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EXTENDED FORMULA FOR THE CRITICAL VELOCITY OF A LOAD MOVING ON A BEAM SUPPORTED BY A FINITE DEPTH FOUNDATION

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Abstract. *In this paper, an extended formula for the critical velocity of a uniformly moving load is derived. It is assumed that the load is traversing an infinite beam supported by finite depth foundation under plane strain condition. The critical velocity is extracted by parametric analysis applied on the analytical solution of the steady state deflection beam shape. Results obtained are compared with the previously published results of this author, where simplified assumptions were implemented on the shear contribution. It is confirmed that there is an interaction between the beam and the foundation and thus the critical velocity is dependent on the mass ratio defined as the square root of the fraction of the foundation mass to the beam mass. Several options for damping are also analysed and results of displacement fields are compared with finite element simulations. In order to obtain steady-state form of the finite element results, the enhanced moving widow method is implemented in software ANSYS.*

1 INTRODUCTION

The response of rails to moving loads is important research topic of high-speed railway transportation. If simple geometries of the track and subsoil are considered, it can be assumed that the track structure acts as a continuously supported beam resting on a uniform layer of springs. Two distinct interpretations are used, either the beam is modelled by the rail and the layer of springs represents the underlying remainder of the track structure, or an equivalent beam encompassing the whole track is used and the spring layer stands for the subgrade or foundation. The stiffness of the spring layer along the length of the beam is named as the track modulus and defines Winkler's model, which is often referred to as a "one-parameter model". Such a simplified model was traditionally used to estimate the critical velocity of moving trains.

The first solution of steady-state dynamic response of an infinite beam on elastic foundation traversed by moving load was presented by Timoshenko [1]. In [2], the moving coordinate system is introduced to convert the governing equation to ordinary differential equation that can be solved by the Fourier integral transform. In [3], the concept of the dynamic stiffness matrix is implemented. Two semi-infinite beams are solved for and connected by continuity equations. Then the critical velocity can be determined as the velocity that ensures the nullity of the determinant of the dynamic stiffness matrix. This concept was extended to finite and infinite beams with sudden change in foundation stiffness [4].

If the beam is modelled as the rail, the classical formula, predicts very high critical velocity, giving impression that is unreachable by high-speed trains and consequently no attention was paid to this fact during expansions of high-speed railway network. Unfortunately, practical experience showed that the realistic critical velocity can be much lower [5] and should be related to the wave-velocity of propagation in the foundation. Therefore, in further investigations the spring layer was replaced by elastic half-space and the critical velocity was determined as the Rayleigh-wave velocity of propagation [6].

Nevertheless, it is important to release, that only finite active depth of the foundation soils should be included in the analysis. Under such assumption, it was shown in [7] that there is an interaction between the beam and the foundation. In [7] only simplified plane models of the foundation were used for analyses of finite and infinite beams, but it was confirmed that the critical velocity is not given either by the classical formula from [2] or by the lowest wave-velocity of propagation in the foundation, but there is a smooth transition between these two extreme values governed by the mass ratio. For a low mass ratio, the critical velocity approaches the classical formula and for a higher mass ratio, it approaches the lowest wave-velocity of propagation in the foundation. In this paper generalizations, in conformity with [8] are derived. It is proven that in such extension, the final results depend on the interface condition between the beam and the foundation.

2 CRITICAL VELOCITY

The critical velocity of the load traversing an infinite Euler-Bernoulli beam on an elastic foundation is given by the classical formula [2]

$$v_{cr} = \sqrt[4]{\frac{4kEI}{m^2}} \tag{1}$$

where m and EI stand for the mass per unit length and the bending stiffness of the beam, respectively, and k for the Winkler constant of the foundation. If the beam is modelled by the rail, then Eq. (1) predicts a critical velocity that is generally much higher than the one observed in reality. The main reason is related to the fact that there is no mass in the foundation and thus no wave propagation is possible. Several works have been published on this subject replacing the spring layer by an elastic half-space and concluding that the critical velocity corresponds to the Rayleigh-wave velocity of propagation, which is the slowest wave-velocity [6].

Nevertheless, better estimation should account for the active finite depth of the foundation, which can either be the actual depth at which a stiff substratum is located or a depth after which no appreciable soil deformations occur. Under simplified assumptions on a plane model under plane strain conditions, it was derived in [7] that the critical velocity is governed by the mass ratio according to approximately

$$V_{cr} = v_{cr}\left[\left(1-\vartheta_s\right)\sqrt{\frac{2}{2+M^{2+\sqrt{\vartheta_s}}}}+\vartheta_s\right] \tag{2}$$

where V_{cr} is the new value of the critical velocity, ϑ_s is the shear ratio defined as $\vartheta_s = v_s / v_{cr}$ with v_s being the shear-wave velocity, and M is the mass ratio defined as the square root of the foundation mass to the beam mass. Thus, for a low mass ratio, the critical velocity approaches the classical value v_{cr} and for a higher mass ratio, it approaches the velocity of propagation of shear waves in the foundation. In this simplified model horizontal displacements were neglected and therefore the Rayleigh velocity could not be detected. Due to the proximity of these two velocities defined by the approximate formula

$$\frac{v_R}{v_s} = \frac{0.87+1.12\upsilon}{1+\upsilon} \tag{3}$$

where υ is the soil Poisson ratio and v_R is the Rayleigh-wave velocity, this does not have to be considered a disadvantage.

In this paper new results are derived in conformity with [8]. Deductions from [8] are simplified by considering only a constant moving force, thus the analytical solution can be restricted only to its steady-state part. Final results of the critical velocity, which can then be extracted by parametric analyses, are dependent on the interface condition between the beam and the foundation. However, there is only a small difference between the three possibilities: the previously published approximate formula (2) and results according to two options for the interface condition. Namely, results with the interface condition in form of zero horizontal displacement give values very similar to Eq. (2) and results obeying the zero shear stress condition have the asymptotic tendency to slightly lower velocity, the velocity of propagation of Rayleigh waves.

Deflection shapes and adequacy of the interface condition are analysed by finite element results. In order to obtain steady-state form of the finite element results, the enhanced moving widow method is implemented in software ANSYS as described in [9].

3 THE MODEL AND ITS SOLUTION

It is assumed that the load is traversing an infinite beam supported by a foundation of finite depth *H*, as depicted on Figure 1. It is further assumed that: (i) the beam obeys linear elastic Euler-Bernoulli theory; (ii) the beam vertical displacement is measured from the equilibrium deflection caused by the beam weight; (iii) the foundation is represented by a finite strip of width *b* under plane strain condition; (iv) the foundation soil is linear elastic homogeneous material.

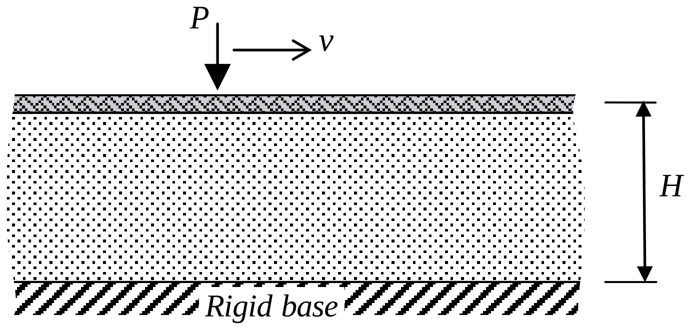


Figure 1: Infinite beam on an elastic foundation of finite depth subjected to a moving load.

The governing equations for determination of the beam deflection shape and displacement fields in the foundation soil are given by

$$\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})=\rho\mathbf{u}_{,tt} \tag{4}$$

$$EIw_{,xxxx}(x,t)+c_b w_{,t}(x,t)+mw_{,tt}(x,t)+p(x,t)=P\delta(x-vt) \tag{5}$$

where $\mathbf{u}=(u_x,u_z)$ is the displacement field in the foundation, λ, μ are Lame's constants of the soil, ρ is the soil density and t is the time. Overall this paper, derivatives will be designated by the corresponding variable symbol in the subscript position, preceded by a comma. Moreover, ∇ is the gradient and Δ is the Laplace operators applied on spatial variables *x*, *z*. Further, *P* is the moving load, *v* is its velocity, *p* is the foundation pressure and c_b is the viscous damping coefficient of the beam. The unknown beam deflection $w(x,t)$, spatial coordinate *z* and *P* are assumed positive when acting downward. Spatial coordinate *x* is positive to the right, the load travels from the left to the right and finally, δ is the Dirac delta function.

Damping in the soil can be assumed as hysteretic

$$\lambda=\lambda_0(1+\mathrm{i}\eta_h),\ \mu=\mu_0(1+\mathrm{i}\eta_h) \tag{6}$$

or viscous

$$\lambda=\lambda_0(1+c_s\bullet_{,t}),\ \mu=\mu_0(1+c_s\bullet_{,t}) \tag{7}$$

where η_h is the loss factor of the soil and c_s is the coefficient of the viscous damping in the soil.

The solution method follows these steps: firstly, the governing equations for the soil layer are expressed in terms of displacement potentials. Then all equations are simplified by introduction of moving coordinates and leaving only the terms that contribute to the steady-state part of the solution. After that several dimensionless variables are introduced to facilitate the equations manipulation and posterior results analyses. The main solution method is the single Fourier transform, which allows analytical solution of all displacement and stress components in the frequency domain. The inverse transform is accomplished numerically. Dimensionless variables allow identifying results for all possible input data combinations. Except for the damping values, which define viscous damping in the beam, viscous and hysteretic damping in the foundation, results depend only on the mass ratio *M*, the velocity ratio $\alpha=v/v_{cr}$, the shear ratio ϑ_s and soil Poisson's ratio υ.

4 THE ENHANCED MOVING WINDOW METHOD

Finite element confirmation of steady-state analytical predictions in infinite media is generally a complicated issue because it is difficult to choose the correct (i) size of the model, (ii) size of the finite elements; (iii) type of the boundary conditions and (iv) a method that would allow deflection fields stabilization. The model itself must be large enough in order to eliminate satisfactorily transient effects due to a sudden placement of the load on the structure and, on the other hand, small enough to be computationally accessible. The edges of the finite elements must be sufficiently small in order to represent adequately propagating waves. The boundary conditions are even more delicate issue. The dynamic analysis of solids of infinite dimensions with discrete methods such as finite elements calls for the use of special boundary that are normally referred to as absorbing, non-reflecting or transmitting boundaries. The purpose of these special boundaries is to prevent wave reflections at the edges of the mathematical models used, which, by necessity, must remain finite in size. A number of these boundaries have been proposed in the past with recourse to various mathematical or physical principles. Unfortunately, none of the transmitting boundaries can fully prevent all possible reflections under the full range of possible incident angles.

Some of the difficulties named above could be overcome by implementation of the moving window method. In the moving window method the load is kept still, and the finite element model of the railway track moves in the direction opposing the originally assumed load movement. This can be achieved by several ways. Either the finite elements are altered in order to implement the effect of the load velocity [10], or results are shifted against the load.

A shift of results is impossible in commercial finite element software, because such an operation is usually protected against inappropriate usage. Implementation of the enhanced moving window method in commercial finite element software ANSYS is described in [9]. The method is tested on one-, two- and three-dimensional models. It is shown that the steady-state response of an infinite structure can be obtained with sufficient accuracy, which significantly reduces the calculation time by reduction of both, the model size and the analysis time.

When the model is large enough, periodic boundary conditions can be used on the front and rear faces of the model. Regarding the bottom face, other considerations must be taken. A reduction of the model depth by representative springs and viscous boundary is used here according to [11]. These distributed elastic springs are defined as:

$$k_n = \frac{\lambda_0 + 2\mu_0}{H - h}, \; k_t = \frac{\mu_0}{H - h} \tag{8}$$

where k_n and k_t are spring stiffnesses in the normal and tangential directions, respectively and h is the depth that is modelled by finite elements. These elastic boundaries allows introduction of absorbing boundaries, which help to stabilization of the results.

5 NUMERICAL RESULTS

Final results are shown in Figure 2 for shear ratio 0.5 and soil Poisson ratio 0.2. It is seen that results from [7] and new results with zero displacement interface condition are quite proximate, especially regarding the asymptotic value for higher mass ratio. When zero shear is admitted at the interface, than the asymptotic tendency directs to lower velocity. In such a case the ratio given by Eq. (2) indicates that this velocity is the Rayleigh-wave velocity of propagation.

Deflection shapes were confirmed by the finite element results exploiting the enhanced moving window method. One case is shown in Figure 3. The case presented considers two rails supported by soft foundation with input data summarized in Table 1. In the first part of Figure 3 finite element solution is compared to the analytical solution with zero horizontal displacement interface condition. It is seen that the coincidence is very good. The other part compares the analytical solutions for the two interface conditions. It can be observed that the zero shear condition allows for higher displacements.

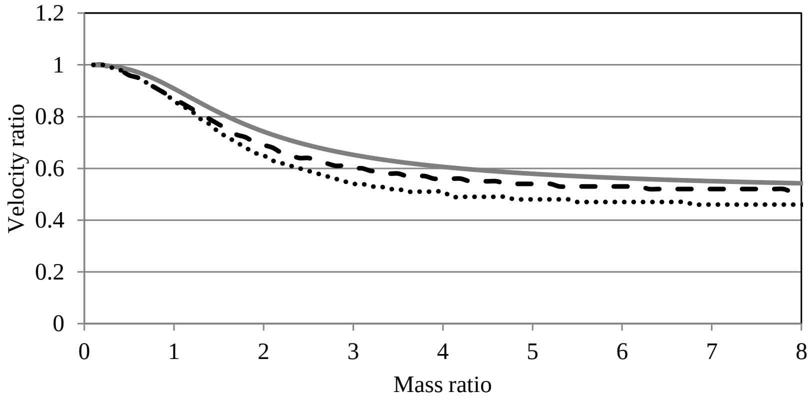
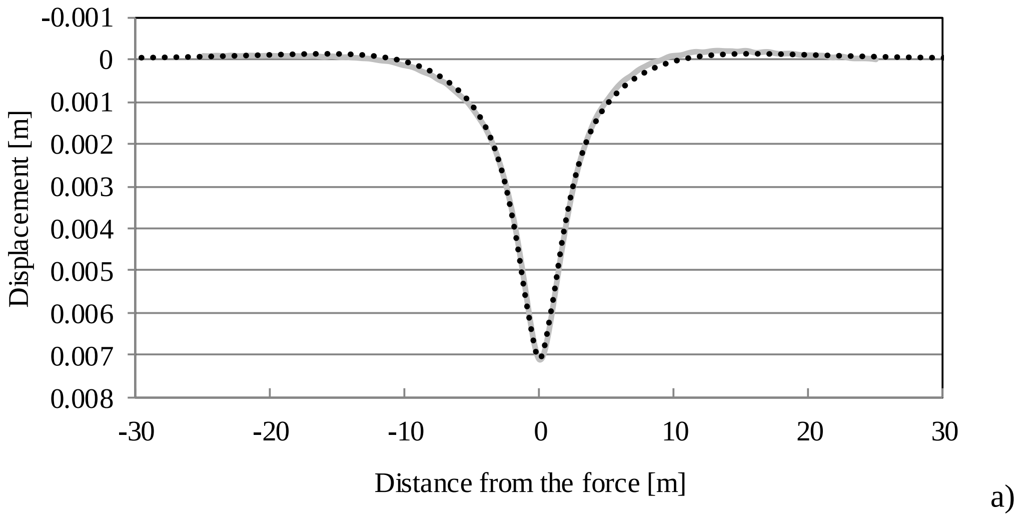


Figure 2: Critical velocity as a function of the mass ratio for shear ratio 0.5: previous estimate according to Eq. (2) (grey), solution with zero displacement interface condition (black dashed), solution with zero shear interface condition (black dotted).

Property	Value
Beam bending stiffness EI (MN·m^2)	12.8
Beam mass per unit length m (kg·m^{-1})	120
Soil Young's modulus E_s (MN·m^{-2})	10
Soil Poisson's ratio υ	0.3
Soil density ρ (kg·m^{-3})	1850
Active depth H (m)	12
Moving force P (kN)	200
Velocity v (m·s^{-1})	50

Table 1: Numerical data used with a unit strip width of the soil.



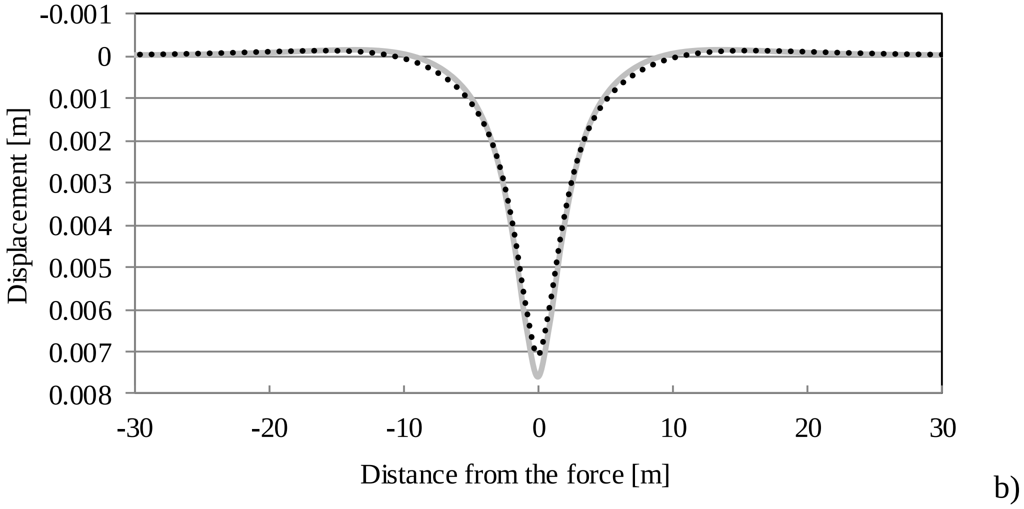


Figure 3: Deflection shapes comparison: a) finite element solution (grey) and solution with zero displacement interface condition (black dotted); b) solution with zero shear interface condition (grey) and solution with zero displacement interface condition (black dotted).

In addition, shear stress and horizontal displacement were extracted in the vertical cut below the load from the finite element solution. It is seen that it is more adequate to assume zero horizontal displacement at the interface between the beam and the foundation.

6 CONCLUSIONS

In this contribution, the critical velocity of a uniformly moving load on a beam supported by a finite depth foundation was analysed. Generalizations of the method published in [7] were derived. The new approach follows developments in [8] but simplifies the analysis by admitting only a constant moving force. Analytical results for the deflection beam shape and displacement fields in the foundation are obtained by the Fourier transform. The inverse transform is accomplished numerically.

Final results of the critical velocity are then extracted by parametric analyses. It is confirmed that there is an interaction between the beam and the foundation, and thus there is a smooth transition between the classical value of the critical velocity and the lowest velocity of wave propagation in the foundation, depending on the assumptions adopted. However, the new results are dependent on the interface condition between the beam and the foundation. Only small differences occur between the three possibilities: the previously published results in [7] and results according to two options for the interface condition. Results with the interface condition in form of zero horizontal displacement give values very similar to the ones published in [7] and results obeying the zero shear stress condition have the asymptotic tendency to slightly lower velocity, the velocity of propagation of Rayleigh waves.

Deflection shapes and adequacy of the interface condition were analysed by finite element results exploiting the enhanced moving widow method implemented in software ANSYS. It was confirmed that, if the model considers the beam axis coincident with the soil upper surface, which is the common finite element approach, then it is more adequate to assume that the horizontal displacements are zero at the interface.

7 ACKNOWLEDGEMENTS

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