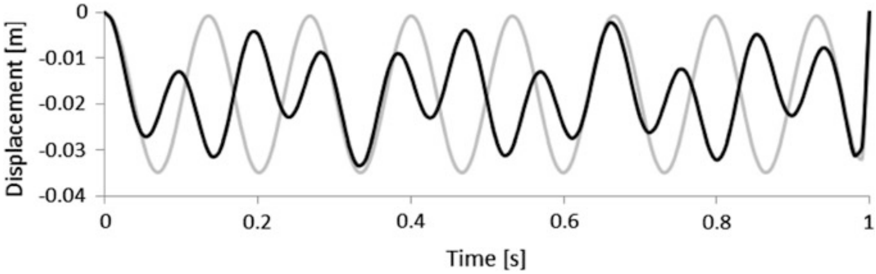
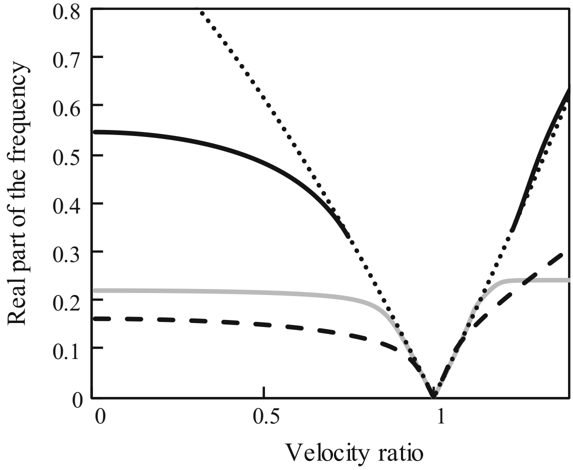


Fig. 5 Beam deflection under the load: moving oscillator (black) moving mass (grey)

Fig. 6 Real part of the dimensionless induced frequencies: two-mass oscillator (black and grey), moving mass (dashed) and cutting frequency (dotted)



4 Conclusions

In this paper, a technique for the semi-analytical determination of the critical velocity of the moving oscillator is proposed. Based on the characteristics of the oscillator, such a critical velocity can be lower that the critical velocity of the moving force and also lower than the velocity inducing instability of the moving mass.

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