Semi-Analytical Solution of a Moving Mass Problem

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Abstract

In this paper a new analytical solution for a moving mass problem is given. The new analytical formula is presented for the deflection shape of an infinite beam that is traversed by a moving mass and supported by a visco-elastic foundation. In such a case the deflection shape resembles the one associated with the moving force with an additional oscillation around it. The frequency of this mass induced oscillation depends on the foundation characteristics and the amplitude can be derived analytically. It is also shown that if the force associated with the mass has a harmonic component, then its frequency is superposed to the one induced by the foundation. The new formula presented accounts also for the effect of the normal force and Pasternak modulus.

Keywords: transverse vibration, moving mass, eigenvalue expansion, Fourier transform, Laplace transform, semi-analytical solution.

1 Introduction

The investigations on moving load problems were initiated with first railway line construction. Since then, numerous studies have been published on this subjected. Among them several are based on analytical or semi-analytical approaches. Regardless the excessive number of published works, there are still some unsolved issues, and, unfortunately, some of the solutions that can be found in the literature are not correct. In this contribution several approaches are reviewed, missing solutions are presented and the errors are pointed out.

Regarding the finite beam, the solution of the problem of the moving mass can be expressed by the eigenvalue expansion method. The first solution of this kind was presented in [1]. This solution suffers from the lack of terms related to the effect of the Coriolis and centrifugal forces. This was pointed out by several other authors [2-3], nevertheless, solutions repeating the same error are still emerging. It is necessary to realize that sometimes these terms are negligible, but often, their omission can completely distort the solution. This is especially true for beams on elastic foundation, as will be shown in this paper. Solution presented in [3] is extended here to account for the effect of the normal force, Pasternak modulus and oscillating force accompanying the constant mass. The disadvantage of the solution method is that the equations in the modal space are coupled, thus even if no discretization is involved and mode shapes are introduced in their analytical form, the generalized coordinate must be solved numerically.

To solve vibrations of infinite homogeneous beams, usually integral transforms are used. Unfortunately, also here, some of the solutions that can be found in the literature, are not correct. Double Fourier transform disregard the foundation induced oscillation and therefore the solution presented *e.g.* in [4] is not correct. The correct approach should be based on an additional calculation that determines this frequency, [5]. Then the problem can be solved by Laplace and Fourier transforms, where the final evaluation can be analytical, based on the residual theorem. In a truly steady state solution the effect of mass is not seen and the deflection shape corresponds to the one induced by the moving force. In other cases the mass oscillates around this position with the frequency induced by the foundation. If the mass is accompanied by a harmonic force, than these two oscillations are superposed.

The new semi-analytical solution presented here was verified by commercial LS-DYNA software. In summary, the new contributions of this paper are the following:

(i) confirmation from analysis of finite beams that the effect of Coriolis and centrifugal forces cannot be omitted in cases with elastic foundation;

(ii) confirmation from analysis of finite beams that additional oscillation is induced by the moving mass;

(iii) confirmation from analysis of finite beams that such an additional oscillation is superposed to the forced oscillation of the harmonic force;

(iv) definition of an iterative procedure for determination of the frequency of the mass induced oscillation;

(ii) new analytical formulas for amplitudes of this oscillation;

(ii) definition of a procedure that defines full deflection shapes from analysis of semi-infinite beams.

The paper is organized in two main sections, one is dedicated to finite beams and the other one to infinite ones. Then the paper is concluded in Section 4.

2 Mass moving on a finite beam

Let a uniform motion of a constant mass and a vertical force with a harmonic component along a horizontal simply supported beam posted on a two-parameter visco-elastic foundation be assumed. The beam material is homogeneous and isotropic, the beam cross-section is uniform and the beam obeys linear elastic Euler-Bernoulli theory. It is assumed that the mass is always in contact with the beam and its horizontal position is determined by the velocity; also, the load proceeds from left to right. The problem at hand is depicted in Figure 1.

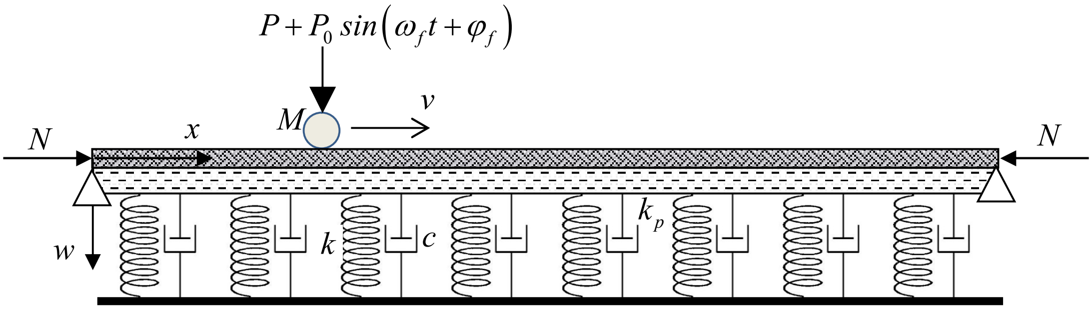

Figure 1: Simply supported beam on a two-parameter visco-elastic foundation subjected to a moving load and a normal force.

The equation of motion for the unknown vertical deflection field $w(x,t)$ can be written as:

$$EIw_{,xxxx}+\left(N-k_p\right)w_{,xx}+mw_{,tt}+cw_{,t}+kw=p\left(x,t\right) \tag{1}$$

where EI, m and N stand for the bending stiffness and mass per unit length of the beam and a normal force acting on the beam (positive when inducing compression). k, k_p and c are Winkler's and Pasternak's moduli of the foundation and the coefficient of viscous damping of the foundation. x designates the spatial coordinate and t is the time. Derivatives are designated by the respective variable in the subscript position, preceded by a comma. For a constant mass M and an associated constant force P with a harmonic component P_0, the loading term $p(x,t)$ can be written as:

$$p\left(x,t\right)=\left(P+P_0\sin\left(\omega_f t+\varphi_f\right)-Mw_{0,tt}\left(t\right)\right)\delta\left(x-vt\right) \tag{2}$$

Function sine is used to define the oscillation with forced frequency ω_f and phase φ_f to keep this term in the real domain, because then, the whole solution can be solved in the real domain. The mass displacement $w_0(t)=w(vt,t)$, *i.e.* the mass is always in contact with the beam, as already stated. Initial conditions are considered as homogeneous. x has its origin at the left extremity of the beam and zero time corresponds to the load position at x=0. For the solution it is necessary to remove the additional unknown $w_0(t)$ and express it in terms of the unknown field $w(x,t)$ as:

$$\left(P+P_0\sin\left(\omega_f t+\varphi_f\right)-M\left(w_{,tt}\left(x,t\right)+2vw_{,xt}\left(x,t\right)+v^2w_{,xx}\left(x,t\right)\right)\right)\delta\left(x-vt\right) \tag{3}$$

Thus

$$
\begin{aligned}
&EIw_{,xxxx} + \left(N - k_p\right) w_{,xx} + mw_{,tt} + cw_{,t} + kw \\
&+ M\delta\left(x - vt\right)\left(w_{,tt}\left(x,t\right) + 2vw_{,xt}\left(x,t\right) + v^2 w_{,xx}\left(x,t\right)\right) \\
&= \left(P + P_0 \sin\left(\omega_f t + \varphi_f\right)\right)\delta\left(x - vt\right)
\end{aligned} \tag{4}
$$

Boundary conditions for simply supported beam are written below, but other conditions could be equally considered, for instance, in Eq. (6) boundary conditions for left cantilever are stated.

$$
w\left(0,t\right) = 0\,,\; w_{,xx}\left(x,t\right)\Big|_{x=0} = 0\,,\; w\left(L,t\right) = 0\,,\; w_{,xx}\left(x,t\right)\Big|_{x=L} = 0 \tag{5}
$$

$$
w\left(0,t\right) = 0\,,\; w_{,x}\left(x,t\right)\Big|_{x=0} = 0\,,\; w_{,xx}\left(x,t\right)\Big|_{x=L} = 0\,,\; w_{,xxx}\left(x,t\right)\Big|_{x=L} = 0 \tag{6}
$$

In Eqs. (5-6) the beam length is designated as L. Solution of the problem can be obtained by eigenvalue expansion:

$$
w\left(x,t\right) = \sum_{j=1}^{\infty} q_j\left(t\right) w_j\left(x\right) \tag{7}
$$

where q_j stand for modal coordinates and w_j for vibration modes. The same designation "w" can be used for the deflection field as well as for the vibration modes, because the vibration modes are distinguished by the corresponding subscript. As usual, the modes are normalized by mass, therefore:

$$
\delta_{jk} = \int_0^L m w_j\left(x\right) w_k\left(x\right) dx \tag{8}
$$

where δ_{jk} is the Kronecker delta. Modal expansion is commonly governed by undamped vibration modes, because this allows their determination within the real domain and completeness of the eigenspace is guaranteed. Unfortunately, equations in modal space are coupled. Assuming only n vibration modes, then the modal equations can be written as:

$$
\mathbf{M}\left(t\right)\cdot\ddot{\mathbf{q}}\left(t\right) + \mathbf{C}\left(t\right)\cdot\dot{\mathbf{q}}\left(t\right) + \mathbf{K}\left(t\right)\cdot\mathbf{q}\left(t\right) = \mathbf{p}\left(t\right) \tag{9}
$$

In the equation above square nxn matrices **M**, **C**, **K** are defined by introduction of vibration modes in their exact analytical form, *i.e.* without any discretization. The system (9) cannot be solved analytically, but numerically. For a numerical solution in Matlab code, the system should be written in the state space form as:

$$
\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \tilde{\mathbf{q}} \end{Bmatrix} \tag{10}
$$

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\tilde{\mathbf{q}} \end{Bmatrix} \tag{11}$$

Computational time increases exponentially with the number of modes involved. Precision of a solution obtained for a certain number of modes cannot be simply increased by including additional modes, but the whole system must be recalculated in its entirety again. If there is no elastic foundation, usually low number of modes is sufficient (around 5-10). With the foundation included, the number of modes must be much higher, depending on several factors and it ranges around 100-200, or more. As an example, the solution of the moving mass and its corresponding weight on a left cantilever is shown in Figure 2. This is one of the examples that are presented in [3]. Numerical data from [3] are slightly adapted to: L=7.62m, P=25.79kN, M=2629kg, EI=9480.6kNm^2, m=46kg/m, v=50.8m/s. It is seen that in this case the effect of the Coriolis and centrifugal forces is significant. This is, however, not a very good example, since the deflection is quite large and the validity of the Euler-Bernoulli beam theory is compromised.

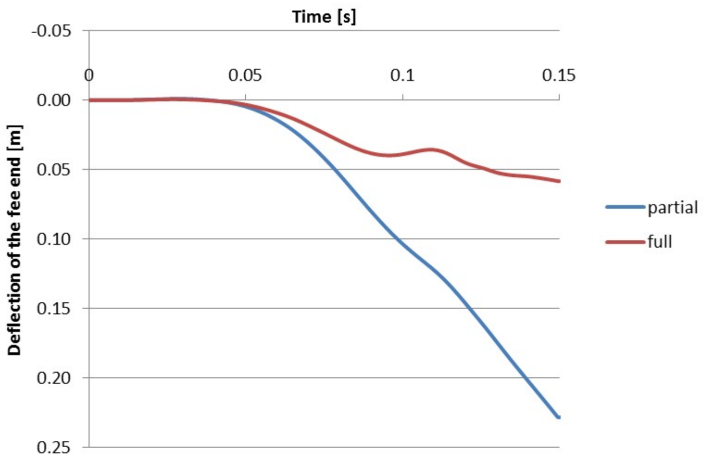

Figure 2: Deflection of the cantilever free end, “partial” means that some terms were omitted as in [3], “full” means that all terms are included.

Another application is a simply supported beam on an elastic foundation. The input data are: L=100m, P=100kN, M=10ton, EI=6.4MNm^2, m=60kg/m, k=4MN/m^2, v=100m/s. The beam and foundation data are related to railway applications, where the beam stands for one single rail. Deflection shapes are shown in Figure 3. In this case 150 modes were necessary for a good accuracy of the solution. Convergence study is shown on the same case, but with stiffer foundation of k=30MN/m^2 . In Figure 4 deflection under the load is plotted along the beam length for 50 and 100 modes. It can thus be concluded that the convergence is slow, because the difference between 50 and 100 modes is still quite large, in terms of the maximum displacement as well as the frequency of the additional oscillations. It is also seen that these oscillations are nicely formed and such a result can be used for verification of the frequency determined for infinite beams.

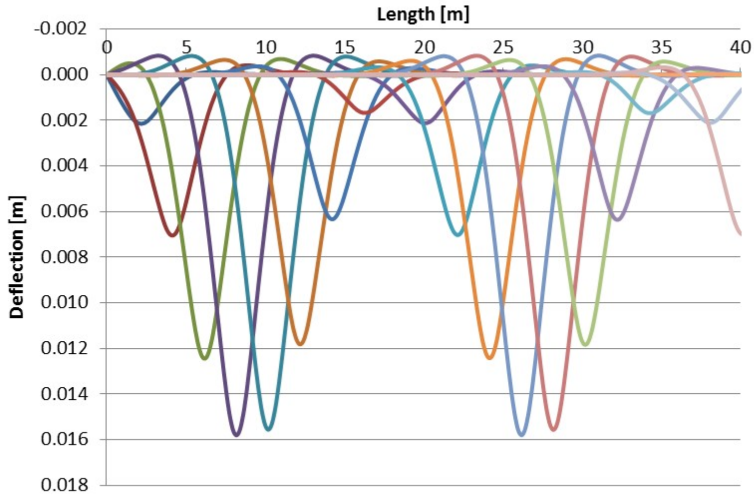

Figure 3: Deflection of the simply supported beam on an elastic foundation, initial 40m of the full length, deflections related to mass position at each 2m.

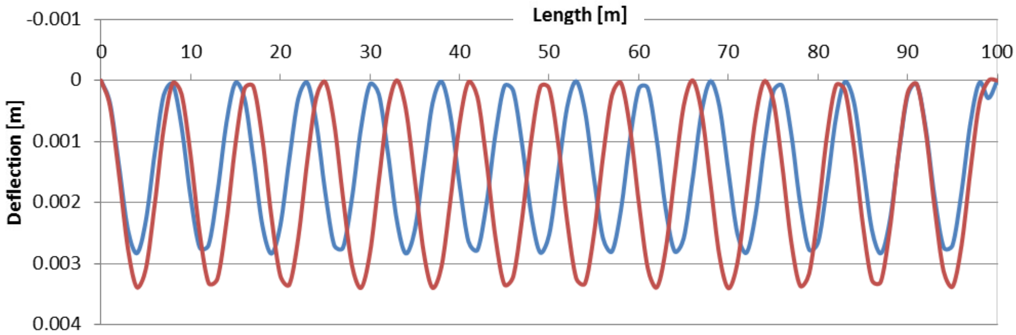

Figure 4: Convergence analysis: deflection under the load along the beam length: 50 modes (blue), 100 modes (red).

Another study intends to confirm the importance of Coriolis and centrifugal forces. Using the same foundation as before, the solution without these terms is unstable. The softest foundation where it is possible to get some results is k=2kN/m^2. The deflection under the load is plotted in Figure 5. Due to the very soft foundation, the displacement values are very high, but nevertheless, completely different from the solution with all terms included.

Next Figure 6 shows the same case, but full deflection shapes are shown for load position at 60, 70, 80 and 90m. It can thus be concluded that the inclusion of Coriolis and centrifugal forces is essential for the beams on elastic foundation, due to the relatively high curvature in the place of the load application. Deflection in Figures 5 and 6 is unphysical, but this example highlights how low the foundation stiffness has to be to avoid instability of the partial solution.

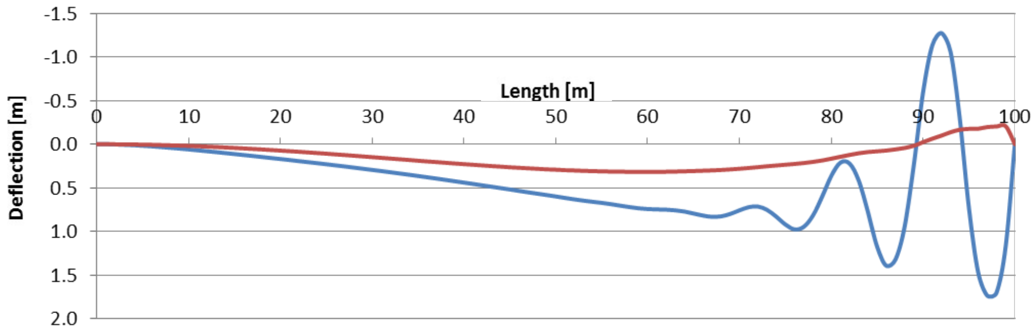

Figure 5: The importance of the effect of Coriolis and centrifugal forces: deflection under the load along the beam length: full solution (red), partial solution (blue).

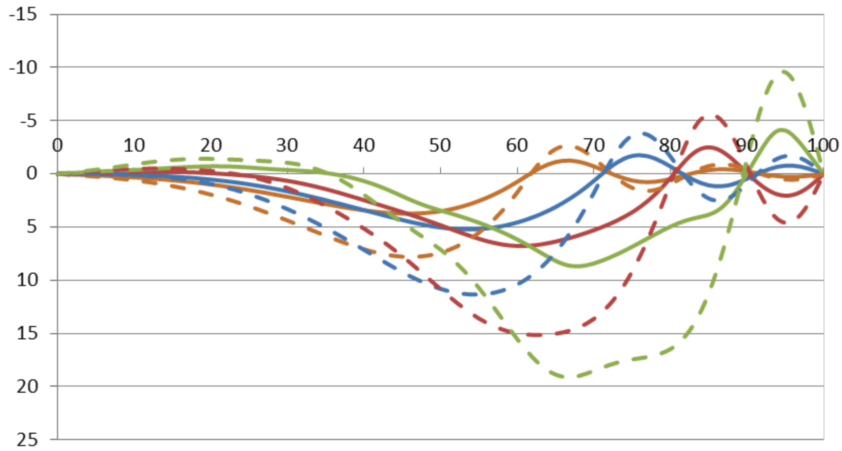

Figure 6: The importance of the effect of Coriolis and centrifugal forces: deflection shapes at load position at 60 (orange), 70 (blue), 80 (red) and 90m (green): full solution (solid), partial solution (dashed).

3 Mass moving on an infinite beam

The governing equation of the vibrations of an infinite beam can be written in a similar way as before:

$$EIw_{,xxxx} + \left(N - k_p\right)w_{,xx} + mw_{,tt} + cw_{,t} + kw = \left(P + P_0 e^{i\omega_f t} - Mw_{0,tt}\left(t\right)\right)\delta\left(x - vt\right) \quad (12)$$

In this formulation there is no problem to express the harmonic component of the force with complex numbers. Similarly as before the additional unknown $w_0(t)$ must be expressed in terms of the unknown field $w(x,t)$ and moreover, it is convenient to introduce the moving coordinate $s=x-vt$. Nevertheless, because some oscillations will be detected in the quasi-steady state solution, the time dependent terms cannot be removed. Then several dimensionless parameters can be introduced in order to facilitate the resolution and analysis of the results. It reads:

$$\widehat{w}_{,\xi\xi\xi\xi}+4\left(\eta_N-\eta_S+\alpha^2\right)\widehat{w}_{,\xi\xi}+4\widehat{w}_{,\tau\tau}-8\alpha\widehat{w}_{,\xi\tau}+8\left(\eta_c\widehat{w}_{,\tau}-\eta_c\alpha\widehat{w}_{,\xi}\right)+4\widehat{w}$$
$$=4\left(2+2\eta_P e^{i\widehat{\omega}_f\tau}-\eta_M\widehat{w}_{,\tau\tau}\right)\delta\left(\xi\right) \tag{13}$$

where

$$\alpha=\frac{v}{v_{cr}},\ \eta_c=\frac{\mathrm{c}}{2\sqrt{mk}},\ \eta_P=\frac{P_0}{P},\ \eta_N=\frac{N}{2\sqrt{kEI}},\ \eta_S=\frac{k_p}{2\sqrt{kEI}},\ \eta_M=\frac{M\chi}{m} \tag{14}$$

$$\chi=\sqrt[4]{\frac{k}{4EI}},\ v_{cr}=\sqrt[4]{\frac{4kEI}{m^2}}=\frac{1}{\chi}\sqrt{\frac{k}{m}},\ w_{st}=\frac{P\chi}{2k},\ \widehat{w}=\frac{w}{w_{st}}\ \text{and}\ \widehat{\omega}_f=\frac{\omega_f}{\chi v_{cr}} \tag{15}$$

There parameters were used together with dimensionless coordinates $\xi=\chi s$ and $\tau=\chi v_{cr}t$.

It can be shown that by application of the double Fourier transform on the governing equation (13) in the form of:

$$F\left(p,q\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(\xi,\tau\right)e^{-i\left(p\xi+q\tau\right)}d\xi d\tau \tag{16}$$

gives solution, where the vibration induced by the foundation is suppressed, which is not correct. In such a case the mass only affects the amplitude of the forced harmonic component. This means that, if there is only a constant force, mass produces no effect on the solution. Such a solution is presented in [4]. On the other hand, solution from [5] indicates that mass induces additional oscillations, which frequency corresponds to the one visualized in figures on finite beams. In order to present such a solution, Laplace and Fourier transforms must be used. Solution in [5] does not develop the amplitudes of the final solution. One of the new contributions of this paper is derivation of an analytical formula for them.

In summary, the solution in [4] can be adapted for the time dependent displacement under the mass as:

$$\widehat{w}_0\left(\tau\right)=\frac{4K\left(0\right)}{\pi}+\frac{4\eta_P K\left(\widehat{\omega}_f\right)}{\pi-2\eta_M\widehat{\omega}_f^2K\left(\widehat{\omega}_f\right)}e^{i\widehat{\omega}_f\tau} \tag{17}$$

which shows that for $\widehat{\omega}_f=0$ there is no harmonic movement. The new formula identifies all harmonics, in the way as:

$$\widehat{w}_0\left(\tau\right)=\frac{4K\left(0\right)}{\pi}+\frac{4\eta_P K\left(\widehat{\omega}_f\right)}{\pi-2\eta_M\widehat{\omega}_f^{\,2}K\left(\widehat{\omega}_f\right)}e^{i\widehat{\omega}_f\tau}+A\left(q_{M_1}\right)e^{iq_{M_1}\tau}+A\left(q_{M_2}\right)e^{iq_{M_2}\tau} \tag{18}$$

where

$$A\left(q_{M_j}\right)=\frac{8\left(q_{M_j}-\hat{\omega}_f+\eta_P q_{M_j}\right)K\left(q_{M_j}\right)}{C\left(q_{M_j}\right)} \tag{19}$$

$$\begin{aligned}C\left(q_{M_j}\right)&=\left(2q_{M_j}-\hat{\omega}_f\right)\left[2\pi-4\eta_M q_{M_j}^2 K\left(q_{M_j}\right)\right]+\\ &q_{M_j}\left(q_{M_j}-\hat{\omega}_f\right)\left[-8\eta_M q_{M_j} K\left(q_{M_j}\right)-4\eta_M q_{M_j}^2 K_{,q}\left(q_{M_j}\right)\right]\end{aligned} \tag{20}$$

$$K\left(q\right)=\int_{-\infty}^{\infty}\frac{dp}{D\left(p,q\right)} \tag{21}$$

$$D\left(p,q\right)=p^4-4p^2\left(\eta_N-\eta_S+\alpha^2\right)-4q^2+8\alpha pq+8iq\eta_c-8ip\alpha\eta_c+4 \tag{22}$$

$K\left(q\right)$ and $K_{,q}\left(q\right)$ can be determined from the residual theorem.

In addition, it is necessary to solve numerically for the frequency that is induced by the foundation. This can be done by simple iterative algorithm, which convergence is secured.

The procedure described above defines the deflection under the load in an analytical form. Full deflection shapes can be derived by joining two semi-infinite beams. If the applied force is constant, then the mass oscillates around the stationary position, as demonstrated in Figures 3 and 4. If the force has a harmonic component, than both harmonic movements are superposed around the stationary position.

4 Conclusions

In this paper, firstly, the semi-analytical solution of the moving mass problem on a finite beam was presented. The importance of the Coriolis and centrifugal forces was highlighted. The solution with elastic foundation indicated the form of the deflection shape on infinite beams. Secondly, a new analytical solution was presented for the deflection shape of an infinite beam that is traversed by a moving mass, that is accompanied by a force with a harmonic component and supported by a visco-elastic foundation. In such a case the deflection shape resembles the one associated with the moving force with an additional oscillation around it. The frequency of this oscillation can be determined by simple iterative method and the amplitude can by given analytically.

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