On semigroups of orientation-preserving transformations with restricted range

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Abstract

Let X_n be a chain with n elements ($n \in \mathbb{N}$) and let $\mathcal{OP}_n$ be the monoid of all orientation-preserving transformations of X_n. In this paper, for any nonempty subset Y of X_n, we consider the subsemigroup $\mathcal{OP}_n(Y)$ of $\mathcal{OP}_n$ of all transformations with range contained in Y: we describe the largest regular subsemigroup of $\mathcal{OP}_n(Y)$, which actually coincides with its subset of all regular elements. Also, we determine when two semigroups of the type $\mathcal{OP}_n(Y)$ are isomorphic and calculate their ranks. Moreover, a parallel study is presented for the correspondent subsemigroups of the monoid $\mathcal{OR}_n$ of all either orientation-preserving or orientation-reversing transformations of X_n.

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Introduction and preliminaries

Let X be a nonempty set and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on X. Let $n \in \mathbb{N}$. Let X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$, and denote the monoid $\mathcal{T}(X_n)$ simply by $\mathcal{T}_n$. Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that a is *cyclic* [respectively, *anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [respectively, $a_i < a_{i+1}$], where a_{t+1} denotes a_1. Let $\alpha \in \mathcal{T}_n$. We say that α is an *orientation-preserving* [respectively, *orientation-reversing*] transformation if the sequence of its images $(1\alpha, \ldots, n\alpha)$ is cyclic [respectively, anti-cyclic]. It is easy to check that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ whose elements are orientation-preserving and by $\mathcal{OR}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either orientation-preserving or orientation-reversing.

The notion of an orientation-preserving transformation was introduced by McAlister in [16] and, independently, by Catarino and Higgins in [4]. Several properties of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ have been investigated in these two papers. A presentation for the monoid $\mathcal{OP}_n$, in terms of $2n-1$ generators, was given by Catarino in [3]. Another presentation for $\mathcal{OP}_n$, in terms of 2 (its rank) generators, was found by Arthur and Ruškuc [2], who also exhibited a presentation for the monoid $\mathcal{OR}_n$, in terms of 3 (its rank) generators. The congruences of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were completely described by Fernandes et al. in [6]. Semigroups of orientation-preserving transformations were also studied in several recent papers (e.g. see [1, 5, 7, 9, 10, 11, 23]).

Let Y be a nonempty subset of X and denote by $\mathcal{T}(X,Y)$ the subsemigroup $\{\alpha \in \mathcal{T}(X) \mid \mathrm{Im}(\alpha) \subseteq Y\}$ of $\mathcal{T}(X)$ of all elements with range (image) restricted to Y.

In 1975, Symons [22] introduced and studied the semigroup $\mathcal{T}(X,Y)$. He described all the automorphisms of $\mathcal{T}(X,Y)$ and also determined when two semigroups of this type are isomorphic. In [18], Nenthein et al. characterized the regular elements of $\mathcal{T}(X,Y)$ and, in [19], Sanwong and Sommanee obtained the largest regular subsemigroup of $\mathcal{T}(X,Y)$ and showed that this subsemigroup determines Green's relations on $\mathcal{T}(X,Y)$. Moreover, they also determined a class of maximal inverse subsemigroups of this semigroup. Later, in 2009, all maximal and minimal congruences on $\mathcal{T}(X,Y)$ were described by Sanwong et al. [20]. Recently, all the ideals of $\mathcal{T}(X,Y)$ were obtained by Mendes-Gonçalves and Sullivan in

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[15] and, for a finite set X, Fernandes and Sanwong computed the rank of $\mathcal{T}(X,Y)$ [12]. On the other hand, in [21], Sullivan considered the linear counterpart of $\mathcal{T}(X,Y)$, that is the semigroup which consists of all linear transformations from a vector space V into a fixed subspace W of V, and described its Green's relations and ideals. If X is a chain then, being $\mathcal{O}(X)$ the monoid of all endomorphisms (i.e. order-preserving mappings) of the chain X, an order-preserving counterpart of the semigroup $\mathcal{T}(X,Y)$ can also be considered, namely the semigroup $\mathcal{O}(X,Y)=\{\alpha\in\mathcal{O}(X)\mid \operatorname{Im}(\alpha)\subseteq Y\}$. If $X=X_n$ then $\mathcal{O}(X,Y)$ is simply denoted by $\mathcal{O}_n(Y)$. A description of the regular elements of $\mathcal{O}(X,Y)$ and a characterization of the regular semigroups of this type were given by Mora and Kemprasit in [17]. This semigroup was also studied by the authors in [8] who described its largest regular subsemigroup and Green's relations. Moreover, for finite chains, also in [8] Fernandes et al. determined when two semigroups of the type $\mathcal{O}_n(Y)$ are isomorphic and calculated their ranks.

In this paper, we consider the semigroups of transformations with restricted range

$$\mathcal{OP}_n(Y)=\{\alpha\in\mathcal{OP}_n\mid \operatorname{Im}(\alpha)\subseteq Y\}\quad\text{and}\quad \mathcal{OR}_n(Y)=\{\alpha\in\mathcal{OR}_n\mid \operatorname{Im}(\alpha)\subseteq Y\},$$

for each nonempty subset Y of X_n. We begin, in Section 1, by characterizing when two semigroups of the type $\mathcal{OP}_n(Y)$ and of the type $\mathcal{OR}_n(Y)$ are isomorphic. Section 2 is dedicated to the study of regularity and Green's relations on $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. Finally, in Section 3, we determine the ranks of these semigroups.

For general background on Semigroup Theory and standard notation, we refer the reader to Howie's book [14].

1 Isomorphism theorems and sizes

In this section we characterize the nonempty subsets Y and Z of X_n such that the semigroups $\mathcal{OP}_n(Y)$ and $\mathcal{OP}_n(Z)$ (respectively, $\mathcal{OR}_n(Y)$ and $\mathcal{OR}_n(Z)$) are isomorphic. In this context, the dihedral group $\mathcal{D}_{2n}$ of order $2n$ plays a relevant role. Recall that

$$\mathcal{D}_{2n}=\langle g,h\mid h^2=g^n=hg^{n-1}hg^{n-1}=1\rangle\ .$$

Moreover, if

$$g=\begin{pmatrix}1&2&\cdots&n-1&n\\2&3&\cdots&n&1\end{pmatrix}\quad\text{and}\quad h=\begin{pmatrix}1&2&\cdots&n-1&n\\n&n-1&\cdots&2&1\end{pmatrix},$$

we can consider the group $\mathcal{D}_{2n}$ as being the subgroup of the symmetric group $\mathcal{S}_n$ on X_n generated by this two permutations:

$$\mathcal{D}_{2n}=\{1,g,g^2,\ldots,g^{n-1},h,hg,hg^2,\ldots,hg^{n-1}\}.$$

Notice that, for $j,k\in\{1,\ldots,n\}$, we have

$$(j)g^k=(j+k)\bmod n\quad\text{and}\quad (j)hg^k=(n-j+1+k)\bmod n. \tag{1}$$

From these relations, it is easy to deduce that any two distinct permutations of $\{1,g,g^2,\ldots,g^{n-1}\}$ do not coincide in any element of X_n. The same is true for any two distinct elements of $\{h,hg,hg^2,\ldots,hg^{n-1}\}$. Moreover, if a permutation of $\{1,g,g^2,\ldots,g^{n-1}\}$ coincides in i and j, for some $1\le i<j\le n$, with a permutation of $\{h,hg,hg^2,\ldots,hg^{n-1}\}$ then $n=2(j-i)$. It follows that any two permutations of $\mathcal{D}_{2n}$ which coincide in three elements of X_n must be equal and, furthermore, for an odd n, any two permutations of $\mathcal{D}_{2n}$ which coincide in two elements of X_n must be equal. On the other hand, if a partial injective transformation α of X_n is a restriction of some element of $\mathcal{D}_{2n}$, then

$$|(j)\alpha-(i)\alpha|\in\{j-i,n-(j-i)\},$$

for $i,j\in\operatorname{Dom}(\alpha)$, with $i<j$. In fact, the converse also holds:

Proposition 1.1 *A partial injective transformation α of X_n is a restriction of a permutation of $\mathcal{D}_{2n}$ if and only if $|(j)\alpha-(i)\alpha|\in\{j-i,n-(j-i)\}$, for all $i,j\in\operatorname{Dom}(\alpha)$ such that $i<j$.*

Proof The direct implication is an immediate consequence of the equalities (1). Conversely, take a partial injective transformation α of X_n such that $|(j)\alpha-(i)\alpha|\in\{j-i,n-(j-i)\}$, for all $i,j\in\operatorname{Dom}(\alpha)$ with $i<j$.

Let $i\in\operatorname{Dom}(\alpha)$. It is easy to check that $(i)g^{n+i\alpha-i}=(i)\alpha=(i)hg^{i\alpha+i-1}$. Clearly, if $|\operatorname{Dom}(\alpha)|\le 1$ then the result follows immediately. Thus, we may assume that $|\operatorname{Dom}(\alpha)|\ge 2$.

First, suppose that $\operatorname{Dom}(\alpha)=\{i<j\}$. Then, it is easy to show: if $(j)\alpha-(i)\alpha\in\{j-i,-n+j-i\}$ then we also have $(j)g^{n+i\alpha-i}=(j)\alpha$, whence α is a restriction of $g^{n+i\alpha-i}$; on the other hand, if $(j)\alpha-(i)\alpha\in\{i-j,n-j+i\}$ then also $(j)hg^{i\alpha+i-1}=(j)\alpha$ and so α is a restriction of $hg^{i\alpha+i-1}$.

Secondly, consider $\operatorname{Dom}(\alpha)=\{i<j<k\}$. Notice that $(k)\alpha-(i)\alpha=((k)\alpha-(j)\alpha)+((j)\alpha-(i)\alpha)$. Now, it requires only routine calculations to show that:

1. If $(j)\alpha-(i)\alpha=j-i$ then α is a restriction of $g^{n+i\alpha-i}$, unless $(k)\alpha-(j)\alpha=j-k$ and $n=2(j-i)$, in which case α is a restriction of $hg^{i\alpha+i-1}$;

2. If $(j)\alpha-(i)\alpha=i-j$ then α is a restriction of $hg^{i\alpha+i-1}$, unless $(k)\alpha-(j)\alpha=k-j$ and $n=2(j-i)$, in which case α is a restriction of $g^{n+i\alpha-i}$;

3. If $(j)\alpha-(i)\alpha=n-j+i$ then α is a restriction of $hg^{i\alpha+i-1}$, unless either $(k)\alpha-(j)\alpha=k-j$ or $(k)\alpha-(j)\alpha=-n+k-j$ and $n=2(j-i)$, in which cases α is a restriction of $g^{n+i\alpha-i}$;

4. If $(j)\alpha-(i)\alpha=-n+j-i$ then α is a restriction of $g^{n+i\alpha-i}$, unless either $(k)\alpha-(j)\alpha=j-k$ or $(k)\alpha-(j)\alpha=n-k+j$ and $n=2(j-i)$, in which cases α is a restriction of $hg^{i\alpha+i-1}$.

Next, take $\mathrm{Dom}(\alpha)=\{i<j<k<\ell\}$. By the previous case, we may find permutations $\sigma_1,\sigma_2,\sigma_3\in\mathcal{D}_{2n}$ such that the restrictions of α to $\{i<j<k\}$, $\{i<k<\ell\}$ and $\{j<k<\ell\}$ are restrictions of σ_1, σ_2 and σ_3, respectively. If $\sigma_1\neq\sigma_2$ and $\sigma_1\neq\sigma_3$ then, since σ_1 and σ_2 coincide in $\{i<k\}$ and σ_1 and σ_3 coincide in $\{j<k\}$, we have $n=2(k-i)$ and $n=2(k-j)$, whence $i=j$, which is a contradiction. Thus $\sigma_1=\sigma_2$ or $\sigma_1=\sigma_3$ and so α is a restriction of σ_1.

Finally, suppose that $|\mathrm{Dom}(\alpha)|=m\geq 5$ and admit, by induction hypothesis, that any partial injective transformation β of X_n such that $|\mathrm{Dom}(\beta)|=m-1$ and $|(j)\beta-(i)\beta|\in\{j-i,n-(j-i)\}$, for all $i,j\in\mathrm{Dom}(\beta)$ with $i<j$, is a restriction of a permutation of $\mathcal{D}_{2n}$. Let $\sigma_1,\sigma_2\in\mathcal{D}_{2n}$ be such that the restrictions of α to $\mathrm{Dom}(\alpha)\setminus\{\min(\mathrm{Dom}(\alpha))\}$ and $\mathrm{Dom}(\alpha)\setminus\{\max(\mathrm{Dom}(\alpha))\}$ are restrictions of σ_1 and σ_2, respectively. Then σ_1 and σ_2 coincide in $\mathrm{Dom}(\alpha)\setminus\{\min(\mathrm{Dom}(\alpha)),\max(\mathrm{Dom}(\alpha))\}$, a set with $m-2\geq 3$ elements, whence $\sigma_1=\sigma_2$. Thus α is a restriction of a permutation of $\mathcal{D}_{2n}$, as required. □

Let $x\in X_n$. We denote by $\mathcal{C}_x$ the constant transformation of $\mathcal{T}_n$ with image $\{x\}$. Observe that, given $\alpha\in\mathcal{T}_n$, we have

$$\mathcal{C}_x\alpha=\mathcal{C}_{x\alpha} \tag{2}$$

and

$$\alpha\mathcal{C}_x=\mathcal{C}_x. \tag{3}$$

These immediate equalities allow us to easily deduce the following properties.

Lemma 1.2 *Let Y and Z be nonempty subsets of X_n and let $\Theta:\mathcal{OP}_n(Y)\longrightarrow\mathcal{OP}_n(Z)$ be an isomorphism. Then:*

1. *For all $y\in Y$ there exists (a unique) $z\in Z$ such that $\mathcal{C}_y\Theta=\mathcal{C}_z$;*

2. *Θ induces a bijection $\theta:Y\longrightarrow Z$ defined by $\mathcal{C}_y\Theta=\mathcal{C}_{y\theta}$, for all $y\in Y$;*

3. *$(y\theta)(\alpha\Theta)=(y\alpha)\theta$, for all $y\in Y$ and $\alpha\in\mathcal{OP}_n(Y)$;*

4. *$\mathrm{Im}(\alpha\Theta)=(\mathrm{Im}(\alpha))\theta$, for any idempotent $\alpha\in\mathcal{OP}_n(Y)$.*

Proof Let $y\in Y$. Then, by (3) we have $\alpha\mathcal{C}_y=\mathcal{C}_y$, for all $\alpha\in\mathcal{OP}_n(Y)$, whence $(\alpha\Theta)(\mathcal{C}_y\Theta)=\alpha\mathcal{C}_y\Theta=\mathcal{C}_y\Theta$, for all $\alpha\in\mathcal{OP}_n(Y)$. Since Θ is surjective, it follows that $\beta(\mathcal{C}_y\Theta)=\mathcal{C}_y\Theta$, for all $\beta\in\mathcal{OP}_n(Z)$. In particular, given (any) $x\in Z$, we obtain $\mathcal{C}_y\Theta=\mathcal{C}_x(\mathcal{C}_y\Theta)=\mathcal{C}_{(x)(\mathcal{C}_y\Theta)}$, using also equality (2). Thus $\mathcal{C}_y\Theta=\mathcal{C}_z$, with $z=(x)(\mathcal{C}_y\Theta)\in Z$ (which does not depend of the taken $x\in Z$, since Θ is a function).

Therefore, we proved property 1 and so we have a well defined function $\theta:Y\longrightarrow Z$ satisfying the equality $\mathcal{C}_y\Theta=\mathcal{C}_{y\theta}$, for all $y\in Y$. A similar reasoning applied to the inverse isomorphism $\Theta^{-1}:\mathcal{OP}_n(Z)\longrightarrow\mathcal{OP}_n(Y)$ allows us to deduce the existence of a function $\theta':Z\longrightarrow Y$ satisfying the equality $\mathcal{C}_z\Theta^{-1}=\mathcal{C}_{z\theta'}$, for all $z\in Z$. Moreover, we have $\mathcal{C}_y=\mathcal{C}_y\Theta\Theta^{-1}=\mathcal{C}_{y\theta}\Theta^{-1}=\mathcal{C}_{y\theta\theta'}$, for all $y\in Y$, and similarly $\mathcal{C}_z=\mathcal{C}_{z\theta'\theta}$, for all $z\in Z$, which shows that θ and θ' are mutually inverse bijections. Thus, we just proved 2.

Next, we prove property 3. Let $y\in Y$ and $\alpha\in\mathcal{OP}_n(Y)$. Then

$$\mathcal{C}_{(y\theta)(\alpha\Theta)}=\mathcal{C}_{y\theta}(\alpha\Theta)=(\mathcal{C}_y\Theta)(\alpha\Theta)=(\mathcal{C}_y\alpha)\Theta=\mathcal{C}_{y\alpha}\Theta=\mathcal{C}_{(y\alpha)\theta},$$

by using equality (2), the definition of θ and the fact that Θ is a homomorphism, whence $(y\theta)(\alpha\Theta)=(y\alpha)\theta$.

It remains to prove 4. Let α be an idempotent of $\mathcal{OP}_n(Y)$. Then $\alpha\Theta$ is an idempotent of $\mathcal{OP}_n(Z)$ and so $\mathrm{Im}(\alpha\Theta)=\mathrm{Fix}(\alpha\Theta)$. Let $z\in\mathrm{Im}(\alpha\Theta)\subseteq Z$. Then $z=z(\alpha\Theta)$ and, on the other hand, $z=y\theta$, for some $y\in Y$. Hence, by property 3, $z=z(\alpha\Theta)=(y\theta)(\alpha\Theta)=(y\alpha)\theta\in(\mathrm{Im}(\alpha))\theta$. Conversely, let $z\in(\mathrm{Im}(\alpha))\theta$. Then $z=y\theta$, for some $y\in\mathrm{Im}(\alpha)\subseteq Y$. As α is an idempotent (hence $\mathrm{Im}(\alpha)=\mathrm{Fix}(\alpha)$), we have $y=y\alpha$ and so, by using property 3, we obtain $z=y\theta=(y\alpha)\theta=(y\theta)(\alpha\Theta)=z(\alpha\Theta)\in\mathrm{Im}(\alpha\Theta)$. Thus $\mathrm{Im}(\alpha\Theta)=(\mathrm{Im}(\alpha))\theta$, as required. □

Theorem 1.3 *Let Y and Z be nonempty subsets of X_n. Then $\mathcal{OP}_n(Y)$ and $\mathcal{OP}_n(Z)$ are isomorphic if and only if there exists $\sigma \in \mathcal{D}_{2n}$ such that $Y\sigma = Z$.*

Proof If there exists $\sigma \in \mathcal{D}_{2n}$ such that $Y\sigma = Z$, then it is easy to show that the mapping $\mathcal{OP}_n(Y) \to \mathcal{OP}_n(Z)$, $\alpha \mapsto \sigma^{-1}\alpha\sigma$, is an isomorphism. Conversely, suppose there exists an isomorphism $\Theta : \mathcal{OP}_n(Y) \longrightarrow \mathcal{OP}_n(Z)$ and let $\theta : Y \longrightarrow Z$ be the bijection induced by Θ given by Lemma 1.2. Let $i, j \in Y$ be such that $i < j$. Take

$$A(i,j) = \{\alpha \in \mathcal{OP}_n(Y) \mid \alpha = \alpha^2 \text{ and } \operatorname{Im}(\alpha) = \{i < j\}\}.$$

Then, by Lemma 1.2, we have

$$A(i,j)\Theta = \{\beta \in \mathcal{OP}_n(Z) \mid \beta = \beta^2 \text{ and } \operatorname{Im}(\beta) = \{i\theta, j\theta\}\}.$$

Moreover, by enumerating their elements, it is not difficult to conclude that $|A(i,j)| = (j-i)(n-(j-i))$ and $|A(i,j)\Theta| = |j\theta - i\theta|(n - |j\theta - i\theta|)$. As Θ is an isomorphism, we get $|A(i,j)\Theta| = |A(i,j)|$, i.e. $(j-i)(n-(j-i)) = |j\theta-i\theta|(n-|j\theta-i\theta|)$, whence $|j\theta - i\theta| = j - i$ or $|j\theta - i\theta| = n - (j-i)$. Therefore, by Proposition 1.1, it follows that θ is a restriction of some permutation σ of $\mathcal{D}_{2n}$ and so $Z = Y\theta = Y\sigma$, as required. $\square$

Now, observing that we can replace $\mathcal{OP}_n$ by $\mathcal{OR}_n$ in the proof of both Lemma 1.2 and Theorem 1.3 (also observe that all the idempotents of $\mathcal{OR}_n$ belong to $\mathcal{OP}_n$), we obtain an analogous characterization:

Theorem 1.4 *Let Y and Z be nonempty subsets of X_n. Then $\mathcal{OR}_n(Y)$ and $\mathcal{OR}_n(Z)$ are isomorphic if and only if there exists $\sigma \in \mathcal{D}_{2n}$ such that $Y\sigma = Z$.*

We finish this section by determining the cardinality of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, for each nonempty subset Y of X_n.

Recall that Catarino and Higgins showed in [4] that $|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n(n-1)$ and $|\mathcal{OR}_n| = n\binom{2n}{n} - \frac{n^2}{2}(n^2-2n+5)+n$. One of the methods used by Catarino and Higgins to find this formula for $|\mathcal{OP}_n|$ consisted of counting the number of transformations of rank k, for $1 \le k \le n$. In fact, they showed that, for $2 \le k \le n$, the number of transformations with a fixed image of size k is $k\binom{n}{k}$. In addition, $\mathcal{OP}_n$ has n constant transformations.

Now, let Y be a subset of X_n, with $1 \le |Y| = r \le n$. Since the number of distinct images contained in Y of size k is $\binom{r}{k}$, we get

$$|\{\alpha \in \mathcal{OP}_n(Y) \mid |\operatorname{Im}(\alpha)| = k\}| = k\binom{n}{k}\binom{r}{k},$$

for $2 \le k \le r$. As $\mathcal{OP}_n(Y)$ also contains r constant transformations, it follows that

$$|\mathcal{OP}_n(Y)| = r + \sum_{k=2}^{r} k\binom{r}{k}\binom{n}{k} = \sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} - r(n-1).$$

Noticing that $\sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} = r\binom{n+r-1}{r}$ (Combinatorial Identity (3.30) of [13, page 25]), we obtain:

Theorem 1.5 *Let Y be a subset of X_n of size r, for $1 \le r \le n$. Then $|\mathcal{OP}_n(Y)| = r\binom{n+r-1}{r} - r(n-1)$.*

Regarding $\mathcal{OR}_n$, Catarino and Higgins [4] proved that, for $3 \le k \le n$, the number of transformations with a fixed image of size k is $2k\binom{n}{k}$. Moreover, they also observed that $\{\alpha \in \mathcal{OR}_n \mid |\operatorname{Im}(\alpha)| \le 2\} = \{\alpha \in \mathcal{OP}_n \mid |\operatorname{Im}(\alpha)| \le 2\}$. Therefore

$$|\mathcal{OR}_n(Y)| = r + 2\binom{r}{2}\binom{n}{2} + \sum_{k=3}^{r} 2k\binom{r}{k}\binom{n}{k} = 2\sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} - 2\binom{r}{2}\binom{n}{2} - 2rn + r$$

and so, by using again the above combinatorial identity, we obtain:

Theorem 1.6 *Let Y be a subset of X_n of size r, for $1 \le r \le n$. Then $|\mathcal{OR}_n(Y)| = 2r\binom{n+r-1}{r} - \frac{rn}{2}(rn - r - n + 5) + r$.*

Notice that the sizes of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$ just depend of the size of Y.

2 Regularity and Green's relations

Recall that Catarino and Higgins showed in [4] that both $\mathcal{OP}_n$ and $\mathcal{OR}_n$ are regular semigroups. Thus, Green's relations $\mathcal{L}$ and $\mathcal{R}$ in $\mathcal{OP}_n$ and in $\mathcal{OR}_n$ are just restrictions of the correspondent relations in $\mathcal{T}_n$. This is also the case for Green's relation $\mathcal{D}$, as proved by Catarino and Higgins in [4]. Therefore, if $\alpha, \beta \in \mathcal{OP}_n$ [respectively, $\alpha, \beta \in \mathcal{OR}_n$], we have

1. $\alpha \mathcal{L} \beta$ if and only if $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$,
2. $\alpha \mathcal{R} \beta$ if and only if $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$ and
3. $\alpha \mathcal{D} \beta$ if and only if $|\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)|$,

in $\mathcal{OP}_n$ [respectively, $\mathcal{OR}_n$]. Regarding Green's relation $\mathcal{H}$, if α is an element of $\mathcal{OP}_n$ of rank k, for $1 \le k \le n$, then the $\mathcal{H}$-class in $\mathcal{OP}_n$ of α has k elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OP}_n$ of an idempotent of rank k, for $1 \le k \le n$, is a cycle group of order k; if α is an element of $\mathcal{OR}_n$ of rank k, for $3 \le k \le n$, then the $\mathcal{H}$-class in $\mathcal{OR}_n$ of α has $2k$ elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n$ of an idempotent of rank k, for $3 \le k \le n$, is a dihedral group of order $2k$; if α is an element of $\mathcal{OR}_n$ of rank 1 or 2, then the $\mathcal{H}$-class in $\mathcal{OR}_n$ of α has 1 or 2 elements (all with the same domain and image), respectively; in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n$ of an idempotent of rank 1 or 2 is a cycle group of order 1 or 2, respectively. These facts were proved in [4]. See also [16].

Let Y be a nonempty subset of X_n of size r ($1 \le r \le n$). In this section we discuss regularity and give descriptions for Green's relations of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$.

Let

$$F\mathcal{OP}_n(Y) = \{\alpha \in \mathcal{OP}_n(Y) \mid \mathrm{Im}(\alpha) = Y\alpha\}$$

and

$$F\mathcal{OR}_n(Y) = \{\alpha \in \mathcal{OR}_n(Y) \mid \mathrm{Im}(\alpha) = Y\alpha\}.$$

Clearly $F\mathcal{OP}_n(Y)$ and $F\mathcal{OR}_n(Y)$ are right ideals of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, respectively. In fact, we will show that these subsets determine relevant aspects of the structure of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. Regarding the regularity, we have:

Theorem 2.1 *Let Y be a nonempty subset of X_n. Then $F\mathcal{OP}_n(Y)$ [respectively, $F\mathcal{OR}_n(Y)$] is the set of all regular elements of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$]. Furthermore:*

1. *$F\mathcal{OP}_n(Y)$ [respectively, $F\mathcal{OR}_n(Y)$] is the largest regular subsemigroup of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$];*
2. *$\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$] is regular if and only if $|Y| = 1$ or $Y = X_n$.*

Proof If α is a regular element of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$], then $\alpha = \alpha\beta\alpha$, for some $\beta \in \mathcal{OP}_n(Y)$ [respectively, $\beta \in \mathcal{OR}_n(Y)$], whence $X\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha$ and so $X\alpha = Y\alpha$, i.e. $\alpha \in F\mathcal{OP}_n(Y)$ [respectively, $\alpha \in F\mathcal{OR}_n(Y)$].

Conversely, take $\alpha \in F\mathcal{OP}_n(Y)$ [respectively, $\alpha \in F\mathcal{OR}_n(Y)$] and suppose that $\mathrm{Im}(\alpha) = \{a_1 < \cdots < a_k\}$, for some $a_1, \ldots, a_k \in Y$, with $1 \le k \le |Y|$. Then there exist $b_1, \ldots, b_k \in Y$ such that $b_i\alpha = a_i$, for $1 \le i \le k$. Let β be the transformation of X_n defined by

$$x\beta = \begin{cases} b_k & \text{if } \ 1 \le x < a_1 \ \text{ or } \ a_k \le x \le n \\ b_j & \text{if } \ a_j \le x < a_{j+1} \ \text{ and } \ 1 \le j \le k-1 \ . \end{cases}$$

It is easy to check that $\beta \in \mathcal{OP}_n(Y)$ [respectively, $\beta \in \mathcal{OR}_n(Y)$] and $\alpha = \alpha\beta\alpha$. Thus α is a regular element of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$].

Hence, we proved that $F\mathcal{OP}_n(Y)$ [respectively, $F\mathcal{OR}_n(Y)$] is the set of all regular elements of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$].

Statement 1 is obvious, since $F\mathcal{OP}_n(Y)$ and $F\mathcal{OR}_n(Y)$ are subsemigroups of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, respectively.

Regarding statement 2, if Y is a proper subset of X_n such that $|Y| \ge 2$ then we may consider two distinct elements $y, y' \in Y$ and an element $z \in X_n \setminus Y$. Thus, we define a transformation α on X_n by $x\alpha = y$, if $x = z$, and $x\alpha = y'$, if $x \in X_n \setminus \{z\}$. Clearly, $\alpha \in \mathcal{OP}_n(Y)$ and, since $\mathrm{Im}(\alpha) = \{y, y'\}$ and $Y\alpha = \{y'\}$, α is not regular both in $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. For the converse, we already recalled that $\mathcal{OP}_n = \mathcal{OP}_n(X_n)$ and $\mathcal{OR}_n = \mathcal{OR}_n(X_n)$ are regular semigroups and, on the other hand, if $|Y| = 1$ then $\mathcal{OP}_n(Y) = \mathcal{OR}_n(Y)$ is trivial (it is just formed by the constant mapping with range Y), whence a regular semigroup, as required. $\square$

We finish this section with the following characterization of Green's relations in $\mathcal{OP}_n(Y)$ and in $\mathcal{OR}_n(Y)$:

Theorem 2.2 *Let Y be a nonempty subset of X_n of size r. Let $S = \mathcal{OP}_n(Y)$ or $S = \mathcal{OR}_n(Y)$. Let $\alpha, \beta \in S$. Then:*

1. *$\alpha\mathcal{L}\beta$ in S if and only if either $\alpha = \beta$ or both α and β are regular and $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$;*
2. *$\alpha\mathcal{R}\beta$ in S if and only if $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$;*
3. *$\alpha\mathcal{D}\beta$ in S if and only if either (i) both α and β are regular and $|\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)|$ or (ii) both α and β are not regular and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$;*
4. *If α is a non regular element of $\mathcal{OP}_n(Y)$ then the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of α is trivial; if α is a regular element of $\mathcal{OP}_n(Y)$ of rank k, for $1 \le k \le r$, then the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of α has k elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of an idempotent of rank k, for $1 \le k \le r$, is a cycle group of order k;*
5. *If α is a non regular element of $\mathcal{OR}_n(Y)$ then the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of α is trivial; if α is a regular element of $\mathcal{OR}_n(Y)$ of rank k, for $3 \le k \le r$, then the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of α has $2k$ elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of an idempotent of rank k, for $3 \le k \le r$, is a dihedral group of order $2k$; if α is a regular element of $\mathcal{OR}_n(Y)$ of rank 1 or 2, then the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of α has 1 or 2 elements (all with the same domain and image), respectively; in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of an idempotent of rank 1 or 2 is a cycle group of order 1 or 2, respectively.*

Proof 1. First, suppose that $\alpha\mathcal{L}\beta$ in S. Then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$, for some $\gamma, \lambda \in S^1$. If $\alpha \ne \beta$ then $\gamma \ne 1$ and $\lambda \ne 1$, whence $X\alpha = X\gamma\beta \subseteq Y\beta \subseteq X\beta = X\lambda\alpha \subseteq Y\alpha \subseteq X\alpha$ and so $\mathrm{Im}(\alpha) = Y\alpha = Y\beta = \mathrm{Im}(\beta)$, i.e. α and β are regular and $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$.

The converse is obvious, since $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ implies $\alpha\mathcal{L}\beta$ in $\mathcal{T}_n$ and so, being either $\alpha = \beta$ or α and β are regular elements in S, it follows that $\alpha\mathcal{L}\beta$ in S (in fact, no matter who is the subsemigroup S of $\mathcal{T}_n$).

2. If $\alpha\mathcal{R}\beta$ in S then $\alpha\mathcal{R}\beta$ in $\mathcal{T}_n$ and so $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$.

In order to prove the converse, it is useful to introduce the following notation. Let $I = \{i_1 < \cdots < i_k\}$ be a nonempty subset of X_n ($1 \le k \le n$). Define the transformation ι_I of $\mathcal{T}_n$ by

$$x\iota_I = \begin{cases} i_k & \text{if } \ 1 \le x < i_1 \ \text{ or } \ i_k \le x \le n \\ i_j & \text{if } \ i_j \le x < i_{j+1} \ \text{ and } \ 1 \le j \le k-1 \ . \end{cases}$$

Clearly, ι_I is an idempotent of $\mathcal{OP}_n$ with image (fixed points) I. In particular, if $I \subseteq Y$ then $\iota_I \in \mathcal{OP}_n(Y)$.

Now, take $T = \mathcal{OP}_n$ for $S = \mathcal{OP}_n(Y)$ and $T = \mathcal{OR}_n$ for $S = \mathcal{OR}_n(Y)$. Suppose that $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$. Then $\alpha\mathcal{R}\beta$ in T and so $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$, for some $\gamma, \lambda \in T$. Thus, it is clear that we also have $\alpha = \beta\iota_{\mathrm{Im}(\beta)}\gamma$ and $\beta = \alpha\iota_{\mathrm{Im}(\alpha)}\lambda$. Moreover, since $\iota_{\mathrm{Im}(\beta)}\gamma$ and $\iota_{\mathrm{Im}(\alpha)}\lambda$ have the same rank of α (and β), it follows that $\mathrm{Im}(\iota_{\mathrm{Im}(\beta)}\gamma) = \mathrm{Im}(\alpha)$ and $\mathrm{Im}(\iota_{\mathrm{Im}(\alpha)}\lambda) = \mathrm{Im}(\beta)$, whence $\iota_{\mathrm{Im}(\beta)}\gamma, \iota_{\mathrm{Im}(\alpha)}\lambda \in S$ and so $\alpha\mathcal{R}\beta$ in S.

3. If $\alpha\mathcal{D}\beta$ in S then there exists $\gamma \in S$ such that $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$ in S. On the other hand, we also have $\alpha\mathcal{D}\beta$ in $\mathcal{T}_n$ and so $|\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)|$. Moreover, as in any semigroup, either both α and β are regular or both α and β are non regular. Hence, in case both α and β are non regular, it follows that γ is also non regular and thus $\alpha = \gamma$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\gamma) = \mathrm{Ker}(\beta)$.

Conversely, if (both α and β are not regular and) $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$ then $\alpha\mathcal{R}\beta$ in S and so $\alpha\mathcal{D}\beta$ in S. On the other hand, suppose that both α and β are regular and $|\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)|$. Then $\alpha\mathcal{D}\beta$ in T (with T as defined above) and so $\alpha = \gamma_1\beta\gamma_2$ and $\beta = \lambda_1\alpha\lambda_2$, for some $\gamma_1, \gamma_2, \lambda_1, \lambda_2 \in T$. Let $\beta' \in S$ be an inverse of β. Then $\alpha = (\gamma_1\beta\beta')\beta(\beta'\beta\gamma_2)$. Clearly, $\gamma_1\beta\beta' \in S$. On the other hand, since $\beta'\beta\gamma_2$ has the same rank as α, it should have the same image as α, whence also $\beta'\beta\gamma_2 \in S$. Similarly, if $\alpha' \in S$ is an inverse of α then $\beta = (\lambda_1\alpha\alpha')\alpha(\alpha'\alpha\lambda_2)$ and $\lambda_1\alpha\alpha', \alpha'\alpha\lambda_2 \in S$. Thus, also in this case, $\alpha\mathcal{D}\beta$ in S.

Properties 4 and 5 are immediate, since $\mathcal{L}$-classes in S of non regular elements are trivial and it is clear that $\mathcal{H}$-classes in T (with T as defined above) of regular elements of S must coincide with the respective $\mathcal{H}$-classes in S. $\square$

3 Ranks

Let Y be a nonempty subset of X_n. In this section we determine the ranks of the semigroups $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. Surprisingly, as opposed to the case of $\mathcal{O}_n(Y)$ [8], we will show that the ranks of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$ only depend of the size of Y.

It is well known that $\mathcal{OP}_n$ and $\mathcal{OR}_n$ have, respectively, ranks 2 and 3 (see [2, 4]). Therefore, in what follows, we suppose that Y is a proper subset of X_n. Let $r = |Y|$.

We begin by showing that $\mathcal{OP}_n(Y)$ is generated by its elements of rank r.

Lemma 3.1 *Any transformation of $\mathcal{OP}_n(Y)$ of rank k is a product of two elements of $\mathcal{OP}_n(Y)$ of rank $k+1$, for $1 \le k < r$.*

Proof Let α be an element of $\mathcal{OP}_n(Y)$ of rank k. By [4, Theorem 2.6], there exist $\beta \in \mathcal{O}_n$ and $0 \le t < n$ such that $\alpha = g^t\beta$, where g is the permutation of X_n defined in Section 1. Since g^t is a permutation, then $\mathrm{Im}(\beta) = \mathrm{Im}(\alpha)$ and so, in addition, $\beta \in \mathcal{O}_n(Y)$. Now, by [8, Lemma 3.5], there exist transformations $\beta_1, \beta_2 \in \mathcal{O}_n(Y)$ of rank $k+1$ such that $\beta = \beta_1\beta_2$. Let $\alpha_1 = g^t\beta_1$. Then $\beta_1 \in \mathcal{OP}_n$. Again, since g^t is a permutation, we have $\mathrm{Im}(\alpha_1) = \mathrm{Im}(\beta_1)$, whence α_1 is an element of $\mathcal{OP}_n(Y)$ with rank $k+1$. Thus, $\alpha = \alpha_1\beta_2$, with both α_1 and β_2 elements of $\mathcal{OP}_n(Y)$ of rank $k+1$, as required. $\square$

From this lemma, it follows immediately that $\mathcal{OP}_n(Y)$ is generated by its elements of rank r.

Now, suppose that $Y = \{y_1 < y_2 < \cdots < y_r\}$. Let

$$\widehat{g}_Y = \left(\begin{array}{c|c|c|c|c} A_1 & A_2 & \cdots & A_{r-1} & A_r \\ y_2 & y_3 & \cdots & y_r & y_1 \end{array}\right) \in \mathcal{OP}_n(Y),$$

where $A_j = \{y_j, \ldots, y_{j+1}-1\}$, $1 \le j \le r-1$, and $A_r = \{y_r, \ldots, n, 1, \ldots, y_1 - 1\}$.

Lemma 3.2 *Let $\alpha, \beta \in \mathcal{OP}_n(Y)$ be two elements of rank r such that* $\mathrm{Ker}(\beta) = \mathrm{Ker}(\alpha)$. *Then $\beta = \alpha\widehat{g}_Y^k$, for some $k \in \{0, \ldots, r-1\}$.*

Proof Suppose that $I_1, I_2, \ldots, I_k$ are the kernel classes of α (and β) in order $\max I_i < \max I_{i+1}$, for $i = 1, \ldots, r-1$. Then

$$\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c} I_1 & I_2 & \cdots & I_{r-i} & I_{r-i+1} & I_{r-i+2} & \cdots & I_r \\ y_{i+1} & y_{i+2} & \cdots & y_r & y_1 & y_2 & \cdots & y_i \end{array}\right)$$

and

$$\beta = \left(\begin{array}{c|c|c|c|c|c|c|c} I_1 & I_2 & \cdots & I_{r-j} & I_{r-j+1} & I_{r-j+2} & \cdots & I_r \\ y_{j+1} & y_{j+2} & \cdots & y_r & y_1 & y_2 & \cdots & y_j \end{array}\right),$$

for some $1 \le i, j \le r$. Take $k = j - i$, if $i \le j$, and $k = r - i + j$, otherwise. Hence, it is a routine matter to prove that $\beta = \alpha\widehat{g}_Y^k$, as required. $\square$

Now, notice that any generating set of $\mathcal{OP}_n(Y)$ (and of $\mathcal{OR}_n(Y)$) must contain at least one element from each distinct kernel of transformations of rank r. On the other hand, the number of distinct kernels of transformations of $\mathcal{OP}_n(Y)$ (and of $\mathcal{OR}_n(Y)$) of rank r coincides with the number of distinct kernels of transformations of $\mathcal{OP}_n$ of rank r, which is precisely $\binom{n}{r}$ (see [4]). These observations, together with the previous two lemmas, prove the following result.

Theorem 3.3 *The semigroup $\mathcal{OP}_n(Y)$ is generated by any subset of transformations of rank r containing $\widehat{g}_Y$ and at least one element from each distinct kernel. Furthermore, $\mathcal{OP}_n(Y)$ has rank equal to $\binom{n}{r}$.*

Next, let

$$\widetilde{h}_Y = \left(\begin{array}{c|c|c|c|c} B_1 & B_2 & \cdots & B_{r-1} & B_r \\ y_r & y_{r-1} & \cdots & y_2 & y_1 \end{array}\right) \in \mathcal{OR}_n(Y),$$

where $B_1 = \{1, \ldots, y_1, y_r + 1, \ldots, n\}$ and $B_j = \{y_{j-1} + 1, \ldots, y_j\}$, $2 \le j \le r$.

Notice that

$$\mathrm{Ker}(\widetilde{h}_Y) = \mathrm{Ker}(\widehat{g}_Y) \iff B_j = A_j,\ 1 \le j \le r \iff B_j = A_j = \{y_j\},\ 1 \le j \le r \iff r = n,$$

whence $\widetilde{h}_Y$ and $\widehat{g}_Y$ have distinct kernels. On the other hand, clearly

$$\widetilde{h}_Y^2 = \left(\begin{array}{c|c|c|c|c} B_1 & B_2 & \cdots & B_{r-1} & B_r \\ y_1 & y_2 & \cdots & y_{r-1} & y_r \end{array}\right) \in \mathcal{OP}_n(Y)$$

is a right identity of $\mathcal{OR}_n(Y)$. Thus, if $\alpha \in \mathcal{OR}_n(Y) \setminus \mathcal{OP}_n(Y)$ then $\alpha = (\alpha\widetilde{h}_Y)\widetilde{h}_Y$, with $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha\widetilde{h}_Y)$ and $\alpha\widetilde{h}_Y \in \mathcal{OP}_n(Y)$. Therefore, it is easy to conclude:

Theorem 3.4 *The semigroup $\mathcal{OR}_n(Y)$ is generated by any subset of transformations of rank r containing both $\widehat{g}_Y$ and $\widetilde{h}_Y$ and at least one element from each distinct kernel. Furthermore, $\mathcal{OR}_n(Y)$ has rank equal to $\binom{n}{r}$.*

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