The ranks of ideals in various transformation monoids

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Abstract

In this paper we consider various classes of monoids of transformations of a finite chain, including those of transformations that preserve or reverse either the order or the orientation. In line with Howie and McFadden [24], we complete the study of the ranks (and of idempotent ranks, when applicable) of all their ideals.

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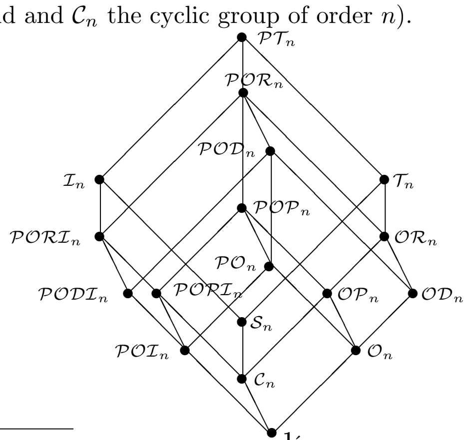
Introduction and preliminaries

As usual we denote by $\mathcal{PT}_n$ the monoid of all partial transformations of a finite set X_n with n elements (under composition), by $\mathcal{T}_n$ the submonoid of $\mathcal{PT}_n$ of all full transformations of X_n, by $\mathcal{I}_n$ the symmetric inverse semigroup on X_n, i.e. the submonoid of $\mathcal{PT}_n$ of all injective partial transformations of X_n, and by $\mathcal{S}_n$ the symmetric group on X_n, i.e. the subgroup of $\mathcal{PT}_n$ of all injective full transformations (permutations) of X_n.

Let now X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. We say that a transformation $s \in \mathcal{PT}_n$ is order-preserving (respectively, order-reversing) if $x \leq y$ implies $xs \leq ys$ (respectively, $xs \geq ys$), for all $x, y \in \mathrm{Dom}(s)$. Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all order-preserving transformations and by $\mathcal{O}_n$ the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations of X_n that preserve the order. The injective counterpart of $\mathcal{O}_n$, i.e. the inverse monoid $\mathcal{PO}_n \cap \mathcal{I}_n$ of all order-preserving injective partial transformations is denoted by $\mathcal{POI}_n$. Further classes of monoids are obtained by considering transformations that either preserve or reverse the order. In this way, we get the submonoid $\mathcal{POD}_n$ of $\mathcal{PT}_n$ of all partial transformations that preserve or reverse the order, as well as its submonoids $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$ and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$.

The order-preserving and order-reversing notions can be generalized as follows. Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that c is cyclic (respectively, anti-cyclic) if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$ (respectively, $c_i < c_{i+1}$), where c_{t+1} denotes c_1. Let $s \in \mathcal{PT}_n$ and suppose that $\mathrm{Dom}(s) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that s is orientation-preserving (respectively, orientation-reversing) if the sequence of its image $(a_1 s, \ldots, a_t s)$ is cyclic (respectively, anti-cyclic). Denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations and by $\mathcal{POR}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations together with all orientation-reversing transformations. Also, let $\mathcal{OP}_n = \mathcal{POP}_n \cap \mathcal{T}_n$, $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$, $\mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$.

The relationship between these various monoids, with respect to the inclusion relation, is represented by the diagram bellow (where $\mathbf{1}$ denotes the trivial monoid and $\mathcal{C}_n$ the cyclic group of order n).



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Let S be a semigroup. Denote by S^1 the monoid obtained from S through the adjoining of an identity if S has none and exactly S otherwise. Recall the definition of Green's equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{J}$: for all $u, v \in S$,

$$\begin{array}{rll} u\mathcal{R}v & \text{if and only if} & uS^1 = vS^1; \\ u\mathcal{L}v & \text{if and only if} & S^1u = S^1v; \\ u\mathcal{H}v & \text{if and only if} & u\mathcal{R}v \text{ and } u\mathcal{L}v; \\ u\mathcal{J}v & \text{if and only if} & S^1uS^1 = S^1vS^1. \end{array}$$

Associated to Green's relation $\mathcal{J}$ there is a quasi-order $\leq_{\mathcal{J}}$ on S defined by

$$u \leq_{\mathcal{J}} v \quad \text{if and only if} \quad S^1uS^1 \subseteq S^1vS^1,$$

for all $u, v \in S$. Notice that, for every $u, v \in S$, we have $u \mathcal{J} v$ if and only if $u \leq_{\mathcal{J}} v$ and $v \leq_{\mathcal{J}} u$. Denote by J_u^S the $\mathcal{J}$-class of the element $u \in S$. As usual, a partial order relation $\leq_{\mathcal{J}}$ is defined on the quotient set $S/\mathcal{J}$ by putting $J_u^S \leq_{\mathcal{J}} J_v^S$ if and only if $u \leq_{\mathcal{J}} v$, for all $u, v \in S$. Given a subset A of S and $u \in S$, we denote by $E(A)$ set of idempotents of S belonging to A and by L_u^S, R_u^S and H_u^S the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class of u, respectively. For general background on Semigroup Theory, we refer the reader to Howie's book [23].

Let M_n denotes any of the monoids $\mathcal{T}_n$, $\mathcal{O}_n$, $\mathcal{OD}_n$, $\mathcal{OP}_n$, $\mathcal{OR}_n$, $\mathcal{PT}_n$, $\mathcal{PO}_n$, $\mathcal{POD}_n$, $\mathcal{POP}_n$, $\mathcal{POR}_n$, $\mathcal{I}_n$, $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ or $\mathcal{PORI}_n$. Then M_n is regular and, in particular, if $M_n \in \{\mathcal{I}_n, \mathcal{POI}_n, \mathcal{PODI}_n, \mathcal{POPI}_n, \mathcal{PORI}_n\}$, it is an inverse monoid. On the other hand, Green's relations $\mathcal{L}$ and $\mathcal{R}$ of M_n can be characterized by $s\mathcal{L}t$ if and only if $\operatorname{Im}(s) = \operatorname{Im}(t)$, for all $s, t \in M_n$, and $s\mathcal{R}t$ if and only if $\operatorname{Ker}(s) = \operatorname{Ker}(t)$, for all $s, t \in M_n$. For $M_n \in \{\mathcal{I}_n, \mathcal{POI}_n, \mathcal{PODI}_n, \mathcal{POPI}_n, \mathcal{PORI}_n\}$, we can more specifically describe $\mathcal{R}$ by $s\mathcal{R}t$ if and only if $\operatorname{Dom}(s) = \operatorname{Dom}(t)$, for all $s, t \in M_n$. Regarding Green's relation $\mathcal{J}$, we have $s \leq_{\mathcal{J}} t$ if and only if $|\operatorname{Im}(s)| \leq |\operatorname{Im}(t)|$ and so $s\mathcal{J}t$ if and only if $|\operatorname{Im}(s)| = |\operatorname{Im}(t)|$, for all $s, t \in M_n$. It follows that the partial order $\leq_{\mathcal{J}}$ on the quotient $M_n/\mathcal{J}$ is linear. More precisely, letting

$$J_r^{M_n} = \{s \in M_n \mid |\operatorname{Im}(s)| = r\},$$

i.e. the $\mathcal{J}$-class of the transformations of image size r (called the *rank* of the transformations) of M_n, for $0 \leq r \leq n$, we have

$$M_n/\mathcal{J} = \{J_1^{M_n} \leq_{\mathcal{J}} J_2^{M_n} \leq_{\mathcal{J}} \cdots \leq_{\mathcal{J}} J_n^{M_n}\},$$

for $M_n \in \{\mathcal{T}_n, \mathcal{O}_n, \mathcal{OD}_n, \mathcal{OP}_n, \mathcal{OR}_n\}$, and

$$M_n/\mathcal{J} = \{J_0^{M_n} \leq_{\mathcal{J}} J_1^{M_n} \leq_{\mathcal{J}} \cdots \leq_{\mathcal{J}} J_n^{M_n}\},$$

for the remaining cases. See [3, 7, 8, 9, 11, 12, 18, 23] for more details.

Since $M_n/\mathcal{J}$ is a chain, the sets

$$M(n, r) = \{s \in M_n \mid |\operatorname{Im}(s)| \leq r\} = J_0^{M_n} \cup J_1^{M_n} \cup \cdots \cup J_r^{M_n},$$

with $1 \leq r \leq n$, constitute all the non-null ideals of M_n (see [9, Note of page 181]). Notice that, if M_n is a full transformation monoid then $J_0^{M_n}$ just represents the empty set, while in the other cases $J_0^{M_n}$ contains only the empty transformation.

As usual, the rank of a finite semigroup S is defined by $\operatorname{rank} S = \min\{|A| \mid A \subseteq S, \langle A \rangle = S\}$. If S is generated by its set E of idempotents, then the idempotent rank of S is defined by $\operatorname{idrank} S = \min\{|A| \mid A \subseteq E, \langle A \rangle = S\}$. Clearly, $\operatorname{rank} S \leq \operatorname{idrank} S$. Throughout this paper we consider always generators of semigroups, i.e. generators of algebras of type 2, even in the cases we are dealing with inverse semigroups. The notions of rank and idempotent rank are taken accordingly.

Notice that the rank (and the idempotent rank, when applicable) of a finite semigroup S is always greater than or equal to the number of its $\mathcal{R}$-classes contained in maximal $\mathcal{J}$-classes and to the number of its $\mathcal{L}$-classes contained in maximal $\mathcal{J}$-classes. This follows immediately from the fact that $s \leq_{\mathcal{R}} t$ (respectively, $s \leq_{\mathcal{L}} t$) and $s\mathcal{J}t$ if and only if $s\mathcal{R}t$ (respectively, $s\mathcal{L}t$), for all $s, t \in S$. Thus, in particular, the rank (and the idempotent rank, when applicable) of the semigroup $M(n, r)$ is, in any case, at least equal to the number of $\mathcal{R}$-classes of rank r of M_n and to the number of $\mathcal{L}$-classes of rank r of M_n, for $1 \leq r \leq n$.

For $n \geq 3$, the ranks of $\mathcal{PT}_n = \mathcal{PT}(n, n)$, $\mathcal{I}_n = \mathcal{I}(n, n)$ and $\mathcal{T}_n = \mathcal{T}(n, n)$ are equal to 4, 3 and 3, respectively. These are well known results and all of them have reasonably easy proofs. See [23, pages 39, 41 and 211] for example.

On the other hand, the rank of the semigroup of singular mappings $\operatorname{Sing}_n = \{\alpha \in \mathcal{T}_n \mid |\operatorname{Im}(\alpha)| \leq n - 1\}$ is more difficult to determine. In [17], Gomes and Howie proved that both the rank and the idempotent rank of Sing_n are equal to $n(n-1)/2$. This result was later generalized by Howie and McFadden [24] who showed that the rank and idempotent rank of the semigroups $\mathcal{T}(n,r) = \{\alpha \in \mathcal{T}_n \mid |\mathrm{Im}(\alpha)| \le r\}$ are both equal to $S(n,r)$, the Stirling number of the second kind, for $2 \le r \le n-1$.

In [14], Garba considered the semigroup $\mathcal{PT}(n,r) = \{\alpha \in \mathcal{PT}_n \mid |\mathrm{Im}(\alpha)| \le r\}$ and showed that, for $2 \le r \le n-1$, both its rank and idempotent rank are equal to $S(n+1,r+1)$.

Regarding the partial injective counterparts, Gomes and Howie [17] showed that the rank of the semigroup $\mathcal{SP}_n = \{\alpha \in \mathcal{I}_n \mid |\mathrm{Im}(\alpha)| \le n-1\}$, as inverse semigroup, is $n+1$. Garba [16] generalized this result by showing that the rank (as inverse semigroup) of $\mathcal{I}(n,r) = \{\alpha \in \mathcal{I}_n \mid |\mathrm{Im}(\alpha)| \le r\}$ is $\binom{n}{r}+1$ for $3 \le r \le n-1$, $\binom{n}{2}$ for $r=2$, and $n-1$ for $r=1$.

The ranks of the remaining semigroups represented in the above diagram have also been studied. In fact, Gomes and Howie [18] calculated the ranks of the monoids $\mathcal{PO}_n$ and $\mathcal{O}_n$. In [2] Catarino calculated the rank of the monoid $\mathcal{OP}_n$ and the rank of the monoid $\mathcal{OR}_n$ was determined by Arthur and Ruškuc in [1]. The ranks of the monoids $\mathcal{OD}_n$,$\mathcal{POD}_n$, $\mathcal{POP}_n$ and $\mathcal{POR}_n$ were determined by Fernandes, Gomes and Jesus in [11, 12], and the ranks of the monoids $\mathcal{POI}_n$ and $\mathcal{POPI}_n$ were determined by Fernandes in [8, 9].

Concerning the ranks of their ideals, in same cases they are completely studied, in others just partially. In fact, in addition to the ideals of $\mathcal{T}_n$ and $\mathcal{PT}_n$ above mentioned, these are the cases of the semigroups $\mathcal{O}(n,r)$ and $\mathcal{PO}(n,r)$ [18, 15], for $2 \le r \le n-1$, and of the semigroups $\mathcal{POI}(n,n-1)$ [9] and $\mathcal{OP}(n,n-1)$ [27]. The remaining cases are considered here for the first time, including the study of the ranks of $\mathcal{I}(n,r)$, for $1 \le r \le n-1$. Notice that, for these last semigroups, Gomes and Howie [17] and Garba [16] have studied their ranks as inverse semigroups.

Observe that, being $\mathcal{O}_n$, $\mathcal{PO}_n$ and $\mathcal{POI}_n$ aperiodic monoids (i.e. $\mathcal{H}$-trivial), their ranks as monoids, i.e. as algebras of type (2,0), coincide with the ranks of the semigroups $\mathcal{O}(n,n-1)$, $\mathcal{PO}(n,n-1)$ and $\mathcal{POI}(n,n-1)$, respectively.

This paper is organized as follows. In Section 1, we calculate the ranks of $\mathcal{OD}(n,r)$ and $\mathcal{POD}(n,r)$, while in Section 2, we establish the ranks of $\mathcal{OP}(n,r)$ and $\mathcal{OR}(n,r)$. In Section 3, we determine the ranks of $\mathcal{POP}(n,r)$ and $\mathcal{POR}(n,r)$. Finally, Section 4 is dedicated to the study of the ranks of $\mathcal{POI}(n,r)$, $\mathcal{PODI}(n,r)$, $\mathcal{POPI}(n,r)$, $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$. A summary of all results presented in this paper is given in Section 5.

Throughout this paper we always assume that $n \ge 4$.

We would like to acknowledge the use of GAP [13], a system for computational discrete algebra.

1 The ranks of $\mathcal{OD}(n,r)$ and $\mathcal{POD}(n,r)$

Recall that the ranks and idempotent ranks of the semigroups $\mathcal{O}(n,r)$ and $\mathcal{PO}(n,r)$ were computed by Garba [15] for $2 \le r < n-1$ and by Gomes and Howie [18] for $r = n-1$. They showed that $\mathcal{O}(n,r)$ and $\mathcal{PO}(n,r)$, with $2 \le r \le n-1$, have ranks $\binom{n}{r}$ and $\sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$, respectively. It is easy to show that these formulas also hold for $r=1$. Notice that $\mathcal{O}(n,1) = \mathcal{T}(n,1)$ and $\mathcal{PO}(n,1) = \mathcal{PT}(n,1)$. On the other hand, for $1 \le r \le n-2$, the idempotent ranks of $\mathcal{O}(n,r)$ and $\mathcal{PO}(n,r)$ coincide with their respective ranks. However, for $r = n-1$ this is not the case. In fact, the idempotent ranks of $\mathcal{O}(n,n-1)$ and $\mathcal{PO}(n,n-1)$ are $2n-2$ and $3n-2$, respectively.

The ranks of the monoids $\mathcal{OD}_n$ and $\mathcal{POD}_n$ were established by Fernandes, Gomes and Jesus in [11]. These monoids are not idempotent generated. In fact, we have $E(\mathcal{OD}_n) = E(\mathcal{O}_n)$ and $E(\mathcal{POD}_n) = E(\mathcal{PO}_n)$. Likewise, for $r \ge 2$, since $\mathcal{O}(n,r) \subsetneq \mathcal{OD}(n,r)$ and $\mathcal{PO}(n,r) \subsetneq \mathcal{POD}(n,r)$, the semigroups $\mathcal{OD}(n,r)$ and $\mathcal{POD}(n,r)$ are not idempotent generated.

In this section we aim to show that the ranks of $\mathcal{OD}(n,r)$ and $\mathcal{POD}(n,r)$ coincide with the ranks of $\mathcal{O}(n,r)$ and $\mathcal{PO}(n,r)$, respectively.

We begin by recalling that Dimitrova and Koppitz [5, Corollary 4] proved:

Lemma 1.1 *For $1 \le r \le n-1$, the semigroup $\mathcal{OD}(n,r)$ is generated by its idempotents of rank r together with any single order-reversing transformation of rank r.*

We must observe that Corollary 4 of [5] was stated *for some* order-reversing transformation of rank r instead of *for any*, as we presented above. However, they indeed proved the stronger version we are stating in Lemma 1.1. See also [19, Lemma 1].

Next, for $\mathcal{POD}_n$, we aim to prove a result analogous to Lemma 1.1.

First, recall that Garba [15], for $2 \le r < n-1$, and Gomes and Howie [18], for $r = n-1$, proved:

Lemma 1.2 *For $2 \le r \le n-1$, the semigroup $\mathcal{PO}(n,r)$ is generated by its idempotents of rank r.*

Secondly, let $2 \le r \le n-1$ and take $\alpha \in \mathcal{POD}(n,r)$ with rank $1 \le k \le r$. Suppose that $\alpha = \begin{pmatrix} I_1 & I_2 & \cdots & I_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$. Notice that $\{I_1,\ldots,I_k\}$ forms a partition into intervals of $\mathrm{Dom}(\alpha)$. For $1 \le \ell \le k$, let $i_\ell = \min I_\ell$ (naturally, we are assuming that $i_1 < i_2 < \cdots < i_k$). Define

$$\varepsilon_\alpha = \begin{pmatrix} I_1 & I_2 & \cdots & I_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} \;, \quad \overline{\alpha} = \begin{pmatrix} \{1,\ldots,i_2-1\} & \{i_2,\ldots,i_3-1\} & \cdots & \{i_{k-1},\ldots,i_k-1\} & \{i_k,\ldots,n\} \\ j_1 & j_2 & \cdots & j_{k-1} & j_k \end{pmatrix}$$

and

$$\overline{\varepsilon}_\alpha = \begin{pmatrix} \{1,\ldots,i_2-1\} & \{i_2,\ldots,i_3-1\} & \cdots & \{i_{k-1},\ldots,i_k-1\} & \{i_k,\ldots,n\} \\ i_1 & i_2 & \cdots & i_{k-1} & i_k \end{pmatrix} \;.$$

Then, we have

$$\varepsilon_\alpha \in \mathcal{PO}(n,r) \;, \quad \overline{\alpha} \in J_k^{\mathcal{OD}_n} \subset \mathcal{OD}(n,r) \;, \quad \overline{\varepsilon}_\alpha \in E(J_k^{\mathcal{O}_n}) \;, \quad \alpha = \varepsilon_\alpha \overline{\alpha} \quad \text{and} \quad \overline{\alpha} = \overline{\varepsilon}_\alpha \alpha \;.$$

Notice that, by Lemma 1.2, we also have $\varepsilon_\alpha \in \langle E(J_r^{\mathcal{PO}_n})\rangle$. Further, if α is an order-reversing transformation then so is $\overline{\alpha}$.

Now, let γ be an order-reversing element of $J_r^{\mathcal{POD}_n}$ and let $\overline{\gamma}$ and $\overline{\varepsilon}_\gamma$ be defined as above (taking γ in place of α). Then $\overline{\gamma}$ is an order-reversing full transformation of rank r and so, by Lemma 1.1, we have $\mathcal{OD}(n,r) = \langle E(J_r^{\mathcal{O}_n}), \overline{\gamma}\rangle$. Moreover, $\overline{\gamma} = \overline{\varepsilon}_\gamma \gamma$, with $\overline{\varepsilon}_\gamma \in E(J_r^{\mathcal{O}_n})$, whence $\overline{\gamma} \in \langle E(J_r^{\mathcal{O}_n}), \gamma\rangle$.

Therefore, returning to α, we have $\alpha = \varepsilon_\alpha \overline{\alpha}$, with $\varepsilon_\alpha \in \langle E(J_r^{\mathcal{PO}_n})\rangle$ and $\overline{\alpha} \in \mathcal{OD}(n,r) = \langle E(J_r^{\mathcal{O}_n}), \overline{\gamma}\rangle \subset \langle E(J_r^{\mathcal{O}_n}), \gamma\rangle \subset \langle E(J_r^{\mathcal{PO}_n}), \gamma\rangle$, and so $\alpha \in \langle E(J_r^{\mathcal{PO}_n}), \gamma\rangle$.

Thus, we have proved:

Lemma 1.3 *For $2 \le r \le n-1$, the semigroup $\mathcal{POD}(n,r)$ is generated by its idempotents of rank r together with any single order-reversing partial transformation of rank r.*

Now, let M_n be either the monoid $\mathcal{O}_n$ or the monoid $\mathcal{PO}_n$. Similarly, let MD_n be either the monoid $\mathcal{OD}_n$ or the monoid $\mathcal{POD}_n$. Taking into account the above cited results of Garba [15] and Gomes and Howie [18], for $r \ge 2$ we may take a generating set $\{s_1, s_2, \ldots, s_p\}$ of $M(n,r)$ with $p = \binom{n}{r}$, if $M_n = \mathcal{O}_n$, and $p = \sum_{k=r}^n \binom{n}{k}\binom{k-1}{r-1}$, if $M_n = \mathcal{PO}_n$. Moreover, in addition, we may suppose that s_1 is an idempotent of rank r (in fact, if $r \le n-2$ then the same may be assumed for the remaining $s_2, \ldots, s_p$ generators). Let γ be the order-reversing transformation with the same kernel and image of s_1. Then $\gamma^2 = s_1$ and, by applying Lemmas 1.1 and 1.3, we may deduce that $MD(n,r) = \langle M(n,r), \gamma\rangle$. Since $M(n,r) = \langle s_1, s_2, \ldots, s_p\rangle \subset \langle \gamma, s_2, \ldots, s_p\rangle$, it follows that $\{\gamma, s_2, \ldots, s_p\}$ is a generating set of $MD(n,r)$.

Finally, by noticing that, like $\mathcal{O}_n$, the monoid $\mathcal{OD}_n$ has $\binom{n}{r}$ $\mathcal{L}$-classes of rank r and, like $\mathcal{PO}_n$, the monoid $\mathcal{POD}_n$ has $\sum_{k=r}^n \binom{n}{k}\binom{k-1}{r-1}$ $\mathcal{R}$-classes of rank r, we conclude our aim of this section.

Theorem 1.4 *For $1 \le r \le n-1$, we have* $\operatorname{rank}\mathcal{OD}(n,r) = \binom{n}{r}$ *and* $\operatorname{rank}\mathcal{POD}(n,r) = \sum_{k=r}^n \binom{n}{k}\binom{k-1}{r-1}$.

Observe that, for $r = 1$ the above result is simply justified by the equalities $\mathcal{OD}(n,1) = \mathcal{O}(n,1) = \mathcal{T}(n,1)$ and $\mathcal{POD}(n,1) = \mathcal{PO}(n,1) = \mathcal{PT}(n,1)$.

We finish this section by proving the above statements on the ranks and the idempotents ranks of $\mathcal{T}(n,1)$ and $\mathcal{PT}(n,1)$, which are not mentioned by the referred authors (or by others, as far as we know).

Regarding the semigroup $\mathcal{T}(n,1)$, since it is a right-zero semigroup of size n, its rank and idempotent rank have to match its size, i.e. $\operatorname{idrank}\mathcal{T}(n,1) = \operatorname{rank}\mathcal{T}(n,1) = n = \binom{n}{1}$. The case of $\mathcal{PT}(n,1)$ is not so trivial. Nevertheless, it is a routine matter to show that (for instance) the set formed by the (idempotent) elements

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 2 & 2 & \cdots & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & \cdots & i-2 & i & \cdots & n \\ i & \cdots & i & i & \cdots & i \end{pmatrix}, \quad \text{for } i = 3, \ldots, n,$$

and any other single idempotent element from each of the remaining non-zero $\mathcal{R}$-classes of $\mathcal{PT}(n,1)$ generates $\mathcal{PT}(n,1)$. Since $\mathcal{PT}(n,1)$ has $2^n - 1$ $\mathcal{R}$-classes of rank 1, it follows that $\operatorname{idrank}\mathcal{PT}(n,1) = \operatorname{rank}\mathcal{PT}(n,1) = 2^n - 1 = \sum_{k=1}^n \binom{n}{k}\binom{k-1}{0}$.

2 The ranks of $\mathcal{OP}(n,r)$ and $\mathcal{OR}(n,r)$

It is well known that the ranks of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ are equal to 2 and 3, respectively; see [2, 25] and also [1]. On the other hand, both the rank and the idempotent rank of the semigroup $\mathcal{OP}(n,n-1)$ are equal to n and were calculated by Zhao in [27] (see also [28]).

In this section we complete the study of the ranks and idempotent ranks of the ideals of $\mathcal{OP}_n$ and we determine the ranks of the ideals of $\mathcal{OR}_n$. Notice that, for $r \ge 3$, the semigroup $\mathcal{OR}(n,r)$ is not idempotent generated. In fact, $E(\mathcal{OP}_n) = E(\mathcal{OR}_n)$ and, for $r \ge 3$, we have $\mathcal{OP}(n,r) \subsetneq \mathcal{OR}(n,r)$. Observe also that $\mathcal{OP}(n,1) = \mathcal{OR}(n,1) = \mathcal{O}(n,1) = \mathcal{T}(n,1)$ and $\mathcal{OP}(n,2) = \mathcal{OR}(n,2)$.

Let $2 \leq r \leq n-1$. Then, given a transformation $\alpha \in \mathcal{OP}_n$ of rank r, it is easy to show that if $(1,n) \notin \operatorname{Ker}\alpha$ then all kernel classes of α are intervals of X_n and, on the other hand, if $(1,n) \in \operatorname{Ker}\alpha$ then all kernel classes of α are intervals of X_n, except for the class containing 1 and n which is a union of two intervals of X_n (one containing 1 and the other n). Therefore, we may establish a one-to-one correspondence between the collection of all subsets of X_n of cardinality r (which consists of all possible images of elements of rank r of $\mathcal{OP}_n$) and the collection of all possible kernels of elements of rank r of $\mathcal{OP}_n$ as follows: associate to each r-set $\{a_1, a_2, \ldots, a_r\}$, with $1 \leq a_1 < a_2 < \cdots < a_r \leq n$, the r-partition $\{A_1, \ldots, A_r\}$ of X_n defined by

$$A_i = \{a_i, a_i + 1, \ldots, a_{i+1} - 1\}, \text{ for } i = 1, \ldots, r-1, \quad \text{and} \quad A_r = \{a_r, \ldots, n\} \cup \{1, \ldots, a_1 - 1\} \tag{1}$$

(if $a_1 = 1$ then, naturally, $\{1, \ldots, a_1 - 1\}$ denotes the empty set). Thus, the $\mathcal{J}$-class $J_r^{\mathcal{OP}_n}$ of $\mathcal{OP}_n$ (and of $\mathcal{OP}(n,r)$) contains $\binom{n}{r}$ $\mathcal{R}$-classes and $\binom{n}{r}$ $\mathcal{L}$-classes. See [3] for more details.

Now, for $1 \leq a_1 < a_2 < \cdots < a_r \leq n$, define

$$\varepsilon_{a_1,a_2,\ldots,a_r} = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

where $\{A_1, \ldots, A_r\}$ is the r-partition of X_n associated to $\{a_1, a_2, \ldots, a_r\}$ as in (1). Clearly, $\varepsilon_{a_1,a_2,\ldots,a_r} \in E(J_r^{\mathcal{OP}_n})$. Moreover, the set

$$E_r = \{\varepsilon_{a_1,a_2,\ldots,a_r} \mid 1 \leq a_1 < a_2 < \cdots < a_r \leq n\}$$

contains exactly one (idempotent) element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{OP}_n$ of rank r.

It is easy to check the following lemma.

Lemma 2.1 *Let $0 = a_0 < 1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and $a \in X_n$.*

1. *If $a_{i-1} < a \leq a_i$, for some $i = 1, \ldots, r$, then $\{a_1, a_2, \ldots, a_r\}$ is a transversal of $\varepsilon_{a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_r}$.*
2. *If $a_r < a \leq n$ then $\{a_1, a_2, \ldots, a_r\}$ is a transversal of $\varepsilon_{a_2,\ldots,a_r,a}$.*

Notice that, given any two transformations $\alpha, \beta \in \mathcal{PT}_n$ of rank r, the product $\alpha\beta$ is a transformation of rank r if and only if $\operatorname{Im}(\alpha)$ is a transversal of $\operatorname{Ker}(\beta)$, in which case $\operatorname{Ker}(\alpha\beta) = \operatorname{Ker}(\alpha)$ and $\operatorname{Im}(\alpha\beta) = \operatorname{Im}(\beta)$. This observation and Lemma 2.1 allow us to prove:

Lemma 2.2 *Let $2 \leq r \leq n-1$. Then $E(J_r^{\mathcal{OP}_n}) \subseteq \langle E_r \rangle$.*

Proof. Take $\varepsilon \in E(J_r^{\mathcal{OP}_n})$. Let $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and $1 \leq b_{n-r+1} < b_{n-r+2} < \cdots < b_n \leq n$ be such that $\operatorname{Ker}(\varepsilon) = \operatorname{Ker}(\varepsilon_{a_1,a_2,\ldots,a_r})$ and $\operatorname{Im}(\varepsilon) = \{b_{n-r+1}, b_{n-r+2}, \ldots, b_n\}$.

Observe that $i \leq a_i$, for $i = 1, \ldots, r$, and $j \geq b_j$, for $j = n-r+1, \ldots, n$. Then, by Lemma 2.1,

$$\alpha_1 = \varepsilon_{a_1,a_2,\ldots,a_r}\varepsilon_{1,a_2,\ldots,a_r}\varepsilon_{1,2,a_3,\ldots,a_r}\cdots\varepsilon_{1,2,\ldots,r} \in J_r^{\mathcal{OP}_n},$$

$$\alpha_2 = \varepsilon_{1,2,\ldots,r}\varepsilon_{2,\ldots,r,r+1}\cdots\varepsilon_{n-r,\ldots,n-2,n-1}\varepsilon_{n-r+1,n-r+2,\ldots,n} \in J_r^{\mathcal{OP}_n},$$

$$\alpha_3 = \varepsilon_{n-r+1,n-r+2,\ldots,n}\varepsilon_{b_{n-r+1},n-r+2,\ldots,n}\varepsilon_{b_{n-r+1},b_{n-r+2},n-r+3,\ldots,n}\cdots\varepsilon_{b_{n-r+1},b_{n-r+2},\ldots,b_n} \in J_r^{\mathcal{OP}_n}$$

and

$$\alpha = \alpha_1\alpha_2\alpha_3 \in J_r^{\mathcal{OP}_n}.$$

Moreover, $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\varepsilon)$ and $\operatorname{Im}(\alpha) = \operatorname{Im}(\varepsilon)$. Since ε in an idempotent, it follows that $\varepsilon = \alpha^\omega$, for some $\omega \in \mathbb{N}$, whence $\varepsilon \in \langle E_r \rangle$, as required. $\square$

Now, recall that Zhao [27, Theorems 1.1 and 2.1] (and, independently, Dimitrova, Fernandes and Koppitz [6, Corollary 1.9]) proved:

Lemma 2.3 *For $1 \leq r \leq n-1$, the semigroup $\mathcal{OP}(n,r)$ is generated by its idempotents of rank r.*

Thus, combining Lemmas 2.2 and 2.3, we immediately deduce:

Proposition 2.4 *For $2 \leq r \leq n-1$, the set E_r generates $\mathcal{OP}(n,r)$.*

The following lemma was also proved by Dimitrova, Fernandes and Koppitz [6, Corollary 2.10].

Lemma 2.5 *For $3 \le r \le n-1$, the semigroup $\mathcal{OR}(n,r)$ is generated by its idempotents of rank r together with any single orientation-reversing transformation of rank r.*

Now, consider the following elements of $\mathcal{OR}_n$ of rank r:

$$\varepsilon_{1,2,\ldots,r} = \begin{pmatrix} 1 & 2 & \cdots & r-1 & \{r, r+1, \ldots, n\} \\ 1 & 2 & \cdots & r-1 & r \end{pmatrix} \in E_r \quad \text{and} \quad \widetilde{\varepsilon}_{1,2,\ldots,r} = \begin{pmatrix} 1 & 2 & \cdots & r-1 & \{r, r+1, \ldots, n\} \\ r & r-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Clearly, $\widetilde{\varepsilon}_{1,2,\ldots,r} \in J_r^{\mathcal{OR}_n} \setminus J_r^{\mathcal{OP}_n}$. Let

$$\widetilde{E}_r = (E_r \setminus \{\varepsilon_{1,2,\ldots,r}\}) \cup \{\widetilde{\varepsilon}_{1,2,\ldots,r}\}.$$

Then $E_r \subseteq \langle \widetilde{E}_r \rangle$ (since $\widetilde{\varepsilon}_{1,2,\ldots,r}^{\,2} = \varepsilon_{1,2,\ldots,r}$). Therefore, by Proposition 2.4 and Lemma 2.5, it follows:

Proposition 2.6 *For $3 \le r \le n-1$, the set $\widetilde{E}_r$ generates $\mathcal{OR}(n,r)$.*

Since the sets E_r and $\widetilde{E}_r$ have cardinality $\binom{n}{r}$, which equals the number of $\mathcal{R}$-classes (and, in fact, also of $\mathcal{L}$-classes) of rank r of both semigroups $\mathcal{OP}(n,r)$ and $\mathcal{OR}(n,r)$, from Propositions 2.4 and 2.6, our main result of this section follows immediately.

Theorem 2.7 *For $1 \le r \le n-1$, we have* $\operatorname{idrank} \mathcal{OP}(n,r) = \operatorname{rank} \mathcal{OP}(n,r) = \operatorname{rank} \mathcal{OR}(n,r) = \binom{n}{r}$.

Observe that Propositions 2.4 and 2.6 were stated for $r \ge 2$ and $r \ge 3$, respectively. However, we also have $\mathcal{OR}(n,2) = \mathcal{OP}(n,2)$ and, on the other hand, $\mathcal{OR}(n,1) = \mathcal{OP}(n,1) = \mathcal{T}(n,1)$. This justifies the full range $1 \le r \le n-1$ in the previous theorem.

3 The ranks of $\mathcal{POP}(n,r)$ and $\mathcal{POR}(n,r)$

The ranks of the monoids $\mathcal{POP}_n$ and $\mathcal{POR}_n$ were determined by Fernandes, Gomes and Jesus in [12]. Regarding their ideals, only the rank and the idempotent rank of the semigroup $\mathcal{POP}(n,n-1)$ are known up to now and these numbers were calculated by Zhao in [28].

Notice that we have $\mathcal{PO}(n,1) = \mathcal{POP}(n,1) = \mathcal{POR}(n,1) = \mathcal{PT}(n,1)$, $\mathcal{POP}(n,2) = \mathcal{POR}(n,2)$ and, for $3 \le r \le n$, $\mathcal{POP}(n,r) \subsetneq \mathcal{POR}(n,r)$. Since $E(\mathcal{POR}_n) = E(\mathcal{POP}_n)$, it follows that $\mathcal{POR}(n,r)$ is not idempotent generated for $r \ge 3$.

In this section we aim to finish the study of the ranks and idempotent ranks of $\mathcal{POP}(n,r)$ and to determine the ranks of the ideals of $\mathcal{POR}_n$.

The structures of the monoids $\mathcal{POP}_n$ and $\mathcal{POR}_n$ were also studied by Fernandes, Gomes and Jesus in [12]. In particular, they showed that, for $2 \le r \le n$, the number of $\mathcal{R}$-classes of rank r and of $\mathcal{L}$-classes of rank r of these two monoids are $\binom{n}{r}2^{n-r}$ and $\binom{n}{r}$, respectively. Observe also that each $\mathcal{R}$-class of $\mathcal{OP}_n$ is also an entire $\mathcal{R}$-class of $\mathcal{POP}_n$ and, on the other hand, each $\mathcal{L}$-class of $\mathcal{POP}_n$ contains exactly one $\mathcal{L}$-class of $\mathcal{OP}_n$. The same properties are valid for $\mathcal{OR}_n$ with respect to $\mathcal{POR}_n$.

Consider the element $g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$ of $\mathcal{POP}_n$ and let $s \in \mathcal{POP}(n,r)$, with $2 \le r \le n-1$. Then, by [12, Proposition 1.2], we have $s = g^i u$, for some $0 \le i \le n-1$ and $u \in \mathcal{PO}_n$. Clearly, we must also have $u \in \mathcal{PO}(n,r)$. Now, recall again that Garba [15], for $2 \le r < n-1$, and Gomes and Howie [18], for $r = n-1$, proved that $\mathcal{PO}(n,r) = \langle E(J_r^{\mathcal{PO}_n}) \rangle$. Then, $u = e_1 e_2 \cdots e_k$, for some $k \in \mathbb{N}$ and $e_1, e_2, \ldots, e_k \in E(J_r^{\mathcal{PO}_n})$. Hence $s = (g^i e_1) e_2 \cdots e_k$, with $g^i e_1, e_2, \ldots, e_k \in J_r^{\mathcal{POP}_n}$. Thus, we proved:

Lemma 3.1 *Let $2 \le r \le n-1$. Then $\mathcal{POP}(n,r) = \langle J_r^{\mathcal{POP}_n} \rangle$.*

Let $2 \le r \le n-1$ and let E_r be as defined in Section 2. Recall that E_r is formed by $\binom{n}{r}$ idempotents and contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{OP}_n$ of rank r. Consequently, the set E_r also has exactly one element from each $\mathcal{L}$-class of $\mathcal{POP}_n$ of rank r.

Let $m = \binom{n}{r}$ and $p = \binom{n}{r}2^{n-r}$. Since $\mathcal{POP}_n$ is regular (see [8, Proposition 2.3]), we may choose idempotents $\varepsilon_{m+1}, \ldots, \varepsilon_p$ of $\mathcal{POP}_n$ covering all $\mathcal{R}$-classes of $\mathcal{POP}_n$ contained in $J_r^{\mathcal{POP}_n} \setminus J_r^{\mathcal{OP}_n}$. Take

$$F_r = E_r \cup \{\varepsilon_{m+1}, \ldots, \varepsilon_p\}.$$

Notice that F_r has exactly one (idempotent) element from each $\mathcal{R}$-class of $\mathcal{POP}_n$ of rank r.

Lemma 3.2 *Let $2 \le r \le n-1$. Then $\mathcal{POP}(n,r) = \langle F_r \rangle$.*

Proof. Let $k \in \{m+1, \ldots, p\}$. Since E_r has (exactly) one idempotent from each $\mathcal{L}$-class of $\mathcal{POP}_n$ of rank r, then there exists $\widetilde{\varepsilon}_k \in E_r$ such that $\varepsilon_k \mathcal{L} \widetilde{\varepsilon}_k$. It follows that $\varepsilon_k \widetilde{\varepsilon}_k = \varepsilon_k$ and, by Green's Lemma, the map $x \mapsto \varepsilon_k x$ is a bijection from $R^{\mathcal{POP}_n}_{\widetilde{\varepsilon}_k}$ onto $R^{\mathcal{POP}_n}_{\varepsilon_k \widetilde{\varepsilon}_k} = R^{\mathcal{POP}_n}_{\varepsilon_k}$. Hence $R^{\mathcal{POP}_n}_{\varepsilon_k} = \varepsilon_k R^{\mathcal{POP}_n}_{\widetilde{\varepsilon}_k}$.

Therefore, we have

$$J_r^{\mathcal{POP}_n} = J_r^{\mathcal{OP}_n} \cup \bigcup_{k=m+1}^{p} R^{\mathcal{POP}_n}_{\varepsilon_k} = J_r^{\mathcal{OP}_n} \cup \bigcup_{k=m+1}^{p} \varepsilon_k R^{\mathcal{POP}_n}_{\widetilde{\varepsilon}_k} = J_r^{\mathcal{OP}_n} \cup \bigcup_{k=m+1}^{p} \varepsilon_k R^{\mathcal{OP}_n}_{\widetilde{\varepsilon}_k}.$$

Now, as $R^{\mathcal{OP}_n}_{\widetilde{\varepsilon}_k} \subset J_r^{\mathcal{OP}_n} \subset \mathcal{OP}(n,r)$, for all $k \in \{m+1, \ldots, p\}$, and $\mathcal{OP}(n,r) = \langle E_r \rangle$, by Proposition 2.4, it follows that $J_r^{\mathcal{POP}_n} \subset \langle F_r \rangle$. Finally, by Lemma 3.1, we conclude that $\mathcal{POP}(n,r) = \langle F_r \rangle$, as required. $\square$

Next, let $3 \le r \le n-1$ and take $\alpha \in \mathcal{POR}(n,r)$ an element of rank $k = 1, \ldots, r$. Then, we may suppose that $\alpha = \begin{pmatrix} I_0 & I_1 & \cdots & I_k \\ j_k & j_1 & \cdots & j_k \end{pmatrix}$, where $\{I_0, I_1, \ldots, I_k\}$ is a partition into intervals of $\mathrm{Dom}(\alpha)$, admitting that I_0 may be empty. See [12], for more details. For $1 \le \ell \le k$, let $i_\ell = \min I_\ell$ and $i_0 = \max I_0$, if $I_0 \neq \emptyset$, or $i_0 = 0$, otherwise (naturally, we are assuming that $i_0 < i_1 < i_2 < \cdots < i_k$). Define

$$\varepsilon_\alpha = \begin{pmatrix} I_0 & I_1 & I_2 & \cdots & I_k \\ i_k & i_1 & i_2 & \cdots & i_k \end{pmatrix} ,$$

$$\overline{\alpha} = \begin{pmatrix} \{1, \ldots, i_0\} & \{i_0+1, \ldots, i_2-1\} & \{i_2, \ldots, i_3-1\} & \cdots & \{i_{k-1}, \ldots, i_k-1\} & \{i_k, \ldots, n\} \\ j_k & j_1 & j_2 & \cdots & j_{k-1} & j_k \end{pmatrix}$$

and

$$\overline{\varepsilon}_\alpha = \begin{pmatrix} \{1, \ldots, i_0\} & \{i_0+1, \ldots, i_2-1\} & \{i_2, \ldots, i_3-1\} & \cdots & \{i_{k-1}, \ldots, i_k-1\} & \{i_k, \ldots, n\} \\ i_k & i_1 & i_2 & \cdots & i_{k-1} & i_k \end{pmatrix} .$$

Then, we have

$$\varepsilon_\alpha \in \mathcal{POP}(n,r) , \quad \overline{\alpha} \in J_k^{\mathcal{OR}_n} \subset \mathcal{OR}(n,r) , \quad \overline{\varepsilon}_\alpha \in E(J_k^{\mathcal{OP}_n}) , \quad \alpha = \varepsilon_\alpha \overline{\alpha} \quad \text{and} \quad \overline{\alpha} = \overline{\varepsilon}_\alpha \alpha .$$

Moreover, if α is an orientation-reversing transformation then so is $\overline{\alpha}$.

Now, a similar reasoning to the proof of Lemma 1.3, using this time Lemmas 3.2 and 2.5 instead of Lemmas 1.2 and 1.1, respectively, allows us to prove:

Lemma 3.3 *For $3 \le r \le n-1$, the semigroup $\mathcal{POR}(n,r)$ is generated by its idempotents of rank r together with any single orientation-reversing partial transformation of rank r.*

Consider again the transformations $\varepsilon_{1,2,\ldots,r}$ and $\widetilde{\varepsilon}_{1,2,\ldots,r}$ defined in Section 2. Then, we have $\varepsilon_{1,2,\ldots,r} \in E_r \subset F_r$ and $\widetilde{\varepsilon}_{1,2,\ldots,r} \in J_r^{\mathcal{POR}_n} \setminus J_r^{\mathcal{POP}_n}$. Let

$$\widetilde{F}_r = (F_r \setminus \{\varepsilon_{1,2,\ldots,r}\}) \cup \{\widetilde{\varepsilon}_{1,2,\ldots,r}\}.$$

Since $\widetilde{\varepsilon}^{\,2}_{1,2,\ldots,r} = \varepsilon_{1,2,\ldots,r}$, we have $F_r \subseteq \langle \widetilde{F}_r \rangle$ and so, by Lemmas 3.2 and 3.3, we have:

Lemma 3.4 *Let $3 \le r \le n-1$. Then $\mathcal{POR}(n,r) = \langle \widetilde{F}_r \rangle$.*

Since $\mathcal{POR}(n,2) = \mathcal{POP}(n,2)$ and the sets F_r and $\widetilde{F}_r$ have cardinality $\binom{n}{r} 2^{n-r}$, i.e. the number of $\mathcal{R}$-classes of rank r of the semigroups $\mathcal{POP}(n,r)$ and $\mathcal{POR}(n,r)$, our main theorem of this section follows from Lemmas 3.2 and 3.4.

Theorem 3.5 *For $2 \le r \le n-1$, we have* $\operatorname{idrank} \mathcal{POP}(n,r) = \operatorname{rank} \mathcal{POP}(n,r) = \operatorname{rank} \mathcal{POR}(n,r) = \binom{n}{r} 2^{n-r}$.

4 The ranks of $\mathcal{POI}(n,r)$, $\mathcal{PODI}(n,r)$, $\mathcal{POPI}(n,r)$, $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$

The structure of the symmetric inverse monoid $\mathcal{I}_n$ is well known (see [23], for instance). The structures and the ranks of the monoids $\mathcal{POI}_n$ and $\mathcal{POPI}_n$ were studied by Fernandes (see [7, 8, 9]) and of the monoids $\mathcal{PODI}_n$ and $\mathcal{PORI}_n$ by Fernandes, Gomes and Jesus (see [10] and also [4]).

In any of these five monoids, the number of $\mathcal{L}$-classes of rank r is $\binom{n}{r}$, which (being inverse semigroups) coincides with the number of $\mathcal{R}$-classes of rank r, for $1 \le r \le n$. Also, all the five share the same set (semilattice) of idempotents formed by all partial identities. Consequently, none of them is idempotent generated.

The monoid $\mathcal{POI}_n$ is aperiodic. The maximal subgroups of the $\mathcal{J}$-class of rank r of the monoid $\mathcal{PODI}_n$ are cyclic of order two, for $2 \le r \le n$, and trivial, for $r = 1$. Regarding the monoid $\mathcal{POPI}_n$, the maximal subgroups of its $\mathcal{J}$-class of rank r are cyclic of order r, for $1 \le r \le n$. The maximal subgroups of the $\mathcal{J}$-class of rank r of the monoid $\mathcal{PORI}_n$ are dihedral of order $2r$, for $3 \le r \le n$, cyclic of order two, for $r = 2$, and trivial, for $r = 1$. Finally, the maximal subgroups of the $\mathcal{J}$-class of rank r of the monoid $\mathcal{I}_n$ are isomorphic to the symmetric group $\mathcal{S}_r$, for $1 \le r \le n$.

It is clear that $\mathcal{POI}(n,1) = \mathcal{PODI}(n,1) = \mathcal{POPI}(n,1) = \mathcal{PORI}(n,1) = \mathcal{I}(n,1)$, $\mathcal{PODI}(n,2) = \mathcal{POPI}(n,2) = \mathcal{PORI}(n,2) = \mathcal{I}(n,2)$ and $\mathcal{PORI}(n,3) = \mathcal{I}(n,3)$.

Our objective in this section is to determine the ranks of the semigroups $\mathcal{POI}(n,r)$, $\mathcal{PODI}(n,r)$, $\mathcal{POPI}(n,r)$, $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$, for $1 \le r \le n-1$. Recall that by *rank* we mean always the rank as semigroup, i.e. as algebra of type 2. Notice that, Gomes and Howie [17] and Garba [16] determined the *ranks* of $\mathcal{I}(n,r)$, for $1 \le r \le n-1$, as inverse semigroups, i.e. as algebras of type (2,1).

Let $1 \le r \le n-1$.

Recall that Fernandes proved in [9, Proposition 2.8] that rank of the semigroup $\mathcal{POI}(n,n-1)$ is n (the same as $\mathcal{POI}_n$ as monoid). Also in [9] (see Proof of Lemma 2.7) the following result is proved:

Lemma 4.1 *Any element of $\mathcal{POI}_n$ of rank k is a product of elements of $\mathcal{POI}_n$ of rank $k+1$, for $0 \le k \le n-2$.*

As an immediate consequence, we have:

Corollary 4.2 *The semigroup $\mathcal{POI}(n,r)$ is generated by its elements of rank r.*

Now, consider the permutation (reflection) $h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$ of X_n.

Recall that the semigroup $\mathcal{PODI}_n$ is generated by $\mathcal{POI}_n \cup \{h\}$ (see [10, Proposition 3.4]). Using a similar argument, we show:

Corollary 4.3 *The semigroup $\mathcal{PODI}(n,r)$ is generated by its elements of rank r.*

Proof. Let $s \in \mathcal{PODI}(n,r)$. Then $sh \in \mathcal{POI}(n,r)$ and so, by Corollary 4.2, we have $sh = s_1 \cdots s_{\ell-1}s_\ell$, for some elements $s_1, \ldots, s_\ell \in \mathcal{POI}_n$ ($\ell \in \mathbb{N}$) of rank r. Since $h^2 = 1$ then $s = s_1 \cdots s_{\ell-1}(s_\ell h)$, a product of elements of $\mathcal{PODI}_n$ of rank r, as required. $\square$

Next, consider again the permutation (n-cycle) $g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$ of X_n. Recall that the semigroup $\mathcal{POPI}_n$ is generated by $\mathcal{POI}_n \cup \{g\}$ [8, Corollary 3.2]. This result is an immediate consequence of the following lemma [8, Proposition 3.1]:

Lemma 4.4 *Let $s \in \mathcal{POPI}_n$. Then there exist $i \in \{0, 1, \ldots, n-1\}$ and $t \in \mathcal{POI}_n$ such that $s = g^i t$.*

Combining this lemma with Corollary 4.3, we have:

Corollary 4.5 *The semigroup $\mathcal{POPI}(n,r)$ is generated by its elements of rank r.*

Proof. Let $s \in \mathcal{POPI}(n,r)$. Then, by the previous lemma, $s = g^i t$, for some $i \in \{0, 1, \ldots, n-1\}$ and $t \in \mathcal{POI}_n$. As g is a permutation, we have $t \in \mathcal{POI}(n,r)$. Hence, $t = s_1 s_2 \cdots s_\ell$, for some elements $s_1, \ldots, s_\ell \in \mathcal{POI}_n$ ($\ell \in \mathbb{N}$) of rank r. Thus $s = (g^i s_1)s_2 \cdots s_\ell$, a product of elements of $\mathcal{POPI}_n$ of rank r, as required. $\square$

Now, recall also that the semigroup $\mathcal{PORI}_n$ is generated by $\mathcal{POPI}_n \cup \{h\}$ (see [10, Theorem 5.5]). Again, by using a similar argument, we show:

Corollary 4.6 *The semigroup $\mathcal{PORI}(n,r)$ is generated by its elements of rank r.*

Proof. Let $s \in \mathcal{PORI}(n,r)$. Then $sh \in \mathcal{POPI}(n,r)$ and so, by Corollary 4.5, we have $sh = s_1 \cdots s_{\ell-1}s_\ell$, for some elements $s_1, \ldots, s_\ell \in \mathcal{POPI}_n$ ($\ell \in \mathbb{N}$) of rank r. Since $h^2 = 1$ then $s = s_1 \cdots s_{\ell-1}(s_\ell h)$, a product of elements of $\mathcal{PORI}_n$ of rank r, as required. $\square$

Finally, we consider the ideals of the symmetric inverse monoid $\mathcal{I}_n$. As an immediate consequence of Gomes and Howie [17] and Garba [16] results on the rank (as inverse semigroup) of $\mathcal{I}(n,r)$, we have:

Lemma 4.7 *The semigroup $\mathcal{I}(n,r)$ is generated by its elements of rank r.*

In fact, it is very easy to show that any element of $\mathcal{I}_n$ of rank k is a product of two elements of $\mathcal{I}_n$ of rank $k+1$, for $0 \le k \le n-2$, from which the previous lemma also follows immediately.

Now, let $m = \binom{n}{r}$ and $\binom{X_n}{r} = \{Y_i \mid i = 1, \dots, m\}$ (the family of all subsets of X_n of size r). For $1 \le i, j \le m$, denote by $\binom{Y_i}{Y_j}$ the unique element s of $\mathcal{POI}_n$ such that $\mathrm{Dom}(s) = Y_i$ and $\mathrm{Im}(s) = Y_j$ (see [7, Lemma 2.1]). Notice that

$$\left\{ \binom{Y_i}{Y_j} \mid 1 \le i, j \le m \right\}$$

is the set of all elements of $\mathcal{POI}(n,r)$ of rank r. Define $s_i = \binom{Y_i}{Y_{i+1}}$, for $1 \le i \le m-1$, and $s_m = \binom{Y_m}{Y_1}$. Thus, we have:

Proposition 4.8 *The set $\{s_1, \dots, s_m\}$ generates the semigroup $\mathcal{POI}(n,r)$.*

Proof. By Corollary 4.2, it suffices to show that any element of $\mathcal{POI}_n$ of rank r is a product of the elements $s_1, \dots, s_m$. Let $i, j \in \{1, \dots, m\}$. Then $\binom{Y_i}{Y_j} = \binom{Y_i}{Y_{i+1}} \cdots \binom{Y_{m-1}}{Y_m}\binom{Y_m}{Y_1}\binom{Y_1}{Y_2} \cdots \binom{Y_{j-1}}{Y_j} = s_i \cdots s_{m-1} s_m s_1 \cdots s_{j-1}$, as required. $\square$

Our next lemma, which is a simple consequence of Green's Lemma, gives us a general framework that we will apply to several cases.

Lemma 4.9 *Let S be a finite semigroup and let J be a (regular) $\mathcal{J}$-class of S. Let G be a maximal subgroup of S contained in J and let W be a subset of S having at least one element from each $\mathcal{H}$-class of S contained in J. Then $J \subseteq \langle W, G \rangle$.*

Proof. Let e be the identity of G.

First, we show that $R_e^S \subseteq \langle W, G \rangle$. Let $s \in R_e^S$ and take $c \in H_s^S \cap W$. Then $ec = c$ and so, by Green's Lemma, the mapping $x \mapsto xc$ is a bijection from $G = H_e^S$ onto H_c^S. Hence $H_s^S = H_c^S = Gc \subseteq \langle W, G \rangle$ and thus $R_e^S \subseteq \langle W, G \rangle$.

Now, let s be any element of J and take $c \in R_s^S \cap L_e^S \cap W$. Then $ce = c$ and so, by Green's Lemma, the mapping $x \mapsto cx$ is a bijection from R_e^S onto R_c^S. Hence $R_s^S = R_c^S = cR_e^S \subseteq \langle W, G \rangle$ and thus $J \subseteq \langle W, G \rangle$, as required. $\square$

For convenience, from now on, we assume $Y_1 = \{1, \dots, r\}$ and $Y_m = \{n-r+1, \dots, n\}$. Define:

$$\widetilde{s}_m = \begin{pmatrix} n-r+1 & n-r+2 & \cdots & n-1 & n \\ r & r-1 & \cdots & 2 & 1 \end{pmatrix}, \quad 2 \le r \le n-1$$

(observe that $\widetilde{s}_m \in \mathcal{PODI}(n,r)$, $\mathrm{Dom}(\widetilde{s}_m) = Y_m$ and $\mathrm{Im}(\widetilde{s}_m) = Y_1$);

$$\widehat{s}_m = \begin{pmatrix} n-r+1 & n-r+2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & r & 1 \end{pmatrix}, \quad 3 \le r \le n-1$$

(observe that $\widehat{s}_m \in \mathcal{POPI}(n,r)$, $\mathrm{Dom}(\widehat{s}_m) = Y_m$ and $\mathrm{Im}(\widehat{s}_m) = Y_1$); and

$$s_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & r \\ 2 & 1 & 3 & \cdots & r \end{pmatrix}, \quad 3 \le r \le n-1$$

(observe that $s_0 \in \mathcal{I}(n,r)$, $\mathrm{Dom}(s_0) = Y_1$ and $\mathrm{Im}(s_0) = Y_1$). Then, we have:

Proposition 4.10 *The set $\{s_1, \dots, s_{m-1}, \widetilde{s}_m\}$ generates the semigroup $\mathcal{PODI}(n,r)$, for $2 \le r \le n-1$, and the sets $\{s_1, \dots, s_{m-1}, \widehat{s}_m\}$, $\{s_1, \dots, s_{m-1}, \widetilde{s}_m, \widehat{s}_m\}$ and $\{s_1, \dots, s_{m-1}, \widehat{s}_m, s_0\}$ generate the semigroups $\mathcal{POPI}(n,r)$, $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$, respectively, for $3 \le r \le n-1$.*

Proof. Let $2 \le r \le n-1$ and take $\widetilde{s} = \begin{pmatrix} 1 & 2 & \cdots & r \\ r & r-1 & \cdots & 1 \end{pmatrix}$. Then, since $\widetilde{s}$ generates a maximal subgroup of $\mathcal{PODI}_n$ contained in $J_r^{\mathcal{PODI}_n}$, by Lemmas 4.8 and 4.9, we have $J_r^{\mathcal{PODI}_n} \subseteq \langle J_r^{\mathcal{POI}_n}, \widetilde{s} \rangle = \langle s_1, \dots, s_m, \widetilde{s} \rangle$. Hence, by Corollary 4.3, the set $\{s_1, \dots, s_m, \widetilde{s}\}$ generates $\mathcal{PODI}(n,r)$. Now, as

$$\widetilde{s} = \binom{Y_1}{Y_2} \cdots \binom{Y_{m-1}}{Y_m} \begin{pmatrix} n-r+1 & n-r+2 & \cdots & n \\ r & r-1 & \cdots & 1 \end{pmatrix} = s_1 \cdots s_{m-1} \widetilde{s}_m$$

and

$$s_m = \binom{Y_m}{Y_1} = \begin{pmatrix} n-r+1 & n-r+2 & \cdots & n \\ r & r-1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & r \\ r & r-1 & \cdots & 1 \end{pmatrix} = \widetilde{s}_m \widetilde{s},$$

it follows that $\{s_1,\ldots,s_{m-1},\widetilde{s}_m\}$ also generates $\mathcal{PODI}(n,r)$.

From now on, consider $3 \le r \le n-1$.

Let $g_r = \begin{pmatrix} 1 & 2 & \cdots & r-1 & r \\ 2 & 3 & \cdots & r & 1 \end{pmatrix}$. Then g_r generates a cyclic group of order r, i.e. a maximal subgroup of $\mathcal{POPI}_n$ contained in $J_r^{\mathcal{POPI}_n}$. Again, by Lemmas 4.8 and 4.9, it follows that $J_r^{\mathcal{POPI}_n} \subseteq \langle J_r^{\mathcal{POI}_n}, g_r\rangle = \langle s_1,\ldots,s_m,g_r\rangle$ and, by Corollary 4.5, that the set $\{s_1,\ldots,s_m,g_r\}$ generates $\mathcal{POPI}(n,r)$. Now, since

$$g_r = \binom{Y_1}{Y_2}\cdots\binom{Y_{m-1}}{Y_m}\begin{pmatrix} n-r+1 & \cdots & n-1 & n \\ 2 & \cdots & r & 1 \end{pmatrix} = s_1\cdots s_{m-1}\widehat{s}_m$$

and

$$s_m = \binom{Y_m}{Y_1} = \begin{pmatrix} n-r+1 & \cdots & n-1 & n \\ 2 & \cdots & r & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \cdots & r-1 & r \\ r & 1 & \cdots & r-2 & r-1 \end{pmatrix} = \widehat{s}_m g_r^{r-1},$$

we also have that $\{s_1,\ldots,s_{m-1},\widehat{s}_m\}$ generates $\mathcal{POPI}(n,r)$.

Next, by noticing that $\{\widetilde{s},g_r\}$ generates a dihedral group of order $2r$, whence a maximal subgroup of $\mathcal{PORI}_n$ contained in $J_r^{\mathcal{PORI}_n}$, once again, by Lemmas 4.8 and 4.9, it follows that $J_r^{\mathcal{PORI}_n} \subseteq \langle J_r^{\mathcal{POI}_n},\widetilde{s},g_r\rangle = \langle s_1,\ldots,s_m,\widetilde{s},g_r\rangle = \langle s_1,\ldots,s_{m-1},\widetilde{s}_m,\widehat{s}_m\rangle$. Thus, by Corollary 4.6, the set $\{s_1,\ldots,s_{m-1},\widetilde{s}_m,\widehat{s}_m\}$ generates $\mathcal{PORI}(n,r)$.

Finally, observe that $\{s_0,g_r\}$ generates the symmetric group $\mathcal{S}_r$, i.e. a maximal subgroup of $\mathcal{I}_n$ contained in $J_r^{\mathcal{I}_n}$. Once again, by Lemmas 4.8 and 4.9, we have $J_r^{\mathcal{I}_n} \subseteq \langle J_r^{\mathcal{POI}_n}, g_r, s_0\rangle = \langle s_1,\ldots,s_m,g_r,s_0\rangle = \langle s_1,\ldots,s_{m-1},\widehat{s}_m,s_0\rangle$. Then, by Corollary 4.7, the set $\{s_1,\ldots,s_{m-1},\widehat{s}_m,s_0\}$ generates $\mathcal{I}(n,r)$, as required. $\square$

Before presenting our last main result, we prove the following lemma.

Lemma 4.11 *Let S be a finite inverse semigroup and let J be a maximal $\mathcal{J}$-class of S with k $\mathcal{L}$-classes and containing a non-cyclic subgroup. Then* $\operatorname{rank} S \ge k+1$.

Proof. Since J is maximal and has k $\mathcal{L}$-classes (and, consequently, k $\mathcal{R}$-classes), then $\operatorname{rank} S \ge k$. Suppose by contradiction that $\operatorname{rank} S = k$ and let $A = \{a_1,\ldots,a_k\}$ be a generating set of S with k elements. Then $J = L^S_{a_1}\cup\cdots\cup L^S_{a_k} = R^S_{a_1}\cup\cdots\cup R^S_{a_k}$.

Take $i \in \{1,\ldots,k\}$. Since each $\mathcal{L}$-class of S contains exactly one idempotent and, on the other hand, $a_ia_j \in J$ if and only if $a_ia_j \in R^S_{a_i}\cap L^S_{a_j}$ if and only if $E(L^S_{a_i}\cap R^S_{a_j}) \ne \emptyset$, for all $j \in \{1,\ldots,k\}$, then there exists a unique $j \in \{1,\ldots,k\}$ such that $a_ia_j \in J$. Thus A induces a permutation σ of $\{1,\ldots,k\}$ (defined by $i\sigma = j$ if and only if $a_ia_j \in J$, for $1 \le i,j \le k$).

Now, let $1 \le i,j \le k$. Since an element $s \in R^S_{a_i}\cap L^S_{a_j}$ has to be of the form $s = a_iua_j$, for some product u of elements of A, then i and j must be in the same orbit of σ. Therefore σ is a k-cycle.

Next, for instance, take an element $s \in R^S_{a_1}\cap L^S_{a_{1\sigma^{k-1}}}$. Then, $s = (a_1a_{1\sigma}\cdots a_{1\sigma^{k-1}})^\ell$, for some $\ell \ge 1$. (Observe that the maximal subgroups of J are the $\mathcal{H}$-classes $R^S_{a_i}\cap L^S_{a_{i\sigma^{k-1}}}$, for $i \in \{1,\ldots,k\}$.) It follows that $R^S_{a_1}\cap L^S_{a_{1\sigma^{k-1}}}$ is a cyclic group (generated by $a_1a_{1\sigma}\cdots a_{1\sigma^{k-1}}$) and so all maximal subgroups of S contained in J are cyclic groups, which is a contradiction. Thus $\operatorname{rank} S \ge k+1$, as required. $\square$

Regarding that $\mathcal{POI}(n,r)$, $\mathcal{PODI}(n,r)$, $\mathcal{POPI}(n,r)$, $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$ have $\binom{n}{r}$ $\mathcal{L}$-classes of rank r, and $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$, for $r \ge 3$, have non-cyclic subgroups inside their maximal $\mathcal{J}$-class, in view of Propositions 4.8 and 4.10 and of Lemma 4.11, we immediately have:

Theorem 4.12 *For $r \in \{1,\ldots,n-1\}$, the semigroups $\mathcal{POI}(n,r)$, $\mathcal{PODI}(n,r)$ and $\mathcal{POPI}(n,r)$ have rank $\binom{n}{r}$. For $r \in \{3,\ldots,n-1\}$, the rank of both semigroups $\mathcal{PORI}(n,r)$ and $\mathcal{I}(n,r)$ is $\binom{n}{r}+1$.*

Observe that the rank of the semigroup $\mathcal{I}(n,r)$ coincides with its *rank* as inverse semigroup, for $2 \le r \le n$. However, this is not the case for $r = 1$. The first is n, by Theorem 4.12, while the second is $n-1$, by [16, Theorem 2.2].

Finally, notice that, regarding an inverse semigroup, since the rank as semigroup is always less than or equal to the *rank* as inverse semigroup, in fact to prove Theorem 4.12, by the results of Gomes and Howie [17] and Garba [16], we would only require Lemma 4.11 for the case of $\mathcal{PORI}(n,r)$.

5 Summary

The following tables summarize the results presented in this paper, including the ones previously proved by several authors and mentioned along the text. Notice that all ranks and idempotent ranks presented in the tables are *semigroup ranks*, i.e. we are always considering algebras of type (2).

For semigroups of partial transformations, we have:

	rank	idrank
$\mathcal{PT}_n$	4	n/a
$\mathcal{PT}(n,r),\ 1\le r\le n-1$	$S(n+1,r+1)$	$S(n+1,r+1)$
$\mathcal{POR}_n$	4	n/a
$\mathcal{POR}(n,r),\ 3\le r\le n-1$	$\binom{n}{r}2^{n-r}$	n/a
$\mathcal{POR}(n,2)$	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}2^{n-2}$
$\mathcal{POR}(n,1)$	2^n-1	2^n-1
$\mathcal{POP}_n$	3	n/a
$\mathcal{POP}(n,r),\ 2\le r\le n-1$	$\binom{n}{r}2^{n-r}$	$\binom{n}{r}2^{n-r}$
$\mathcal{POP}(n,1)$	2^n-1	2^n-1
$\mathcal{POD}_n$	$n+1$	n/a
$\mathcal{POD}(n,r),\ 2\le r\le n-1$	$\sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$	n/a
$\mathcal{POD}(n,1)$	2^n-1	2^n-1
$\mathcal{PO}_n$	$2n$	$3n-1$
$\mathcal{PO}(n,n-1)$	$2n-1$	$3n-2$
$\mathcal{PO}(n,r),\ 1\le r\le n-2$	$\sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$	$\sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$

Now, we consider semigroups of full transformations and semigroups of injective partial transformations, respectively:

	rank	idrank
$\mathcal{T}_n$	3	n/a
$\mathcal{T}(n,r),\ 2\le r\le n-1$	$S(n,r)$	$S(n,r)$
$\mathcal{T}(n,1)$	n	n
$\mathcal{OR}_n$	3	n/a
$\mathcal{OR}(n,r),\ 3\le r\le n-1$	$\binom{n}{r}$	n/a
$\mathcal{OR}(n,r),\ r=1,2$	$\binom{n}{r}$	$\binom{n}{r}$
$\mathcal{OP}_n$	2	n/a
$\mathcal{OP}(n,r),\ 1\le r\le n-1$	$\binom{n}{r}$	$\binom{n}{r}$
$\mathcal{OD}_n$	$\lceil\frac{n}{2}\rceil+1$	n/a
$\mathcal{OD}(n,r),\ 2\le r\le n-1$	$\binom{n}{r}$	n/a
$\mathcal{OD}(n,1)$	n	n
$\mathcal{O}_n$	$n+1$	$2n-1$
$\mathcal{O}(n,n-1)$	n	$2n-2$
$\mathcal{O}(n,r),\ 1\le r\le n-2$	$\binom{n}{r}$	$\binom{n}{r}$

	rank
$\mathcal{I}_n$	3
$\mathcal{I}(n,r),\ 3\le r\le n-1$	$\binom{n}{r}+1$
$\mathcal{I}(n,r),\ r=1,2$	$\binom{n}{r}$
$\mathcal{PORI}_n$	3
$\mathcal{PORI}(n,r),\ 3\le r\le n-1$	$\binom{n}{r}+1$
$\mathcal{PORI}(n,r),\ r=1,2$	$\binom{n}{r}$
$\mathcal{POPI}_n$	2
$\mathcal{POPI}(n,r),\ 1\le r\le n-1$	$\binom{n}{r}$
$\mathcal{PODI}_n$	$\lceil\frac{n}{2}\rceil+1$
$\mathcal{PODI}(n,r),\ 1\le r\le n-1$	$\binom{n}{r}$
$\mathcal{POI}_n$	$n+1$
$\mathcal{POI}(n,r),\ 1\le r\le n-1$	$\binom{n}{r}$

Remark

We would like to thank the anonymous referee for his/her helpful comments and suggestions that helped to improve this paper and, in particular, for pointing out the deep work of Gray and Ruškuc on the rank of completely 0-simple semigroups (see [26], [22], [21] and also [20]). After reduction to a principal factor, alternatively, we may use this general framework to deduce our results. In the case of Theorems 1.4, 2.7 and 3.5 this other approach does not allow us to produce significantly shorter proofs. On the other hand, by contrast, the proof of Theorem 4.12 can be substantially reduced by replacing Results 4.8-4.11 by a shorter deduction using Gray and Ruškuc work. Nevertheless, since we constructed explicit sets of generators for the semigroups in question, which may be useful by themselves, we opted for keeping our original proof.

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