ON THE RANKS OF SEMIGROUPS OF TRANSFORMATIONS ON A FINITE SET WITH RESTRICTED RANGE

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ABSTRACT. Let $\mathcal{PT}(X)$ be the semigroup of all partial transformations on X (under composition) and let $\mathcal{T}(X)$ and $\mathcal{I}(X)$ be the subsemigroups of $\mathcal{PT}(X)$ of all full transformations on X and of all injective partial transformations on X, respectively. Given a nonempty subset Y of X, let $\mathcal{PT}(X,Y) = \{\alpha \in \mathcal{PT}(X) \mid X\alpha \subseteq Y\}$, $\mathcal{T}(X,Y) = \mathcal{PT}(X,Y) \cap \mathcal{T}(X)$ and $\mathcal{I}(X,Y) = \mathcal{PT}(X,Y) \cap \mathcal{I}(X)$.

In 2008, Sanwong and Sommanee described the largest regular subsemigroup and determined the Green's relations of $\mathcal{T}(X,Y)$. In this paper, we present analogous results for both $\mathcal{PT}(X,Y)$ and $\mathcal{I}(X,Y)$.

For a finite set X such that $|X| \geq 3$, the ranks of $\mathcal{PT}(X) = \mathcal{PT}(X,X)$, $\mathcal{T}(X) = \mathcal{T}(X,X)$ and $\mathcal{I}(X) = \mathcal{I}(X,X)$ are well known to be 4, 3 and 3, respectively. In this paper, we also compute the ranks of $\mathcal{PT}(X,Y)$, $\mathcal{T}(X,Y)$ and $\mathcal{I}(X,Y)$, for any proper nonempty subset Y of X.

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INTRODUCTION AND PRELIMINARIES

Transformation semigroups play a role in Semigroup Theory corresponding to that of the permutation groups in Group Theory. Two main results parallel to Cayley's theorem for groups are: first of all, the well known result that states that every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup and, secondly, in Inverse Semigroup Theory, the Wagner-Preston Theorem stating that every inverse semigroup is isomorphic to a subsemigroup of a suitable symmetric inverse semigroup. Thereby, in some sense, in order to study all semigroups it suffices to consider transformation semigroups, which yields decisive importance to this kind of semigroups. In this paper we will deal with semigroups of transformations with restricted range. Our purpose is to study, within this context, some well known and very important concepts in Semigroup Theory, such as Green's relations, regularity and rank. Next, we briefly recall this standard notions.

Let S be a semigroup. We say that an element $x \in S$ is *regular* if there exists $y \in S$ such that $x = xyx$. If $y \in S$ verifies both $x = xyx$ and $y = yxy$, then it is called an *inverse* of x. Notice that, if $x = xyx$ then the element yxy is an inverse of x. We say that S is *regular* if every element of S is regular. Furthermore, if each element of S has a unique inverse then we say that S is an *inverse* semigroup.

The Green's relations of S are the natural equivalences associated to the ideals of S. More precisely, being S^1 the monoid obtained from S through the adjoining of an identity if S has none and exactly S otherwise, the Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ of S are defined by

- $x\mathcal{R}y$ if and only if $xS^1 = yS^1$,
- $x\mathcal{L}y$ if and only if $S^1x = S^1y$ and
- $x\mathcal{J}y$ if and only if $S^1xS^1 = S^1yS^1$,

for $x, y \in S$. Moreover, the Green's relation $\mathcal{D}$ of S is the join $\mathcal{R} \vee \mathcal{L}$ (in the lattice of the equivalence relations of S) and, since the relations $\mathcal{R}$ and $\mathcal{L}$ commute (under composition), then $\mathcal{D}$ coincides with $\mathcal{R}\mathcal{L} = \mathcal{L}\mathcal{R}$.

Finally, the *rank* of a semigroup S is the smallest number of elements required to generate S. Notice that, by the *rank* of a transformation we mean the size of its range (or image).

In general, we follow the notations of Howie's book [3].

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Let X be a set and let $\mathcal{PT}(X)$ be the semigroup of all partial transformations on X (under composition). We denote by $\mathcal{T}(X)$ the subsemigroup of $\mathcal{PT}(X)$ of all full transformations on X and by $\mathcal{I}(X)$ the symmetric inverse semigroup on X, that is, the subsemigroup of $\mathcal{PT}(X)$ of all injective partial transformations on X. For a subset Y of X, we consider the subsemigroups with restricted range $\mathcal{PT}(X,Y)=\{\alpha\in\mathcal{PT}(X)\mid X\alpha\subseteq Y\}$ of $\mathcal{PT}(X)$ and $\mathcal{I}(X,Y)=\mathcal{PT}(X,Y)\cap\mathcal{I}(X)$ of $\mathcal{I}(X)$ (and of $\mathcal{PT}(X)$). If Y is nonempty, we also consider the subsemigroup with restricted range $\mathcal{T}(X,Y)=\mathcal{PT}(X,Y)\cap\mathcal{T}(X)$ of $\mathcal{T}(X)$ (and of $\mathcal{PT}(X)$).

Clearly, $\mathcal{PT}(X)=\mathcal{PT}(X,X)$, $\mathcal{T}(X)=\mathcal{T}(X,X)$, $\mathcal{I}(X)=\mathcal{I}(X,X)$ and $\mathcal{PT}(X,\emptyset)=\mathcal{I}(X,\emptyset)=\{\emptyset\}$.

In 1975, Symons [11] introduced and studied the semigroup $\mathcal{T}(X,Y)$. He described all the automorphisms of this semigroup and also determined when $\mathcal{T}(X_1,Y_1)$ is isomorphic to $\mathcal{T}(X_2,Y_2)$. In [6], Nenthein et al. characterized the regular elements of $\mathcal{T}(X,Y)$ and, in [8], Sanwong and Sommanee obtained the largest regular subsemigroup of $\mathcal{T}(X,Y)$ and showed that this subsemigroup determines the Green's relations on $\mathcal{T}(X,Y)$. Moreover, they also gave a class of maximal inverse subsemigroups of this semigroup. In 2007, Sanwong and Sullivan determined all maximal congruences on $I(X)$ in [9]. Later, in 2009, all maximal and minimal congruences on $\mathcal{T}(X,Y)$ were described by Sanwong et al. [7]. Recently, all ideals of $\mathcal{T}(X,Y)$ were obtained by Mendes-Gonçalves and Sullivan in [5]. On the other hand, in [10], Sullivan considered the linear counterpart of $\mathcal{T}(X,Y)$, that is the semigroup $\mathcal{T}(V,W)$ which consists of all linear transformations from a vector space V into a fixed subspace W of V, and described its Green's relations and ideals. Regarding the linear counterparts of $\mathcal{PT}(X,Y)$ and $\mathcal{I}(X,Y)$, that is the semigroups $\mathcal{PT}(V,W)$ and $\mathcal{I}(V,W)$ which consist of all partial linear transformations and of all injective partial linear transformations, respectively, from a vector space V into a fixed subspace W of V, their Green's relations and some partial orders were studied by Sangkhanan and Sanwong.

Recall that Symons [11] proved that if $\mathcal{T}(X_1,Y_1)\cong\mathcal{T}(X_2,Y_2)$ then $|Y_1|=|Y_2|$. Moreover, if $|Y_1|=|Y_2|=1$, then $\mathcal{T}(X_1,Y_1)\cong\mathcal{T}(X_2,Y_2)$; if $|Y_1|=|Y_2|=2$, then $\mathcal{T}(X_1,Y_1)\cong\mathcal{T}(X_2,Y_2)$ if and only if $2^{|X_1\setminus Y_1|}=2^{|X_2\setminus Y_2|}$; and if $|Y_1|=|Y_2|>2$, then $\mathcal{T}(X_1,Y_1)\cong\mathcal{T}(X_2,Y_2)$ if and only if $|X_1\setminus Y_1|=|X_2\setminus Y_2|$. In particular, if X is finite and Y_1 and Y_2 are two nonempty subsets of X, then $\mathcal{T}(X,Y_1)$ and $\mathcal{T}(X,Y_2)$ are isomorphic if and only if $|Y_1|=|Y_2|$. Here, we show that this last result has analogues for the partial and the partial injective counterparts of $\mathcal{T}(X,Y)$. Therefore, regarding transformations on a finite set with restricted range, it suffices to study the semigroups $\mathcal{PT}_{n,r}=\mathcal{PT}(\{1,\ldots,n\},\{1,\ldots,r\})$, $\mathcal{I}_{n,r}=\mathcal{I}(\{1,\ldots,n\},\{1,\ldots,r\})$ and $\mathcal{T}_{n,r}=\mathcal{T}(\{1,\ldots,n\},\{1,\ldots,r\})$, for $1\le r\le n$ and $n\in\mathbb{N}$.

For $n\geq 3$, the ranks of $\mathcal{PT}_n=\mathcal{PT}_{n,n}$, $\mathcal{I}_n=\mathcal{I}_{n,n}$ and $\mathcal{T}_n=\mathcal{T}_{n,n}$ are equal to 4, 3 and 3, respectively. These are well known results (and all of them have reasonably easy proofs). See [3] for example.

On the other hand, the rank of the semigroup of singular mappings $\mathrm{Sing}_n=\{\alpha\in\mathcal{T}_n\mid|\mathrm{Im}(\alpha)|\le n-1\}$ is more difficult to determine. In [2], Gomes and Howie proved that both the rank and the idempotent rank of Sing_n are equal to $n(n-1)/2$. This result was later generalized by Howie and McFadden [4] who showed that the rank and idempotent rank of the semigroups $\mathcal{T}(n,r)=\{\alpha\in\mathcal{T}_n\mid|\mathrm{Im}(\alpha)|\le r\}$ are both equal to $S(n,r)$, the Stirling number of the second kind, for $2\le r\le n-1$. Recall that, for $1\le r\le n$ and $n\in\mathbb{N}$, $S(n,r)$ is the number of r-partitions on a set of n elements, which may be defined by the recurrence relation $S(n,r)=S(n-1,r-1)+rS(n-1,r)$, with $S(n,1)=S(n,n)=1$.

In [1], Garba considered the semigroup $\mathcal{PT}(n,r)=\{\alpha\in\mathcal{PT}_n\mid|\mathrm{Im}(\alpha)|\le r\}$ and showed that, for $2\le r\le n-1$, both its rank and idempotent rank are equal to $S(n+1,r+1)$.

Regarding the partial injective counterparts, Gomes and Howie [2] showed that the rank (as an inverse semigroup) of the inverse semigroup $\mathcal{SP}_n=\{\alpha\in\mathcal{I}_n\mid|\mathrm{Im}(\alpha)|\le n-1\}$ is $n+1$. Garba [1] generalized this result by showing that the rank of $\mathcal{I}(n,r)=\{\alpha\in\mathcal{I}_n\mid|\mathrm{Im}(\alpha)|\le r\}$, for $3\le r\le n-1$, is $\binom{n}{r}+1$.

In this paper, for $1\le r\le n-1$, although the semigroups $\mathcal{PT}_{n,r}$, $\mathcal{I}_{n,r}$ and $\mathcal{T}_{n,r}$ have a quite different structure of the semigroups $\mathcal{PT}(n,r)$, $\mathcal{I}(n,r)$ and $\mathcal{T}(n,r)$ (for example, the first are not regular in general while the latter ones are), respectively, we prove that their ranks coincide.

Now, we recall that Sanwong and Sommanee [8] proved that $F=\{\alpha\in\mathcal{T}(X,Y)\mid X\alpha=Y\alpha\}$ is the largest regular subsemigroup of $\mathcal{T}(X,Y)$ and that, for $\alpha,\beta\in\mathcal{T}(X,Y)$, we have:

- $\alpha\mathcal{L}\beta$ if and only if ($\alpha,\beta\in F$ and $X\alpha=X\beta$) or ($\alpha,\beta\in\mathcal{T}(X,Y)\setminus F$ and $\alpha=\beta$);
- $\alpha\mathcal{R}\beta$ if and only if ($\alpha,\beta\in F$ and $\mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta)$) or ($\alpha,\beta\in\mathcal{T}(X,Y)\setminus F$ and $\mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta)$);
- $\alpha\mathcal{D}\beta$ if and only if ($\alpha,\beta\in F$ and $|X\alpha|=|X\beta|$) or ($\alpha,\beta\in\mathcal{T}(X,Y)\setminus F$ and $\mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta)$); and
- $\alpha\mathcal{J}\beta$ if and only if $\mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta)$ or $|X\alpha|=|Y\alpha|=|Y\beta|=|X\beta|$.

In this paper, Sections 1 and 3 are devoted to determine the largest regular subsemigroups and to describe the Green's relations of $\mathcal{PT}(X,Y)$ and $\mathcal{I}(X,Y)$. The following convenient notation will be used in both these sections: given $\alpha \in \mathcal{PT}(X,Y)$, we write $\alpha = \binom{X_i}{a_i}$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i \mid i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

In Sections 2 and 4, we compute the ranks of the finite semigroups with restricted range $\mathcal{T}_{n,r}$, $\mathcal{PT}_{n,r}$ and $\mathcal{I}_{n,r}$, for $n \geq 2$ and $1 \leq r \leq n-1$.

1. On the semigroup $\mathcal{PT}(X,Y)$

We begin by remarking that, given $\alpha \in \mathcal{PT}(X,Y)$ and $Z \subseteq X$, we have $Z\alpha = \{x\alpha \mid x \in \mathrm{Dom}(\alpha) \cap Z\}$. It follows a simple result that will be used throughout the paper.

Lemma 1.1. *If $A \subseteq B \subseteq X$, then $A\alpha \subseteq B\alpha$ for all $\alpha \in \mathcal{PT}(X,Y)$.*

Consider the subset $\mathcal{P}F = \{\alpha \in \mathcal{PT}(X,Y) \mid X\alpha = Y\alpha\}$ of $\mathcal{PT}(X,Y)$. Notice that, being $\alpha \in \mathcal{P}F$ and $\beta \in \mathcal{PT}(X,Y)$, we have $X\alpha = Y\alpha$ and so $X\alpha\beta = Y\alpha\beta$, by Lemma 1.1. Hence $\mathcal{P}F$ is a right ideal of $\mathcal{PT}(X,Y)$. In particular, $\mathcal{P}F$ is a subsemigroup of $\mathcal{PT}(X,Y)$. Moreover, we have:

Theorem 1.2. *Let $\alpha \in \mathcal{PT}(X,Y)$. Then, α is regular if and only if $\alpha \in \mathcal{P}F$. Consequently, $\mathcal{P}F$ is the largest regular subsemigroup of $\mathcal{PT}(X,Y)$.*

Proof. Let $\alpha \in \mathcal{P}F$. Then for each $x \in Y\alpha = X\alpha$, choose $d_x \in x\alpha^{-1} \cap Y$ and define $\gamma : X\alpha \to \{d_x \mid x \in X\alpha\}$ by $x\gamma = d_x$, for all $x \in X\alpha$. Then $\gamma \in \mathcal{P}F$ and $\alpha\gamma\alpha = \alpha$. Now, let δ be any regular element in $\mathcal{PT}(X,Y)$. Then $\delta = \delta\theta\delta$, for some $\theta \in \mathcal{PT}(X,Y)$, whence $X\delta = X\delta\theta\delta = (X\delta\theta)\delta \subseteq Y\delta$ and so $\delta \in \mathcal{P}F$. Therefore, $\mathcal{P}F$ consists of all regular elements of $\mathcal{PT}(X,Y)$. $\square$

Next, we establish the Green's relations of $\mathcal{PT}(X,Y)$, by beginning with the following lemma.

Lemma 1.3. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. If $\beta \in \mathcal{P}F$, then $X\alpha \subseteq X\beta$ if and only if $\alpha = \gamma\beta$, for some $\gamma \in \mathcal{PT}(X,Y)$.*

Proof. Assume that $\beta \in \mathcal{P}F$ and $\alpha = \gamma\beta$, for some $\gamma \in \mathcal{PT}(X,Y)$. Then, it is clear that $X\alpha \subseteq X\beta$. Conversely, assume that $\beta \in \mathcal{P}F$ and $X\alpha \subseteq X\beta$. Therefore, we can write $\alpha = \binom{A_i}{a_i}$ and $\beta = \binom{B_i\ B_j}{a_i\ a_j}$, where $B_i \cap Y \neq \emptyset$ and $B_j \cap Y \neq \emptyset$. Let $\gamma \in \mathcal{PT}(X,Y)$ be defined as follow: $\gamma = \binom{A_i}{b_i}$, where $b_i \in B_i \cap Y$. Thus $\alpha = \gamma\beta$, as required. $\square$

Theorem 1.4. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. Then $\alpha\mathcal{L}\beta$ if and only if ($\alpha, \beta \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$ and $\alpha = \beta$) or ($\alpha, \beta \in \mathcal{P}F$ and $X\alpha = X\beta$).*

Proof. Assume that $\alpha\mathcal{L}\beta$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, for some $\lambda, \mu \in \mathcal{PT}(X,Y)^1$. Suppose that $\alpha \in \mathcal{P}F$. If $\lambda = 1$ or $\mu = 1$, we get $\beta = \alpha \in \mathcal{P}F$ and $X\alpha = X\beta$. On the other hand, if $\lambda, \mu \in \mathcal{PT}(X,Y)$, then $X\beta = X\mu\alpha = (X\mu\lambda)\beta \subseteq Y\beta$, since $X\mu\lambda \subseteq Y$. Thus $\beta \in \mathcal{P}F$. From $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, we have $X\alpha = X\beta$, by Lemma 1.3. Now, suppose that $\alpha \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$. If $\lambda, \mu \in \mathcal{PT}(X,Y)$, then $X\alpha = X\lambda\beta = (X\lambda\mu)\alpha \subseteq Y\alpha$, which contradicts that $\alpha \notin \mathcal{P}F$. Thus $\lambda = 1$ or $\mu = 1$ and so $\beta = \alpha \in \mathcal{P}F$.

The converse is a direct consequence of Lemma 1.3. $\square$

Theorem 1.5. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. Then $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\beta) \cap (\mathrm{Dom}(\beta) \times \mathrm{Dom}(\alpha)) \subseteq \mathrm{Ker}(\alpha)$ if and only if $\alpha = \beta\gamma$, for some $\gamma \in \mathcal{PT}(X,Y)$. Furthermore, $\alpha\mathcal{R}\beta$ if and only if $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$.*

Proof. If $\alpha = \beta\gamma$, for some $\gamma \in \mathcal{PT}(X,Y)$, then it is clear that $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(\beta)$. On the other hand, consider $(a,b) \in \mathrm{Ker}(\beta) \cap (\mathrm{Dom}(\beta) \times \mathrm{Dom}(\alpha))$. Then we have $a\beta = b\beta$. Since $b \in \mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta\gamma)$, it follows that $b\beta\gamma$ exists, whence $b\beta\gamma = (b\beta)\gamma = (a\beta)\gamma = a\beta\gamma$ and so we also obtain $a \in \mathrm{Dom}(\beta\gamma) = \mathrm{Dom}(\alpha)$. Moreover, $a\alpha = (a\beta)\gamma = (b\beta)\gamma = b\alpha$, whence $(a,b) \in \mathrm{Ker}(\alpha)$. Thus $\mathrm{Ker}(\beta) \cap (\mathrm{Dom}(\beta) \times \mathrm{Dom}(\alpha)) \subseteq \mathrm{Ker}(\alpha)$.

Conversely, assume that the conditions hold. Let $x \in (\mathrm{Dom}(\alpha))\beta$. Then $a\beta = x$, for some $a \in \mathrm{Dom}(\alpha)$. Notice that, if $b \in \mathrm{Dom}(\beta)$ is also such that $b\beta = x$ then $(b,a) \in \mathrm{Ker}(\beta) \cap (\mathrm{Dom}(\beta) \times \mathrm{Dom}(\alpha))$. Hence, by hypothesis, we have $b\alpha = a\alpha$ (and, in particular, $b \in \mathrm{Dom}(\alpha)$). Thus, we may consider the transformation $\gamma \in \mathcal{PT}(X,Y)$ with $\mathrm{Dom}(\gamma) = (\mathrm{Dom}(\alpha))\beta$ and defined, for each $x \in (\mathrm{Dom}(\alpha))\beta$, by $x\gamma = a\alpha$, for some $a \in \mathrm{Dom}(\alpha)$ such that $a\beta = x$. Clearly, $\alpha = \beta\gamma$, as required. $\square$

Corollary 1.6. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$ be such that $\alpha \mathcal{R} \beta$. Then $\alpha, \beta \in \mathcal{P}F$ or $\alpha, \beta \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$.*

Proof. It suffices to show that $\alpha \in \mathcal{P}F$ implies $\beta \in \mathcal{P}F$. Therefore, suppose that $\alpha \in \mathcal{P}F$. Then, we can write $\alpha = \binom{A_i}{a_i}$, where $A_i \cap Y \neq \emptyset$ and $a_i \in Y$, for all i. Since $\alpha \mathcal{R} \beta$, it follows that $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$, by the previous theorem, and so we can write $\beta = \binom{A_i}{b_i}$, where $b_i \in Y$, for all i. Hence $X\beta = Y\beta$ and thus $\beta \in \mathcal{P}F$, as required. $\square$

Theorem 1.7. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. Then $\alpha \mathcal{D} \beta$ if and only if ($\alpha, \beta \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$, $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$) or ($\alpha, \beta \in \mathcal{P}F$ and $|X\alpha| = |X\beta|$).*

Proof. Assume that $\alpha \mathcal{L} \gamma \mathcal{R} \beta$, for some $\gamma \in \mathcal{PT}(X,Y)$. If $\alpha \in \mathcal{P}F$, then $\alpha \mathcal{L} \gamma$ implies that $\gamma \in \mathcal{P}F$ and $|X\alpha| = |X\gamma|$, by Theorem 1.4. Since $\gamma \in \mathcal{P}F$ and $\gamma \mathcal{R} \beta$, it follows that $\beta \in \mathcal{P}F$, $\mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\gamma) = \mathrm{Ker}(\beta)$. Thus $|X\alpha| = |X\gamma| = |\mathrm{Dom}(\gamma)/\mathrm{Ker}(\gamma)| = |\mathrm{Dom}(\beta)/\mathrm{Ker}(\beta)| = |X\beta|$. On the other hand, if $\alpha \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$, then $\gamma \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$ and $\alpha = \gamma$, by Theorem 1.4. Since $\gamma \mathcal{R} \beta$, it follows that $\beta \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$, $\mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\gamma) = \mathrm{Ker}(\beta)$, by Corollary 1.6 and Theorem 1.5. Therefore, $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\gamma) = \mathrm{Ker}(\beta)$.

Conversely, assume that the conditions hold. If $\alpha, \beta \in \mathcal{P}F$ and $|X\alpha| = |X\beta|$, then there exists a bijection $\theta : X\beta \to X\alpha$. Let $\mu = \beta\theta$. Then $\mu \in \mathcal{PT}(X,Y), \mathrm{Dom}(\mu) = \mathrm{Dom}(\beta), X\mu = X\beta\theta = (X\beta)\theta = X\alpha$ and $X\mu = X\beta\theta = (Y\beta)\theta = Y\mu$, since $\beta \in \mathcal{P}F$. Hence $\mu \in \mathcal{P}F$ and $X\mu = X\alpha$ and so, by Theorem 1.4, we have $\alpha \mathcal{L} \mu$. Now, since $\mu = \beta\theta$ and θ is injective on $X\beta$, we also get $\mathrm{Ker}(\mu) = \mathrm{Ker}(\beta)$, whence $\mu \mathcal{R} \beta$, by Theorem 1.5. Therefore, $\alpha \mathcal{D} \beta$. On the other hand, if $\alpha, \beta \in \mathcal{PT}(X,Y) \setminus \mathcal{P}F$, $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$ then, by Theorem 1.5, it follows immediately that $\alpha \mathcal{R} \beta$, whence $\alpha \mathcal{D} \beta$, as required. $\square$

Lemma 1.8. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. If $\alpha = \lambda\beta\mu$, for some $\lambda \in \mathcal{PT}(X,Y)$ and $\mu \in \mathcal{PT}(X,Y)^1$, then $|X\alpha| \leq |Y\beta|$.*

Proof. Since $(X\lambda)\beta \subseteq Y\beta$, it follows that $|(X\lambda)\beta| \leq |Y\beta|$ and so $|X\alpha| = |(X\lambda\beta)\mu| \leq |(X\lambda)\beta| \leq |Y\beta|$, as required. $\square$

Theorem 1.9. *Let $\alpha, \beta \in \mathcal{PT}(X,Y)$. Then $\alpha \mathcal{J} \beta$ if and only if ($\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$) or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.*

Proof. Assume that $\alpha \mathcal{J} \beta$. Then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$, for some $\lambda, \lambda', \mu, \mu' \in \mathcal{PT}(X,Y)^1$. If $\lambda = 1 = \lambda'$, then $\alpha = \beta\mu$ and $\beta = \alpha\mu'$, which imply that $\alpha \mathcal{R} \beta$. Thus $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$. If either λ or λ' is in $\mathcal{PT}(X,Y)$, then we can write $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$, for some $\sigma, \sigma' \in \mathcal{PT}(X,Y)$ and $\delta, \delta' \in \mathcal{PT}(X,Y)^1$. Thus, by Lemma 1.8, it follows that $|Y\beta| \geq |X\alpha| \geq |Y\alpha| \geq |X\beta| \geq |Y\beta|$, whence $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.

Conversely, assume that the conditions hold. If $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$, then $\alpha \mathcal{R} \beta$ and so $\alpha \mathcal{J} \beta$. Now, suppose that $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. Write $\alpha = \binom{A_i}{a_i}$. Then, since $|X\alpha| = |Y\beta|$, we can write $\beta = \begin{pmatrix} B_i & B_j \\ b_i & b_j \end{pmatrix}$, where $B_i \cap Y \neq \emptyset$ and $B_j \cap Y = \emptyset$. Define $\lambda = \binom{A_i}{a_i'}$, where $a_i' \in B_i \cap Y$ and $\mu = \binom{b_i}{a_i}$. Thus $\lambda, \mu \in \mathcal{PT}(X,Y)$ and $\alpha = \lambda\beta\mu$. Similarly, we can show that $\beta = \lambda'\alpha\mu'$, for some $\lambda', \mu' \in \mathcal{PT}(X,Y)$, by using the equality $|X\beta| = |Y\alpha|$, as required. $\square$

Next, we aim to prove an isomorphism theorem for $\mathcal{PT}(X,Y)$.

Lemma 1.10. *If $\alpha \in \mathcal{PT}(X,Y) \setminus \{\emptyset\}$, then the following statements are equivalent:*

(1) $\alpha = \binom{a}{a}$, for some $a \in Y$;
(2) α is an idempotent and, for all $\beta \in \mathcal{PT}(X,Y)$, $(\alpha\beta)^2 = \emptyset$ or $(\alpha\beta)^2 = \alpha$.

Proof. Assume that (1) holds. Then it is clear that α is an idempotent. Let $\beta \in \mathcal{PT}(X,Y)$. Hence $\alpha\beta = \emptyset$, if $a \notin \mathrm{Dom}(\beta)$, and $\alpha\beta = \binom{a}{a\beta}$, if $a \in \mathrm{Dom}(\beta)$. Thus $(\alpha\beta)^2 = \emptyset$ or $(\alpha\beta)^2 = \binom{a}{a} = \alpha$.

Conversely, assume that (2) holds. Since $\alpha \neq \emptyset$, there exists $a \in X\alpha \subseteq Y$. Let $A = a\alpha^{-1}$. Since α is an idempotent, we have $a \in A$, whence $\left(\alpha\binom{a}{a}\right)^2 = \binom{A}{a}^2 = \binom{A}{a} \neq \emptyset$. Thus $\binom{A}{a} = \alpha$, by assumption. Now, let $b \in A$. Then $\left(\binom{A}{a}\binom{a}{b}\right)^2 = \binom{A}{b}^2 = \binom{A}{b} \neq \emptyset$. Again, by assumption, we have $\binom{A}{b} = \alpha = \binom{A}{a}$, whence $b = a$ and so $A = \{a\}$. Therefore $\alpha = \binom{a}{a}$, as required. $\square$

Theorem 1.11. *Let X be a finite set and Y_1 and Y_2 two nonempty subsets of X. Then $\mathcal{PT}(X,Y_1) \cong \mathcal{PT}(X,Y_2)$ if and only if $|Y_1| = |Y_2|$.*

Proof. Assume that $\mathcal{PT}(X,Y_1) \cong \mathcal{PT}(X,Y_2)$ and let $\Psi : \mathcal{PT}(X,Y_1) \to \mathcal{PT}(X,Y_2)$ be an isomorphism. Let

$$M_1 = \{\binom{a}{a} \mid a \in Y_1\} \quad \text{and} \quad M_2 = \{\binom{b}{b} \mid b \in Y_2\}.$$

Then $|M_1| = |Y_1|$ and $|M_2| = |Y_2|$. For each $\alpha \in M_1$, we have that α is an idempotent and $(\alpha\beta)^2 = \emptyset$ or $(\alpha\beta)^2 = \alpha$, for all $\beta \in \mathcal{PT}(X,Y_1)$, by Lemma 1.10. Hence, $\alpha\Psi$ is an idempotent and we have

$$(\alpha\Psi\gamma)^2 = (\alpha\Psi\beta\Psi)^2 = (\alpha\beta)^2\Psi = \begin{cases} \emptyset\Psi = \emptyset \\ \alpha\Psi\ , \end{cases}$$

for all $\gamma = \beta\Psi \in \mathcal{PT}(X,Y_2)$. Thus, $\alpha\Psi = \binom{b}{b}$, for some $b \in Y_2$, and so $M_1\Psi \subseteq M_2$. Since Ψ^{-1} is an isomorphism from $\mathcal{PT}(X,Y_2)$ onto $\mathcal{PT}(X,Y_1)$, similarly, we also get $M_2\Psi^{-1} \subseteq M_1$. Thus $|M_1| = |M_1\Psi| \leq |M_2| = |M_2\Psi^{-1}| \leq |M_1|$ and so $|M_1| = |M_2|$. Therefore, $|Y_1| = |M_1| = |M_2| = |Y_2|$.

Conversely, assume that $|Y_1| = |Y_2|$. Then there exists a bijection $\theta_1 : Y_1 \to Y_2$. Since X is finite, we get $|X \backslash Y_1| = |X \backslash Y_2|$ and so there exists a bijection $\theta_2 : X\backslash Y_1 \to X\backslash Y_2$. Let $\theta = \theta_1 \cup \theta_2$. Clearly, $\theta : X \to X$ is also a bijection. Now, define $\Phi : \mathcal{PT}(X,Y_1) \to \mathcal{PT}(X,Y_2)$ by $\alpha\Phi = \theta^{-1}\alpha\theta$, for all $\alpha \in \mathcal{PT}(X,Y_1)$. It is a routine matter to show that Φ is an isomorphism. Therefore, $\mathcal{PT}(X,Y_1) \cong \mathcal{PT}(X,Y_2)$, as required. □

2. The ranks of $\mathcal{T}_{n,r}$ and $\mathcal{PT}_{n,r}$

Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $1 \leq r \leq n-1$.

Notice that, clearly, $|\mathcal{T}_{n,r}| = r^n$. On the other hand, by considering the natural embedding of $\mathcal{PT}_n$ into $\mathcal{T}_{n+1}$, it is easy to show that $|\mathcal{PT}_{n,r}| = (r+1)^n$.

Recall that two elements in $\mathcal{T}_{n,r}$ or in $\mathcal{PT}_{n,r}$ are $\mathcal{R}$-related if and only if they have the same kernel. Thus, we may easily deduce that $\mathcal{T}_{n,r}$ and $\mathcal{PT}_{n,r}$ have $S(n,r)$ and $S(n+1,r+1)$, respectively, distinct $\mathcal{R}$-classes of maximum rank r (as many as the number of possible distinct kernels of size r). As $\mathrm{Ker}(\alpha) \cap (\mathrm{Dom}(\alpha\beta) \times \mathrm{Dom}(\alpha\beta)) \subseteq \mathrm{Ker}(\alpha\beta)$ and $\mathrm{Dom}(\alpha\beta) \subseteq \mathrm{Dom}(\alpha)$, for all $\alpha, \beta \in \mathcal{PT}_n$, it is easy to show that any generating set of $\mathcal{T}_{n,r}$ must contain at least one element from each of the $S(n,r)$ distinct $\mathcal{R}$-classes of $\mathcal{T}_{n,r}$ of rank r and, similarly, any generating set of $\mathcal{PT}_{n,r}$ must contain at least one element from each of the $S(n+1,r+1)$ distinct $\mathcal{R}$-classes of $\mathcal{PT}_{n,r}$ of rank r. Hence $\mathrm{rank}(\mathcal{T}_{n,r}) \geq S(n,r)$ and $\mathrm{rank}(\mathcal{PT}_{n,r}) \geq S(n+1,r+1)$.

On the other hand, it is clear that each $\mathcal{R}$-class of rank r of $\mathcal{T}_{n,r}$ or of $\mathcal{PT}_{n,r}$ has $r!$ elements (notice that the image of all such elements is precisely $\{1,\dots,r\}$).

Recall that

$$F = \{\alpha \in \mathcal{T}_{n,r} \mid \{1,\dots,n\}\alpha = \{1,\dots,r\}\alpha\}$$

is the largest regular subsemigroup of $\mathcal{T}_{n,r}$ and, moreover, it is the set of all regular elements of $\mathcal{T}_{n,r}$ (see [8, Proof of Theorem 2.4]). As $\{1,\dots,r\}\alpha = \{1,\dots,r\}$, for any element $\alpha \in F$ of rank r, then any two $\mathcal{R}$-related elements of F that coincide in $\{1,\dots,r\}$ must be equal. Therefore, we have r^{n-r} regular $\mathcal{R}$-classes of $\mathcal{T}_{n,r}$ of rank r (as many as the number of distinct functions from $\{r+1,\dots,n\}$ into $\{1,\dots,r\}$), each one being also an $\mathcal{H}$-class and so a subgroup isomorphic to $\mathcal{S}_r$, the symmetric group on $\{1,\dots,r\}$. The remaining $S(n,r) - r^{n-r}$ (non-regular) $\mathcal{R}$-classes of $\mathcal{T}_{n,r}$ of rank r must have $r!$ trivial $\mathcal{H}$-classes.

Regarding the semigroup $\mathcal{PT}_{n,r}$, its largest regular subsemigroup is

$$\mathcal{P}F = \{\alpha \in \mathcal{PT}_{n,r} \mid \{1,\dots,n\}\alpha = \{1,\dots,r\}\alpha\},$$

which also coincides with the set of all regular elements of $\mathcal{PT}_{n,r}$ (see Theorem 1.2). Let α be an element of $\mathcal{P}F$ of rank r. Then $\{1,\dots,r\} = \{1,\dots,n\}\alpha = \{1,\dots,r\}\alpha$ and so $\{1,\dots,r\} \subseteq \mathrm{Dom}(\alpha)$. It follows also that any two $\mathcal{R}$-related elements of $\mathcal{P}F$ that coincide in $\{1,\dots,r\}$ must be equal, whence the number of regular $\mathcal{R}$-classes of $\mathcal{PT}_{n,r}$ of rank r is precisely $(r+1)^{n-r}$ (i.e. the number of distinct partial functions from $\{r+1,\dots,n\}$ into $\{1,\dots,r\}$). Each of these regular $\mathcal{R}$-classes is also a $\mathcal{H}$-class and so a subgroup isomorphic to $\mathcal{S}_r$. The remaining $S(n+1,r+1) - (r+1)^{n-r}$ (non-regular) $\mathcal{R}$-classes of $\mathcal{PT}_{n,r}$ of rank r must contain $r!$ trivial $\mathcal{H}$-classes.

As $\mathcal{T}_{n,1} = \{\binom{12\cdots n}{11\cdots 1}\}$, it is clear that $\mathcal{T}_{n,1}$ has rank equal to $1 = S(n,1)$. On the other hand, the semigroup $\mathcal{PT}_{n,1}$ has precisely $S(n+1,2)$ non-zero elements, which must all be contained in any generating set, by the above observation. It follows that $S(n+1,2) = \mathrm{rank}(\mathcal{PT}_{n,1})$.

From now on, we consider $n \geq 3$ and $2 \leq r \leq n-1$.

Lemma 2.1. *The semigroups $\mathcal{T}_{n,r}$ and $\mathcal{PT}_{n,r}$ are generated by their elements of rank r.*

Proof. Let $\alpha \in \mathcal{PT}_{n,r}$ be an element of rank k, for some $1 \leq k < r$. Let $y_0 \in \{1,\ldots,r\} \setminus \mathrm{Im}(\alpha)$. We will consider two cases. Suppose that α is a full transformation. Hence, we may take $x_0 \in \mathrm{Dom}(\alpha) = \{1,\ldots,n\}$ such that x_0 belongs to a non-trivial kernel class of α. Define $\alpha_1, \alpha_2 \in \mathcal{T}_{n,r}$ by

$$x\alpha_1 = \begin{cases} y_0 & \text{if } x = x_0 \\ x\alpha & \text{otherwise} \end{cases} \quad \text{and} \quad x\alpha_2 = \begin{cases} x & \text{if } x \in \mathrm{Im}(\alpha) \\ x_0\alpha & \text{if } x = y_0 \\ y_0 & \text{otherwise}\ , \end{cases}$$

for all $x \in \{1,\ldots,n\}$. Now, assume that α is a strict partial transformation and let $x_0 \in \{1,\ldots,n\} \setminus \mathrm{Dom}(\alpha)$. Define $\alpha_1, \alpha_2 \in \mathcal{PT}_{n,r}$, with $\mathrm{Dom}(\alpha_1) = \mathrm{Dom}(\alpha) \cup \{x_0\}$ and $\mathrm{Dom}(\alpha_2) = \{1,\ldots,n\} \setminus \{y_0\}$, by

$$x\alpha_1 = \begin{cases} y_0 & \text{if } x = x_0 \\ x\alpha & \text{otherwise} \end{cases} \quad \text{and} \quad x\alpha_2 = \begin{cases} x & \text{if } x \in \mathrm{Im}(\alpha) \\ y_0 & \text{otherwise.} \end{cases}$$

For both cases, we have $\alpha = \alpha_1\alpha_2$. Moreover, $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\alpha_2) = \mathrm{Im}(\alpha) \cup \{y_0\}$ and so $\mathrm{rank}(\alpha_1) = \mathrm{rank}(\alpha_2) = k+1$. This suffices to prove the lemma. ☐

Lemma 2.2. *Let $\alpha, \gamma \in \mathcal{PT}_n$ be such that $\mathrm{Im}(\gamma) \subseteq \mathrm{Im}(\alpha)$ and $\alpha = \alpha\gamma$. Then $\gamma = \gamma^2$.*

Proof. Let $x \in \mathrm{Dom}(\gamma)$. Then $x\gamma \in \mathrm{Im}(\gamma) \subseteq \mathrm{Im}(\alpha)$ and so $x\gamma = a\alpha$, for some $a \in \mathrm{Dom}(\alpha)$. Since $a \in \mathrm{Dom}(\alpha)$, it follows that $a \in \mathrm{Dom}(\alpha\gamma)$, whence $x\gamma = a\alpha \in \mathrm{Dom}(\gamma)$ and $x\gamma^2 = a\alpha\gamma = a\alpha = x\gamma$, as required. ☐

Now, let

$$\varepsilon = \left(\begin{array}{cccc|ccc} 1 & 2 & \cdots & r & r+1 & \cdots & n \\ 1 & 2 & \cdots & r & r & \cdots & r \end{array}\right) \in \mathcal{T}_{n,r}$$

(notice that we can also consider that $\varepsilon \in \mathcal{PT}_{n,r}$). As ε is an idempotent of rank r, then its $\mathcal{R}$-class R_ε is a subgroup of $\mathcal{T}_{n,r}$ (respectively, of $\mathcal{PT}_{n,r}$) with identity ε. Let $\alpha \in \mathcal{T}_{n,r}$ (respectively, $\alpha \in \mathcal{PT}_{n,r}$) be any element of rank r. Then, clearly, $\alpha\varepsilon = \alpha$. Moreover, being $\gamma \in R_\varepsilon$, we have $\gamma^t = \varepsilon$, for some $t \in \mathbb{N}$ (we can take $t = r!$), whence $\alpha = \alpha\varepsilon = \alpha\gamma^t = (\alpha\gamma)\gamma^{t-1}$ and so $\alpha\gamma\mathcal{R}\alpha$. On the other hand, being $\gamma_1, \gamma_2 \in R_\varepsilon$ such that $\alpha\gamma_1 = \alpha\gamma_2$, we have $\alpha = \alpha\varepsilon = \alpha\gamma_1\gamma_1^{-1} = \alpha\gamma_2\gamma_1^{-1}$, with γ_1^{-1} the group inverse of γ_1 taken in R_ε, and so, by Lemma 2.2 (notice that $\mathrm{Im}(\gamma_2\gamma_1^{-1}) = \{1,\ldots,r\} = \mathrm{Im}(\alpha)$), the transformation $\gamma_2\gamma_1^{-1}$ is an idempotent of $\mathcal{T}_{n,r}$ (and of $\mathcal{PT}_{n,r}$). Then, $\gamma_2\gamma_1^{-1} = \varepsilon$ and so $\gamma_1 = \gamma_2$. It follows that the mapping $\gamma \mapsto \alpha\gamma$ from R_ε to R_α is injective. Furthermore, as $|R_\varepsilon| = |R_\alpha| (= r!)$, we obtain $R_\alpha = \alpha R_\varepsilon$. This fact and Lemma 2.1 allow us to conclude that the subset A of $\mathcal{T}_{n,r}$ (respectively, of $\mathcal{PT}_{n,r}$), consisting of R_ε together with a single arbitrary element from each remaining $\mathcal{R}$-class of rank r of $\mathcal{T}_{n,r}$ (respectively, of $\mathcal{PT}_{n,r}$), forms a generating set of $\mathcal{T}_{n,r}$ (respectively, of $\mathcal{PT}_{n,r}$).

Suppose that $r = 2$. Clearly, in this case,

$$R_\varepsilon = \left\{\varepsilon, \varepsilon_b = \left(\begin{array}{cc|ccc} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 1 & \cdots & 1 \end{array}\right)\right\}$$

and so the group R_ε is generated by ε_b. Hence, the set $B = A \setminus \{\varepsilon\} = (A \setminus R_\varepsilon) \cup \{\varepsilon_b\}$ is a generating of $\mathcal{T}_{n,2}$ (respectively, of $\mathcal{PT}_{n,2}$) with $S(n,2)$ (respectively, $S(n+1,3)$) elements.

Finally, admit that $r \geq 3$. It is well known that the symmetric group $\mathcal{S}_r$ can be generated by two elements, in particular, by the transposition $a = (1\,2)$ together with the r-cycle $b = (1\,2\,\cdots\,r)$. The corresponding elements in R_ε (recall that R_ε is a group isomorphic to $\mathcal{S}_r$) are

$$\varepsilon_a = \left(\begin{array}{ccccc|ccc} 1 & 2 & 3 & \cdots & r & r+1 & \cdots & n \\ 2 & 1 & 3 & \cdots & r & r & \cdots & r \end{array}\right) \quad \text{and} \quad \varepsilon_b = \left(\begin{array}{ccccc|ccc} 1 & 2 & \cdots & r-1 & r & r+1 & \cdots & n \\ 2 & 3 & \cdots & r & 1 & 1 & \cdots & 1 \end{array}\right),$$

respectively, and so $\{\varepsilon_a, \varepsilon_b\}$ generates R_ε. Consider also

$$\varepsilon_b' = \left(\begin{array}{ccccc|ccc} 1 & 2 & \cdots & r-1 & r & r+1 & \cdots & n \\ 2 & 3 & \cdots & r & 1 & r & \cdots & r \end{array}\right) \in \mathcal{T}_{n,r}$$

(notice that we can also consider that $\varepsilon_b' \in \mathcal{PT}_{n,r}$). Then, ε_b' has rank r, $\varepsilon_b' \notin R_\varepsilon$ and $\varepsilon_b = \varepsilon\varepsilon_b' = \varepsilon_a^2\varepsilon_b'$. Hence, being ε' the (unique) element of A such that $\varepsilon'\mathcal{R}\varepsilon_b'$, the set $B = (A \setminus (R_\varepsilon \cup \{\varepsilon'\})) \cup \{\varepsilon_a, \varepsilon_b'\}$ is a generating of $\mathcal{T}_{n,r}$ (respectively, of $\mathcal{PT}_{n,r}$) with $S(n,r)$ (respectively, $S(n+1,r+1)$) elements.

Thus, we proved:

Theorem 2.3. *For $n \geq 2$ and $1 \leq r \leq n-1$, the semigroup $\mathcal{T}_{n,r}$ has rank equal to $S(n,r)$.*

And, respectively:

Theorem 2.4. *For $n \geq 2$ and $1 \leq r \leq n-1$, the semigroup $\mathcal{PT}_{n,r}$ has rank equal to $S(n+1,r+1)$.*

3. On the semigroup $\mathcal{I}(X,Y)$

In general $\mathcal{I}(X,Y) = \mathcal{PT}(X,Y) \cap \mathcal{I}(X)$ is not an inverse semigroup. In fact, it is not even regular. For example, let $X = \{1,2,3,4,5\}, Y = \{1,2,3\}$ and $\alpha = \binom{1\,2\,5}{1\,2\,3} \in \mathcal{I}(X,Y)$. Suppose that α is regular in $\mathcal{I}(X,Y)$. Then, there exists $\beta \in \mathcal{I}(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Thus $5\alpha = (5\alpha)\beta\alpha = (3\beta)\alpha$, which implies that $3\beta = 5 \notin Y$ (since α is injective) and this leads to a contradiction.

Observe that the symmetric inverse semigroup $\mathcal{I}(Y)$ on the set Y may be considered as an inverse subsemigroup of $\mathcal{I}(X,Y)$. On the other hand, let α be a regular element of $\mathcal{I}(X,Y)$ and take $\beta \in \mathcal{I}(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Suppose that $\alpha = \binom{x_i}{a_i}$, where $a_i \in Y$, for all i. Then $x_i\alpha = (x_i\alpha)\beta\alpha = (a_i\beta)\alpha$ and so $x_i = a_i\beta \in Y$, since α is injective. Hence $\mathrm{Dom}(\alpha) \subseteq Y$, that is $\alpha \in \mathcal{I}(Y)$. It follows that $\mathcal{I}(Y)$ is the set of all regular elements of $\mathcal{I}(X,Y)$, from which we immediately deduce the following result.

Theorem 3.1. *The symmetric inverse semigroup $\mathcal{I}(Y)$ is the largest regular subsemigroup of $\mathcal{I}(X,Y)$. In particular, $\mathcal{I}(Y)$ is the largest inverse subsemigroup of $\mathcal{I}(X,Y)$.*

Now, we establish the Green's relations of $\mathcal{I}(X,Y)$ by beginning with the following lemma.

Lemma 3.2. *Let $\alpha,\beta \in \mathcal{I}(X,Y)$. If $\beta \in \mathcal{I}(Y)$, then $X\alpha \subseteq X\beta$ if and only if $\alpha = \gamma\beta$, for some $\gamma \in \mathcal{I}(X,Y)$.*

Proof. Assume that $\beta \in \mathcal{I}(Y)$. If $X\alpha \subseteq X\beta$, then we can write $\alpha = \binom{x_i}{a_i}$ and $\beta = \binom{y_i\ y_j}{a_i\ a_j}$, with $a_i, a_j, y_i, y_j \in Y$. Hence, being $\gamma = \binom{x_i}{y_i} \in \mathcal{I}(X,Y)$, we have $\alpha = \gamma\beta$. The converse is clear. □

Theorem 3.3. *Let $\alpha,\beta \in \mathcal{I}(X,Y)$. Then $\alpha\mathcal{L}\beta$ on $\mathcal{I}(X,Y)$ if and only if ($\alpha,\beta \in \mathcal{I}(Y)$ and $X\alpha = X\beta$) or ($\alpha,\beta \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\alpha = \beta$).*

Proof. Assume that $\alpha\mathcal{L}\beta$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, for some $\lambda,\mu \in \mathcal{I}(X,Y)^1$. If $\alpha \in \mathcal{I}(Y)$ and ($\lambda = 1$ or $\mu = 1$), then $\beta = \alpha \in \mathcal{I}(Y)$ and so $X\alpha = X\beta$. On the other hand, suppose that $\alpha \in \mathcal{I}(Y)$ and $\lambda,\mu \in \mathcal{I}(X,Y)$. Let $x \in \mathrm{Dom}(\beta)$. Then, $x\beta = (x\mu\lambda)\beta$ and so $x = x\mu\lambda \in Y$, since β is injective. Thus $\mathrm{Dom}(\beta) \subseteq Y$, that is $\beta \in \mathcal{I}(Y)$. From $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, by Lemma 3.2, we deduce that $X\alpha = X\beta$. Now, suppose that $\alpha \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$. If $\lambda,\mu \in \mathcal{I}(X,Y)$, then $x\alpha = x\lambda\beta = x\lambda\mu\alpha$ and thus $x = x\lambda\mu \in Y$, for all $x \in \mathrm{Dom}(\alpha)$, since α is injective. Hence $\mathrm{Dom}(\alpha) \subseteq Y$, that is $\alpha \in \mathcal{I}(Y)$, which is a contradiction. Therefore $\lambda = 1$ or $\mu = 1$ and so $\beta = \alpha \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$.

The converse is clear by Lemma 3.2. □

Theorem 3.4. *Let $\alpha,\beta \in \mathcal{I}(X,Y)$. Then $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(\beta)$ if and only if $\alpha = \beta\gamma$, for some $\gamma \in \mathcal{I}(X,Y)$. Moreover, $\alpha\mathcal{R}\beta$ on $\mathcal{I}(X,Y)$ if and only if $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$.*

Proof. If $\alpha = \beta\gamma$, for some $\gamma \in \mathcal{I}(X,Y)$, then clearly $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(\beta)$. Conversely, suppose that $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(\beta)$. Then, we can write $\alpha = \binom{x_i}{a_i}$ and $\beta = \binom{x_i\ x_j}{b_i\ b_j}$, where $\{a_i, b_i, b_j\} \subseteq Y$. Now, being $\gamma = \binom{b_i}{a_i} \in \mathcal{I}(X,Y)$, we have $\alpha = \beta\gamma$, as required. □

From the previous theorem, it follows that, if $\alpha\mathcal{R}\beta$ on $\mathcal{I}(X,Y)$ and $\alpha \in \mathcal{I}(Y)$, then $\mathrm{Dom}(\beta) = \mathrm{Dom}(\alpha) \subseteq Y$, which implies that $\beta \in \mathcal{I}(Y)$. On the other hand, if $\alpha \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\alpha\mathcal{R}\beta$ on $\mathcal{I}(X,Y)$, then $\mathrm{Dom}(\beta) = \mathrm{Dom}(\alpha) \nsubseteq Y$ and so $\beta \notin \mathcal{I}(Y)$. Thus, we have the following corollary.

Corollary 3.5. *Let $\alpha,\beta \in \mathcal{I}(X,Y)$. If $\alpha\mathcal{R}\beta$ on $\mathcal{I}(X,Y)$, then $\alpha,\beta \in \mathcal{I}(Y)$ or $\alpha,\beta \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$.*

Theorem 3.6. *Let $\alpha,\beta \in \mathcal{I}(X,Y)$. Then $\alpha\mathcal{D}\beta$ on $\mathcal{I}(X,Y)$ if and only if ($\alpha,\beta \in \mathcal{I}(Y)$ and $|\mathrm{Dom}(\alpha)| = |\mathrm{Dom}(\beta)|$) or ($\alpha,\beta \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$).*

Proof. Assume that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$, for some $\gamma \in \mathcal{I}(X,Y)$. Since $\alpha\mathcal{L}\gamma$, if $\alpha \in \mathcal{I}(Y)$, then $\gamma \in \mathcal{I}(Y)$ and $X\alpha = X\gamma$, by Theorem 3.3. Furthermore, from $\gamma\mathcal{R}\beta$, it follows that $\beta \in \mathcal{I}(Y)$ and $\mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$, by Corollary 3.5 and Theorem 3.4. Hence $|\mathrm{Dom}(\alpha)| = |X\alpha| = |X\gamma| = |\mathrm{Dom}(\gamma)| = |\mathrm{Dom}(\beta)|$. On the other hand, if $\alpha \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$, then $\gamma \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\alpha = \gamma$, by Theorem 3.3. It follows that $\beta \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$, whence $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$.

Conversely, assume that the conditions hold. If $\alpha, \beta \in \mathcal{I}(Y)$ and $|\mathrm{Dom}(\alpha)| = |\mathrm{Dom}(\beta)|$, then we can write $\alpha = \binom{x_i}{a_i}$ and $\beta = \binom{y_i}{b_i}$, where $\{a_i, b_i, x_i, y_i\} \subseteq Y$. Hence, being $\gamma = \binom{y_i}{a_i} \in \mathcal{I}(Y)$, we have $X\alpha = X\gamma$ and $\mathrm{Dom}(\gamma) = \mathrm{Dom}(\beta)$, which implies that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$. On the other hand, if $\alpha, \beta \in \mathcal{I}(X,Y) \setminus \mathcal{I}(Y)$ and $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta))$, then $\alpha\mathcal{R}\beta$ and so $\alpha\mathcal{D}\beta$, as required. ☐

Theorem 3.7. *Let $\alpha, \beta \in \mathcal{I}(X,Y)$. Then $\alpha\mathcal{J}\beta$ on $\mathcal{I}(X,Y)$ if and only if $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.*

Proof. Assume that $\alpha\mathcal{J}\beta$ on $\mathcal{I}(X,Y)$. Then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$, for some $\lambda, \lambda', \mu, \mu' \in \mathcal{I}(X,Y)^1$. If $\lambda = 1 = \lambda'$, then $\alpha = \beta\mu$ and $\beta = \alpha\mu'$ and so $\alpha\mathcal{R}\beta$. Thus $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$. If either λ or λ' belongs to $\mathcal{I}(X,Y)$, then $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$, for some $\sigma, \sigma' \in \mathcal{I}(X,Y)$ and $\delta, \delta' \in \mathcal{I}(X,Y)^1$. Hence, by Lemma 1.8, we have $|Y\beta| \geq |X\alpha| \geq |Y\alpha| \geq |X\beta| \geq |Y\beta|$, whence $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.

Conversely, assume that the conditions hold. First, if $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$, then $\alpha\mathcal{R}\beta$, whence $\alpha\mathcal{J}\beta$. Secondly, if $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$, then we can write $\alpha = \binom{x_i}{a_i}$ and $\beta = \binom{y_i\ y_j}{b_i\ b_j}$, where $\{y_i\} \subseteq Y$ and $\{y_j\} \subseteq X \setminus Y$, since $|X\alpha| = |Y\beta|$. Now, define $\lambda = \binom{x_i}{y_i}$ and $\mu = \binom{b_i}{a_i}$. Thus $\lambda, \mu \in \mathcal{I}(X,Y)$ and $\alpha = \lambda\beta\mu$. Similarly, by using the equality $|X\beta| = |Y\alpha|$, we can find $\lambda', \mu' \in \mathcal{I}(X,Y)$ such that $\beta = \lambda'\alpha\mu'$. Therefore, $\alpha\mathcal{J}\beta$, as required. ☐

Next, we prove an isomorphism theorem for $\mathcal{I}(X,Y)$.

First, we characterize the idempotents in $\mathcal{I}(X,Y)$ and its 0-minimal idempotents.

Denote by id_A the identity map on the set A.

As the idempotents of $\mathcal{I}(X)$ are all the maps of the form id_A, with $A \subseteq X$, it immediately follows:

Lemma 3.8. *The idempotents of $\mathcal{I}(X,Y)$ are precisely the elements of the set $\{\mathrm{id}_A \mid A \subseteq Y\}$.*

Regarding the 0-minimal idempotents, we have:

Lemma 3.9. *$M = \{\binom{a}{a} \mid a \in Y\}$ is the set of all 0-minimal idempotents of $\mathcal{I}(X,Y)$.*

Proof. Let $e = \binom{a}{a} \in M$ and let α be an idempotent of $\mathcal{I}(X,Y)$ such that $\alpha \leq e$, that is $\alpha = e\alpha = \alpha e$. Then $\mathrm{Dom}(\alpha) \subseteq \mathrm{Dom}(e) = \{a\}$ and $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(e) = \{a\}$, whence $\mathrm{Dom}(\alpha) = \mathrm{Im}(\alpha) = \{a\}$ and so $\alpha = e$.

On the other hand, let β be a 0-minimal idempotent of $\mathcal{I}(X,Y)$. Suppose that $\beta = \binom{y_k}{y_k}$, where $\{y_k\} \subseteq Y$. Choosing $y_0 \in \{y_k\}$ and considering $\gamma = \binom{y_0}{y_0} \in M$, we have $\binom{y_0}{y_0}\binom{y_k}{y_k} = \binom{y_0}{y_0} = \binom{y_k}{y_k}\binom{y_0}{y_0}$, which implies that $\gamma \leq \beta$. Since β is 0-minimal, we get $\beta = \gamma \in M$, as required. ☐

Now, we can prove the following result.

Theorem 3.10. *Let X be a finite set and Y_1 and Y_2 two non-empty subsets of X. Then $\mathcal{I}(X,Y_1) \cong \mathcal{I}(X,Y_2)$ if and only if $|Y_1| = |Y_2|$.*

Proof. First, assume that $|Y_1| = |Y_2|$. Let us consider the map Φ given in Theorem 1.11. Notice that, we have $\mathcal{I}(X,Y_1) \subseteq \mathcal{PT}(X,Y_1)$ and $\mathcal{I}(X,Y_2) \subseteq \mathcal{PT}(X,Y_2)$. Moreover, it is clear that $\Phi|_{\mathcal{I}(X,Y_1)}$ is a bijection from $\mathcal{I}(X,Y_1)$ onto $\mathcal{I}(X,Y_2)$. Thus $\mathcal{I}(X,Y_1) \cong \mathcal{I}(X,Y_2)$.

On the other hand, suppose that $\mathcal{I}(X,Y_1) \cong \mathcal{I}(X,Y_2)$ and take an isomorphism $\Psi : \mathcal{I}(X,Y_1) \to \mathcal{I}(X,Y_2)$. Let

$$M_1 = \{\tbinom{a}{a} \mid a \in Y_1\} \quad \text{and} \quad M_2 = \{\tbinom{b}{b} \mid b \in Y_2\}.$$

Then, since by the above lemma M_1 and M_2 are the sets of all 0-minimal idempotents in $\mathcal{I}(X,Y_1)$ and $\mathcal{I}(X,Y_2)$, respectively, we deduce that $M_1\Psi = M_2$. Thus, it follows that $|Y_1| = |M_1| = |M_1\Psi| = |M_2| = |Y_2|$, as required. ☐

4. THE RANK OF $\mathcal{I}_{n,r}$

Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $1 \leq r \leq n-1$.

It is easy to show that $|\mathcal{I}_{n,r}| = \sum_{k=0}^{r} \binom{n}{k}\binom{r}{k} k!$.

Next, we recall that two elements in $\mathcal{I}_{n,r}$ are $\mathcal{R}$-related if and only if they have the same domain. Thus, we may easily deduce that $\mathcal{I}_{n,r}$ has precisely $\binom{n}{r}$ distinct $\mathcal{R}$-classes of maximum rank r (as many as the number of possible distinct domains of size r). As $\mathrm{Dom}(\alpha\beta) \subseteq \mathrm{Dom}(\alpha)$, for all $\alpha, \beta \in \mathcal{PT}_n$, it is easy to show that any generating set of $\mathcal{I}_{n,r}$ contains at least one element from each of the $\binom{n}{r}$ distinct $\mathcal{R}$-classes of $\mathcal{I}_{n,r}$ of rank r. Hence $\mathrm{rank}(\mathcal{I}_{n,r}) \geq \binom{n}{r}$.

On the other hand, it is also clear that each $\mathcal{R}$-class of rank r of $\mathcal{I}_{n,r}$ has $r!$ elements (notice that the image of all such elements is precisely $\{1, \ldots, r\}$).

Since the set of all regular elements of $\mathcal{I}_{n,r}$ coincides with the symmetric inverse semigroup $\mathcal{I}_r$ on $\{1, \ldots, r\}$ (see Theorem 3.1), then $\mathcal{I}_{n,r}$ has a unique regular $\mathcal{R}$-class of rank r, which is also a $\mathcal{H}$-class and, in fact, it is precisely the symmetric group $\mathcal{S}_r$. The remaining $\binom{n}{r} - 1$ (non-regular) $\mathcal{R}$-classes of $\mathcal{I}_{n,r}$ of rank r must contain $r!$ trivial $\mathcal{H}$-classes.

As $\mathcal{I}_{n,1} = \left\{\emptyset, \binom{1}{1}, \ldots, \binom{n}{1}\right\}$, it is clear that $\mathcal{I}_{n,1}$ has rank equal to n.

Next, we consider $n \geq 3$ and $2 \leq r \leq n-1$.

As for $\mathcal{T}_{n,r}$ and $\mathcal{PT}_{n,r}$, we have:

Lemma 4.1. *The semigroup $\mathcal{I}_{n,r}$ is generated by its elements of rank r.*

Proof. Let $\alpha \in \mathcal{I}_{n,r}$ be an element of rank k, for some $1 \leq k < r$. Let $y_0 \in \{1, \ldots, r\} \setminus \mathrm{Im}(\alpha)$ and $x_0 \in \{1, \ldots, n\} \setminus \mathrm{Dom}(\alpha)$. Define $\alpha_1, \alpha_2 \in \mathcal{I}_{n,r}$, with $\mathrm{Dom}(\alpha_1) = \mathrm{Dom}(\alpha) \cup \{x_0\}$ and $\mathrm{Dom}(\alpha_2) = \mathrm{Im}(\alpha) \cup \{n\}$, by

$$x\alpha_1 = \begin{cases} x\alpha & \text{if } x \in \mathrm{Dom}(\alpha) \\ y_0 & \text{if } x = x_0 \end{cases} \qquad \text{and} \qquad x\alpha_2 = \begin{cases} x & \text{if } x \in \mathrm{Im}(\alpha) \\ y_0 & \text{if } x = n. \end{cases}$$

Then, we have $\alpha = \alpha_1\alpha_2$. Moreover, $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\alpha_2) = \mathrm{Im}(\alpha) \cup \{y_0\}$ and so $\mathrm{rank}(\alpha_1) = \mathrm{rank}(\alpha_2) = k+1$. The lemma follows by induction on k. □

Next, observe that, for each element $\alpha \in \mathcal{I}_{n,r}$ of rank r, we clearly have $R_\alpha = \alpha\mathcal{I}_r$. Hence, in view of Lemma 4.1, any subset A of $\mathcal{I}_{n,r}$ consisting of a generating set of $\mathcal{I}_r$ together with a single arbitrary element from each remaining $\mathcal{R}$-class of rank r of $\mathcal{I}_{n,r}$ forms a generating set of $\mathcal{I}_{n,r}$.

On the other hand, notice that, given elements $\alpha_1, \ldots, \alpha_k \in \mathcal{I}_{n,r}$ such that the product $\alpha_1 \cdots \alpha_k$ has rank r, then all the transformations $\alpha_1, \ldots, \alpha_k$ have rank r and $\mathrm{Im}(\alpha_i) = \mathrm{Dom}(\alpha_{i+1})$, for $1 \leq i \leq k-1$. In particular, it follows that, if $\alpha_1 \cdots \alpha_k \in \mathcal{I}_r$ then $\alpha_1, \ldots, \alpha_k \in \mathcal{I}_r$. Therefore, any generating set of $\mathcal{I}_{n,r}$ must contain a generating set of $\mathcal{I}_r$.

Now, if $r = 2$ then $\mathcal{I}_2 = \left\{\binom{12}{12}, \binom{12}{21}\right\}$ and so $\mathcal{I}_2$ is generated by $\binom{12}{21}$. Hence, we may find a generating set A of $\mathcal{I}_{n,2}$ with $\binom{n}{2}$ elements. For $r \geq 3$, as recalled in Section 2, the symmetric group $\mathcal{S}_r$ has rank two and so we may find a generating set A of $\mathcal{I}_{n,r}$ with and no less than $\binom{n}{2} + 1$ elements.

Thus, we have proved:

Theorem 4.2. *Let $n \geq 2$. For $r \in \{1, 2\}$, the semigroup $\mathcal{I}_{n,r}$ has rank equal to $\binom{n}{r}$. For $3 \leq r \leq n-1$, the semigroup $\mathcal{I}_{n,r}$ has rank equal to $\binom{n}{r} + 1$.*

5. SOME RELATED PROBLEMS

In [10] Sullivan considered the semigroup $\mathcal{T}(V, W)$, the linear counterpart of $\mathcal{T}(X, Y)$, and described its Green's relations and ideals. In turn, the Green's relations of the linear counterparts $\mathcal{PT}(V, W)$ and $\mathcal{I}(V, W)$ of $\mathcal{PT}(X, Y)$ and $\mathcal{I}(X, Y)$, respectively, were determined by Sangkhanan and Sanwong. For a finite dimensional vector space V, it is then natural to ask for the ranks of $\mathcal{T}(V, W)$, $\mathcal{PT}(V, W)$ and $\mathcal{I}(V, W)$, which are open questions.

On the other hand, as the notions of order-preserving transformation and orientation-preserving transformation have been widely considered for several classes of transformation semigroups, it is also natural to consider the subsemigroups $\mathcal{O}(X, Y)$ and $\mathcal{OP}(X, Y)$ of $\mathcal{T}(X, Y)$ which consist of all order-preserving transformations and of all orientation-preserving transformations, respectively. In particular, for a finite set X, we may ask for their ranks. Regarding this problem, for $X = \{1, \ldots, n\}$ and $Y = \{1, \ldots, r\}$ $(2 \leq r \leq n-1)$, Fernandes and Quinteiro showed that $\mathcal{OP}_{n,r} = \mathcal{OP}(X,Y)$ has rank equal to $\binom{n}{r}$. All the other cases remain as open problems.

REFERENCES

[1] G.U. Garba, Idempotents in partial transformation semigroups, *Proc. Roy. Soc. Edinburgh A* **116** (1990), 359–366.

[2] G.M.S. Gomes and J.M. Howie, On the ranks of certain finite semigroups of transformations, *Math. Proc. Cambridge Phil. Soc.* **101** (1987) 395–403.

[3] J.M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, 1995.

[4] J.M. Howie and R.B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Royal Soc. Edinburgh A* **114** (1990) 161–167.

[5] S. Mendes-Gonçalves and R.P. Sullivan, The ideal structure of semigroups of transformations with restricted range, *Bull. Austral. Math. Soc.* **83** (2011) 289–300.

[6] S. Nenthein, P. Youngkhong and Y. Kemprasit, Regular elements of some transformation semigroups, *Pure Math. Appl.* **16**(3) (2005) 307–314.

[7] J. Sanwong, B. Singha and R.P. Sullivan, Maximal and minimal congruences on some semigroups, *Acta Math. Sin. (Engl. Ser.)* **25**(3) (2009) 455–466.

[8] J. Sanwong and W. Sommanee, Regularity and Green's Relations on a Semigroup of Transformations with Restricted Range, *Int. J. Math. Math. Sci.* **2008** (2008), Art. ID 794013, 11 pp..

[9] J. Sanwong and R.P. Sullivan, Maximal congruences on some semigroups, *Algebra Colloq.* **14**(2) (2007) 255–263.

[10] R.P. Sullivan, Semigroups of linear transformations with restricted range, *Bull. Austral. Math. Soc.* **77** (2008) 441–453.

[11] J.S.V. Symons, Some results concerning a transformation semigroup, *J. Austral. Math. Soc.* **19** (Series A) (1975) 413–425.

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