Critical velocity of a load moving on a beam supported by a foundation of finite depth

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ABSTRACT: In this paper, the critical velocity of a uniformly moving load is analysed. It is assumed that the load is traversing an infinite beam supported by a finite depth foundation under plane strain condition. Analytical solution of the steady state deflection shape is derived. The critical velocity is then extracted by parametric analysis. Results obtained are compared with the previously published results of this author, where simplified plane models of the foundation were used. It is confirmed that there is an interaction between the beam and the foundation and thus the critical velocity is dependent on the mass ratio defined as the square root of the fraction of the foundation mass to the beam mass. For a low mass ratio, the critical velocity approaches the classical formula and for a higher mass ratio, it approaches the lowest wave-velocity of propagation in the foundation. There is only a small difference with respect to the previously published approximate formula for the critical velocity.

1 INTRODUCTION

Structures subjected to moving loads have several applications in rail, road and bridge engineering. When the load moves over a structure at a critical velocity, induced vibrations rapidly increase and safety and stability of the structure is compromised. The critical velocity depends on many factors and several theoretical models have been developed over the years for its determination. Naturally, as there are differences between the models, there are differences between the conclusions taken. The classical model for railway applications, an infinite beam on the Winkler foundation (Frýba, 1999), predicts the critical velocity that is usually higher than the ones observed in reality, because in this model there is no mass attributed to the foundation and thus no wave propagation in the foundation is possible. If a beam on an elastic half-space is considered, then the critical velocity is derived as the Rayleigh-wave velocity of propagation (Krylov et al. 2000).

More realistic models should be based on a finite-depth foundation. The plane model from (Jaiswal & Iyengar 1993) considers the dynamic equilibrium only in the vertical direction, and thus the critical velocity tends rapidly to zero with increasing mass ratio defined as the square root of the fraction of the foundation mass to the beam mass. Improvement of this model by shear contribution in (Dimitrovová 2015, Dimitrovová 2016) predicts a smooth transition between the classical critical velocity from (Frýba, 1999) and the velocity of propagation of shear waves in the foundation, and thus concludes that the critical velocity results from the beam-foundation dynamic interaction. Also the 3D model from (Dieterman & Metrikine 1997) proves that there is another critical velocity that is not coincident with the Rayleigh-wave propagation and therefore resulting from beam-foundation interaction.

In this contribution, models from (Dimitrovová 2015, Dimitrovová 2016) and (Náprstek & Fischer 2010) are compared. Derivations from (Náprstek & Fischer 2010) are simplified by considering only a constant moving force. Then the steady state part of the solution can be considered. Analytical solution of the steady state displacement shape is derived and the critical velocity is then extracted by parametric analysis. Results obtained are compared with the previously published results in (Dimitrovová 2015, Dimitrovová 2016), where simplified plane models of the foundation were used for the analysis of finite and infinite beams. It is confirmed that there is an interaction between the beam and the foundation and thus the critical velocity is not given either by the classical formula from (Frýba, 1999) or by the lowest wave-velocity of propagation in the foundation, but there is a smooth transition between these two extreme values governed by the mass ratio. For a low mass ratio, the critical velocity approaches the classical formula and for a higher mass ratio, it approaches the lowest wave-velocity of propagation in the foundation. The new results derived in this paper are dependent on the interface condition between the beam and the foundation. There is only a slight difference between the three possibilities: two options for the interface condition and the previously published approximate formula for the critical velocity in (Dimitrovová 2016). Namely, the interface condition in form of zero horizontal displacement gives results very similar to (Dimitrovová 2016) and zero shear stress condition results have the asymptotic tendency to slightly lower velocity, the velocity of propagation of Rayleigh waves.

2 THE MODEL AND ITS SOLUTION

It is assumed that the load is traversing an infinite beam supported by a foundation of finite depth *H* under plane strain condition, as depicted on Figure 1.

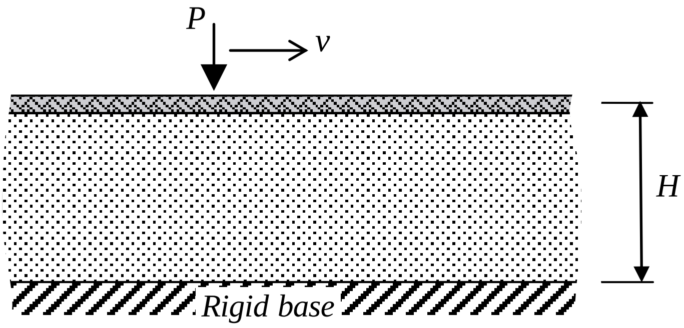

Figure 1: Infinite beam on an elastic foundation of finite depth subjected to a moving load.

It is further assumed that: (i) the beam obeys linear elastic Euler-Bernoulli theory; (ii) the beam vertical displacement is measured from the equilibrium deflection caused by the beam weight; (iii) the foundation is represented by a finite strip of width *b* under plane strain condition; (iv) the soil is modelled by linear elastic homogeneous material. The governing equations are given by

$$\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})=\rho\mathbf{u}_{,tt} \tag{1}$$

$$EIw_{,xxxx}(x,t)+c_b w_{,t}(x,t)+mw_{,tt}(x,t)+p(x,t)=P\delta(x-vt) \tag{2}$$

where $\mathbf{u}=(u_x,u_z)$ is the displacement field in the foundation, λ, μ are Lame's constants of the soil, ρ is the soil density and *t* is the time. Overall this paper, derivatives will be designated by the corresponding variable symbol in the subscript position, preceded by a comma. Moreover, ∇ is the gradient and Δ is the Laplace operators applied on spatial variables. Further, *P* is the moving load, *v* is its velocity and *p* is the foundation pressure. EI, m and c_b are bending stiffness, mass per unit length and viscous damping coefficient of the beam. The unknown beam deflection $w(x,t)$, spatial coordinate *z* and *P* are assumed positive when acting downward. Spatial coordinate *x* is positive to the right, the load travels from the left to the right and finally, δ is the Dirac delta function.

Damping in the soil can be assumed as hysteretic

$$\lambda=\lambda_0(1+\mathrm{i}\eta_h),\ \mu=\mu_0(1+\mathrm{i}\eta_h) \tag{3}$$

or viscous

$$\lambda=\lambda_0(1+c_s\bullet_{,t}),\ \mu=\mu_0(1+c_s\bullet_{,t}) \tag{4}$$

where η_h is the loss factor of the soil and c_s is the coefficient of the viscous damping in the soil.

It is more convenient to transform Equation (1) to a form exploiting the displacement potentials Φ and Ψ linked to the displacement components by

$$u_x=\Phi_{,x}+\Psi_{,z},\ u_z=\Phi_{,z}-\Psi_{,x} \tag{5}$$

Then Equation (1) can be transformed to two independent equations, which e.g. for the case with viscous damping in the soil are written as

$$v_p^2(\Phi_{,xx}+\Phi_{,zz})+c_s v_p^2(\Phi_{,xxt}+\Phi_{,zzt})-\Phi_{,tt}=0 \tag{6}$$

$$v_s^2(\Psi_{,xx}+\Psi_{,zz})+c_s v_s^2(\Psi_{,xxt}+\Psi_{,zzt})-\Psi_{,tt}=0 \tag{7}$$

where v_p and v_s are velocities of propagation of the pressure and shear waves in the soil, respectively, given by the well-known relations

$$v_p=\sqrt{\frac{\lambda_0+2\mu_0}{\rho}},\ v_s=\sqrt{\frac{\mu_0}{\rho}} \tag{8}$$

Firstly, the moving coordinate is introduced

$$r=x-vt,\ z=z,\ t=t \tag{9}$$

and only the steady state part of the solution is left.

$$(\Phi_{,rr}+\Phi_{,zz})-c_s v(\Phi_{,rrr}+\Phi_{,zzr})-\eta_p\Phi_{,rr}=0 \tag{10}$$

$$(\Psi_{,rr}+\Psi_{,zz})-c_s v(\Psi_{,rrr}+\Psi_{,zzr})-\eta_s\Psi_{,rr}=0 \tag{11}$$

$$EIw_{,rrrr}-vc_b w_{,r}+v^2 mw_{,rr}+p=P\delta(r) \tag{12}$$

where

$$\eta_p=\left(\frac{v}{v_p}\right)^2,\ \eta_s=\left(\frac{v}{v_s}\right)^2 \tag{13}$$

Further, it is convenient to introduce several dimensionless parameters. These parameters facilitate the resolution and analysis of the results obtained. They are

$$k=\frac{(\lambda_0+2\mu_0)b}{H},\ \chi=\sqrt[4]{\frac{k}{4EI}},\ w_{st}=\frac{P\chi}{2k} \tag{14}$$

as representation of the Winkler constant k that allows to express the static deflection w_{st} of the beam under the constant force P. Then,

$$v_{cr} = \sqrt[4]{\frac{4kEI}{m^2}} = \frac{1}{\chi}\sqrt{\frac{k}{m}}, \ \alpha = \frac{v}{v_{cr}}, \ \mu = \sqrt{\frac{\rho bH}{m}} \tag{15}$$

stand for the classical value of the critical velocity from (Frýba, 1999), and velocity and mass ratios. It is also convenient to introduce dimensionless displacements

$$\widehat{w} = \frac{w}{w_{st}}, \ \widehat{u}_z = \frac{u_z}{w_{st}}, \ \widehat{u}_x = \frac{u_x}{w_{st}} \tag{16}$$

and spatial coordinates

$$\zeta = \frac{z}{H}, \ \xi = \chi r \tag{17}$$

Finally, damping effects are expressed by

$$\eta_f = c_s\sqrt{\frac{k}{m}}, \ \eta_b = \frac{c_b}{2\sqrt{mk}} \tag{18}$$

With these designations, Equations (10-12) are firstly simplified to

$$\left(\chi^2\Phi_{,\xi\xi} + \frac{\Phi_{,\zeta\zeta}}{H^2}\right) - cv\chi\left(\chi^2\Phi_{,\xi\xi\xi} + \frac{\Phi_{,\zeta\zeta\xi}}{H^2}\right) - \eta_p\chi^2\Phi_{,\xi\xi} = 0 \tag{19}$$

$$\left(\chi^2\Psi_{,\xi\xi} + \frac{\Psi_{,\zeta\zeta}}{H^2}\right) - cv\chi\left(\chi^2\Psi_{,\xi\xi\xi} + \frac{\Psi_{,\zeta\zeta\xi}}{H^2}\right) - \eta_s\chi^2\Psi_{,\xi\xi} = 0 \tag{20}$$

$$\widehat{w}_{,\xi\xi\xi\xi} - 8\alpha\eta_b\widehat{w}_{,\xi} + 4\alpha^2\widehat{w}_{,\xi\xi} + \frac{4p}{kw_{st}} = 8\delta(\xi) \tag{21}$$

and after single Fourier transform of the form

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(\xi)e^{-\mathrm{i}\omega\xi}\mathrm{d}\xi \tag{22}$$

to even more compact from

$$\tilde{\Phi}_{,\zeta\zeta} - \frac{(\alpha\omega\mu)^2}{\eta_p}\left(1 - \frac{\eta_p}{1 - \mathrm{i}\omega\alpha\eta_f}\right)\tilde{\Phi} = 0 \tag{23}$$

$$\tilde{\Psi}_{,\zeta\zeta} - \frac{(\alpha\omega\mu)^2}{\eta_p}\left(1 - \frac{\eta_s}{1 - \mathrm{i}\omega\alpha\eta_f}\right)\tilde{\Psi} = 0 \tag{24}$$

$$\omega^4\tilde{w} - 8\mathrm{i}\omega\alpha\eta_b\tilde{w} - 4\alpha^2\omega^2\tilde{w} + \frac{4\tilde{p}}{kw_{st}} = 8 \tag{25}$$

Equations (23-24) expressed in the frequency domain can be readily solved. Having this solution, displacement field in the soil can be written exploiting Equations (5), also transformed to the frequency domain. Basically, the Fourier images of the displacement potentials are

$$\tilde{\Phi} = A\cosh(\zeta\hat{\eta}_p) + B\sinh(\zeta\hat{\eta}_p) \tag{26}$$

$$\tilde{\Psi} = C\cosh(\zeta\hat{\eta}_s) + D\sinh(\zeta\hat{\eta}_s) \tag{27}$$

where

$$\hat{\eta}_p = \frac{\alpha\omega\mu}{\sqrt{\eta_p}}\sqrt{1 - \frac{\eta_p}{1 - \mathrm{i}\omega\alpha\eta_f}} \tag{28}$$

$$\hat{\eta}_s = \frac{\alpha\omega\mu}{\sqrt{\eta_p}}\sqrt{1 - \frac{\eta_s}{1 - \mathrm{i}\omega\alpha\eta_f}} \tag{29}$$

Thus there are four integration constants to be determined from the boundary and interface conditions. After substitution into Equation (5) in the frequency domain, one obtains

$$\tilde{u}_x = \frac{1}{w_{st}H}\left(\mathrm{i}\frac{\alpha\omega\mu}{\sqrt{\eta_p}}\tilde{\Phi} + \tilde{\Psi}_{,\zeta}\right) \tag{30}$$

$$\tilde{u}_z = \frac{1}{w_{st}H}\left(\tilde{\Phi}_{,\zeta} - \mathrm{i}\frac{\alpha\omega\mu}{\sqrt{\eta_p}}\tilde{\Psi}\right) \tag{31}$$

and the boundary conditions at the rigid base are

$$\tilde{u}_x(\zeta = 1) = 0, \ \tilde{u}_z(\zeta = 1) = 0 \tag{32}$$

Then, stress tensor components can be derived in the frequency domain and the soil pressure acting on the beam can be expressed as

$$\tilde{p} = -b\tilde{\sigma}_z(0) \tag{33}$$

Moreover, the beam deflection is

$$\tilde{w} = \tilde{u}_z(0) \tag{34}$$

Equations (33-34) will introduce the solution obtained for the soil displacements into the beam equation (25). This is actually the third equation for the determination of the integration constants.

Only one more equation can be prescribed, but it is not clear whether it is more appropriate to require zero horizontal displacement at the interface, which would be in conformity with the zero displacement on the beam axis, or zero shear stress representing a smooth contact. In conclusion, the number of integration constants to be solved from the boundary conditions is not sufficient and it is necessary to decide what interface condition is more appropriate. This can be done according to the finite element solution. Using the method described in (Dimitrovová & Rodrigues 2011), steady state solution of the deflection shape can be obtained in few steps of the enhanced moving window algorithm.

For the sake of completeness, both possibilities of interface conditions were analysed analytically. Having the integration constants, the Fourier image of the deflection shape of the beam and of the displacement field in the foundation can be expressed analytically.

The inverse transform was accomplished numerically and after that the critical velocity was extracted by a parametric analysis. Results obtained are compared with the previously published results in Figure 2. The simplified formula from (Dimitrovová, 2016) reads as

$$V_{cr} = v_{cr}\left[\left(1-\vartheta_s\right)\sqrt{\frac{2}{2+\mu^{2+\sqrt{\vartheta_s}}}}+\vartheta_s\right] \quad (35)$$

where the shear ratio is $\vartheta_s = v_s / v_{cr}$. The new value of the critical velocity is distinguished from the classical one by the capital letter V_{cr}. Thus the velocity ratio in Figure 2 is in fact V_{cr} / v_{cr}. Only one particular case of $\vartheta_s = 0.5$ is plotted, but other cases have similar tendencies. Except for the damping coefficients, there are no more dependent variables in the dimensionless solution, only the soil Poisson ratio, which connects the wave-velocities of propagation v_s and v_p, therefore the conclusions presented are valid for all possible cases. Asymptotic tendency is shown in Figure 2 for higher mass ratio, therefore there is no need to examine its further values.

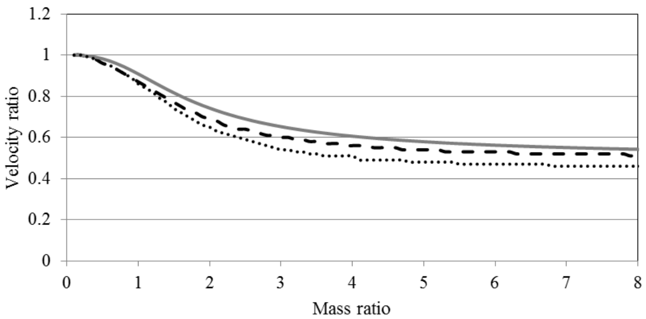

Figure 2. Critical velocity as a function of the mass ratio for shear ratio 0.5: previous estimate (grey), solution with zero displacement interface condition (black dashed), solution with zero shear interface condition (black dotted).

3 CONCLUSIONS

In this paper, the analytical solution of the deflection shape of an infinite beam subjected to a moving constant force supported by finite depth foundation under plane strain condition was derived. Two forms of such results correspond to two extreme interface conditions between the beam and the foundation. The critical velocity of the load was extracted by a parametric analysis. Conclusions previously taken in (Dimitrovová 2016) were confirmed. Thus, there is an interaction between the beam and the foundation and so the critical velocity is not given either by the classical formula from (Frýba, 1999) or by the lowest wave-velocity of propagation in the foundation, but there is a smooth transition between these two extreme values governed by the mass ratio defined as the square root of the fraction of the foundation mass to the beam mass. For a low mass ratio, the critical velocity approaches the classical formula and for a higher mass ratio, it approaches the lowest velocity of propagation of waves that are allowed by the particular model of the foundation.

4 ACKNOWLEDGEMENTS

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5 REFERENCES

Dieterman, H.A. & Metrikine, A.V. 1997. Steady state displacements of a beam on an elastic half-space due to uniformly moving constant load. *European Journal of Mechanics, A/Solids* 16: 295-306.

Dimitrovová, Z. 2015. Uniformly moving load on a beam supported by a foundation with finite depth. *Proc. IFToMM International Conference on Engineering Vibration (ICoEV-2015),* Ljubljana, Slovenia, 7-10 September.

Dimitrovová, Z. 2016. Critical velocity of a uniformly moving load on a beam supported by a finite depth foundation. *Journal of Sound and Vibration* 366: 325–342.

Dimitrovová, Z. & Rodrigues, A.F.S. 2011. An Enhanced Moving Window Method: Applications to High-Speed Tracks. In Topping, B.H.V. & Tsompanakis, Y. (eds) Civil-Comp Press, Stirlingshire, UK. *Proc. 13th International Conference on Civil, Structural and Environmental Engineering Computing (CC2011),* Chania, Crete, Greece, 6-9 September.

Frýba, L. 1999. Vibration of solids and structures under moving loads. Research Institute of Transport, Prague (1972), 3rd edition, Thomas Telford, London.

Jaiswal, O.R. & Iyengar, R.N. 1993. Dynamic response of a beam on elastic foundation of finite depth under a moving force. *Acta Mechanica* 96: 67-83.

Krylov, V.V., Dawson, A.R., Heelis, M.E. & Collop A.C. 2000. Rail movement and ground waves caused by high-speed trains approaching track-soil critical velocities. *Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid Transit* 214: 107-116.

Náprstek, J. & Fischer, C. 2010. Interaction of a Beam and Supporting Continuum under a Moving Periodic Load. In Topping, B.H.V., Adam, J.M., Pallarés, F.J., Bru, R. & Romero, M.L. (eds) Civil-Comp Press, Stirlingshire, UK. *Proc. 10th International Conference on Computational Structures Technology (CST2010),* Valencia, Spain, 14-17 September.