On semigroups of orientation-preserving transformations with restricted range

Vítor H. Fernandes*, Preeyanuch Honyam, Teresa M. Quinteiro† and Boorapa Singha

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Abstract

Let $X_n$ be a chain with $n$ elements ($n \in \mathbb{N}$) and let $\mathcal{OP}_n$ be the monoid of all orientation-preserving transformations of $X_n$. In this paper, for any nonempty subset $Y$ of $X_n$, we consider the subsemigroup $\mathcal{OP}_n(Y)$ of $\mathcal{OP}_n$ of all transformations with range contained in $Y$: we describe the largest regular subsemigroup of $\mathcal{OP}_n(Y)$, which actually coincides with its subset of all regular elements. Also, we determine when two semigroups of the type $\mathcal{OP}_n(Y)$ are isomorphic and calculate their ranks. Moreover, a parallel study is presented for the correspondent subsemigroups of the monoid $\mathcal{OR}_n$ of all either orientation-preserving or orientation-reversing transformations of $X_n$.


Keywords: transformations, orientation-preserving, orientation-reversing, restricted range, rank.

Introduction and preliminaries

Let $X$ be a nonempty set and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on $X$. Let $n \in \mathbb{N}$. Let $X_n$ be a chain with $n$ elements, say $X_n = \{1 < 2 < \cdots < n\}$, and denote the monoid $\mathcal{T}(X_n)$ simply by $\mathcal{T}_n$. Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of $t$ ($t \geq 0$) elements from the chain $X_n$. We say that $a$ is cyclic [respectively, anti-cyclic] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [respectively, $a_i < a_{i+1}$], where $a_{t+1}$ denotes $a_1$. Let $\alpha \in \mathcal{T}_n$. We say that $\alpha$ is an orientation-preserving [respectively, orientation-reversing] transformation if the sequence of its images $(1\alpha, \ldots, n\alpha)$ is cyclic [respectively, anti-cyclic]. It is easy to check that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ whose elements are orientation-preserving and by $\mathcal{OR}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either orientation-preserving or orientation-reversing.

The notion of an orientation-preserving transformation was introduced by McAlister in [16] and, independently, by Catarino and Higgins in [4]. Several properties of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ have been investigated in these two papers. A presentation for the monoid $\mathcal{OP}_n$, in terms of $2n - 1$ generators, was given by Catarino in [3]. Another presentation for $\mathcal{OP}_n$, in terms of $2$ (its rank) generators, was found by Arthur and Ruškuc [2], who also exhibited a presentation for the monoid $\mathcal{OR}_n$, in terms of $3$ (its rank) generators. The congruences of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were completely described by Fernandes et al. in [6]. Semigroups of orientation-preserving transformations were also studied in several recent papers (e.g. see [1, 5, 7, 9, 10, 11, 23]).

Let $Y$ be a nonempty subset of $X$ and denote by $\mathcal{T}(X, Y)$ the subsemigroup $\{\alpha \in \mathcal{T}(X) \mid \text{Im}(\alpha) \subseteq Y\}$ of $\mathcal{T}(X)$ of all elements with range (image) restricted to $Y$.

In 1975, Symons [22] introduced and studied the semigroup $\mathcal{T}(X, Y)$. He described all the automorphisms of $\mathcal{T}(X, Y)$ and also determined when two semigroups of this type are isomorphic. In [18], Neunhein et al. characterized the regular elements of $\mathcal{T}(X, Y)$ and, in [19], Sanwong and Sommanee obtained the largest regular subsemigroup of $\mathcal{T}(X, Y)$ and showed that this subsemigroup determines Green’s relations on $\mathcal{T}(X, Y)$. Moreover, they also determined a class of maximal inverse subsemigroups of this semigroup. Later, in 2009, all maximal and minimal congruences on $\mathcal{T}(X, Y)$ were described by Sanwong et al. [20]. Recently, all the ideals of $\mathcal{T}(X, Y)$ were obtained by Mendes-Gonçalves and Sullivan in

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and, for a finite set \( X \), Fernandes and Sanwong computed the rank of \( T(X, Y) \) [12]. On the other hand, in [21], Sullivan considered the linear counterpart of \( T(X, Y) \), that is the semigroup which consists of all linear transformations from a vector space \( V \) into a fixed subspace \( W \) of \( V \), and described its Green’s relations and ideals. If \( X \) is a chain then, being \( \mathcal{O}(X) \) the monoid of all endomorphisms (i.e. order-preserving mappings) of the chain \( X \), an order-preserving counterpart of the semigroup \( T(X, Y) \) can also be considered, namely the semigroup \( \mathcal{O}(X, Y) = \{ \alpha \in \mathcal{O}(X) \mid \text{Im}(\alpha) \subseteq Y \} \). If \( X = X_n \) then \( \mathcal{O}(X, Y) \) is simply denoted by \( \mathcal{O}_n(Y) \). A description of the regular elements of \( \mathcal{O}(X, Y) \) and a characterization of the regular semigroups of this type were given by Mora and Kemprasit in [17]. This semigroup was also studied by the authors in [8] who described its largest regular subsemigroup and Green’s relations. Moreover, for finite chains, also in [8] Fernandes et al. determined when two semigroups of the type \( \mathcal{O}_n(Y) \) are isomorphic and calculated their ranks.

In this paper, we consider the semigroups of transformations with restricted range

\[
\mathcal{O}_n(Y) = \{ \alpha \in \mathcal{O}_n \mid \text{Im}(\alpha) \subseteq Y \}
\]

and

\[
\mathcal{O}_n(Y) = \{ \alpha \in \mathcal{O}_n \mid \text{Im}(\alpha) \subseteq Y \}
\]

for each nonempty subset \( Y \) of \( X_n \). We begin, in Section 1, by characterizing when two semigroups of the type \( \mathcal{O}_n(Y) \) and of the type \( \mathcal{O}_n(Y) \) are isomorphic. Section 2 is dedicated to the study of regularity and Green’s relations on \( \mathcal{O}_n(Y) \) and \( \mathcal{O}_n(Y) \). Finally, in Section 3, we determine the ranks of these semigroups.

For general background on Semigroup Theory and standard notation, we refer the reader to Howie’s book [14].

1 Isomorphism theorems and sizes

In this section we characterize the nonempty subsets \( Y \) and \( Z \) of \( X_n \) such that the semigroups \( \mathcal{O}_n(Y) \) and \( \mathcal{O}_n(Z) \) (respectively, \( \mathcal{O}_n(Y) \) and \( \mathcal{O}_n(Z) \)) are isomorphic. In this context, the dihedral group \( D_{2n} \) of order \( 2n \) plays a relevant role. Recall that

\[
D_{2n} = \langle g, h \mid h^2 = g^n = hg^{n-1}hg^{n-1} = 1 \rangle.
\]

Moreover, if

\[
g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}
\]

and

\[
h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix},
\]

we can consider the group \( D_{2n} \) as being the subgroup of the symmetric group \( S_n \) on \( X_n \) generated by the two permutations.

\[
D_{2n} = \{1, g, g^2, \ldots, g^{n-1}, h, hg, hg^2, \ldots, hg^{n-1}\}.
\]

Notice that, for \( j, k \in \{1, \ldots, n\} \), we have

\[
(jg^k = (j+k) \mod n \quad \text{and} \quad (j)g^k = (n-j+1+k) \mod n).
\]

From these relations, it is easy to deduce that any two distinct permutations of \( \{1, g, g^2, \ldots, g^{n-1}\} \) do not coincide in any element of \( X_n \). The same is true for any two distinct elements of \( \{h, hg, hg^2, \ldots, hg^{n-1}\} \). Moreover, if a permutation of \( \{1, g, g^2, \ldots, g^{n-1}\} \) coincides in \( i \) and \( j \), for some \( 1 \leq i < j \leq n \), with a permutation of \( \{h, hg, hg^2, \ldots, hg^{n-1}\} \) then \( n = 2(j-i) \). It follows that any two permutations of \( D_{2n} \) which coincide in three elements of \( X_n \) must be equal and, furthermore, for an odd \( n \), any two permutations of \( D_{2n} \) which coincide in two elements of \( X_n \) must be equal. On the other hand, if a partial injective transformation \( \alpha \) of \( X_n \) is a restriction of some element of \( D_{2n} \), then

\[
|(j)\alpha - (i)\alpha| \in \{j-i, n-(j-i)\},
\]

for \( i, j \in \text{Dom}(\alpha) \), with \( i < j \). In fact, the converse also holds:

**Proposition 1.1** A partial injective transformation \( \alpha \) of \( X_n \) is a restriction of a permutation of \( D_{2n} \) if and only if \( |(j)\alpha - (i)\alpha| \in \{j-i, n-(j-i)\} \), for all \( i, j \in \text{Dom}(\alpha) \) such that \( i < j \).

**Proof** The direct implication is an immediate consequence of the equalities (1). Conversely, take a partial injective transformation \( \alpha \) of \( X_n \) such that \( |(j)\alpha - (i)\alpha| \in \{j-i, n-(j-i)\} \), for all \( i, j \in \text{Dom}(\alpha) \) with \( i < j \).

Let \( i \in \text{Dom}(\alpha) \). It is easy to check that \( (i)g^{n+i-j} = (i)\alpha = (i)hg^{i+j-1} \). Clearly, if \( |\text{Dom}(\alpha)| \leq 1 \) then the result follows immediately. Thus, we may assume that \( |\text{Dom}(\alpha)| \geq 2 \).

First, suppose that \( \text{Dom}(\alpha) = \{i < j\} \). Then, it is easy to show: if \( (j)\alpha - (i)\alpha \in \{j-i, n+j-i\} \) then we also have \( (j)\alpha^{n+i-j} = (j)\alpha \), whence \( \alpha \) is a restriction of \( g^{n+i-j} \); on the other hand, if \( (j)\alpha - (i)\alpha \in \{j-i, n+j-i\} \) then also \( (j)\alpha^{n+i-j} = (j)\alpha \) and so \( \alpha \) is a restriction of \( g^{n+i-j} \).

Secondly, consider \( \text{Dom}(\alpha) = \{i < j < k\} \). Notice that \( (k)\alpha - (i)\alpha = ((k)\alpha - (j)\alpha) + ((j)\alpha - (i)\alpha) \). Now, it requires only routine calculations to show that:
1. If \((j)\alpha - (i)\alpha = j - i\) then \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\), unless \((k)\alpha - (j)\alpha = j - k\) and \(n = 2(j - i)\), in which case \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\).

2. If \((j)\alpha - (i)\alpha = n - i\) then \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\), unless \((k)\alpha - (j)\alpha = k - j\) and \(n = 2(j - i)\), in which case \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\).

3. If \((j)\alpha - (i)\alpha = n - j + i\) then \(\alpha\) is a restriction of \(h^{\alpha+i-a-i}\), unless either \((k)\alpha - (j)\alpha = k - j\) or \((k)\alpha - (j)\alpha = -n + k - j\) and \(n = 2(j - i)\), in which case \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\).

4. If \((j)\alpha - (i)\alpha = n - j + i\) then \(\alpha\) is a restriction of \(g^{\alpha+i-a-i}\), unless either \((k)\alpha - (j)\alpha = j - k\) or \((k)\alpha - (j)\alpha = -n + k + j\) and \(n = 2(j - i)\), in which case \(\alpha\) is a restriction of \(h^{\alpha+i-a-i}\).

Next, take \(\operatorname{Dom}(\alpha) = \{i < j < k < \ell\}\). By the previous case, we may find permutations \(\sigma_1, \sigma_2, \sigma_3 \in D_{2n}\) such that the restrictions of \(\alpha\) to \(\{i < j < k\}\), \(\{i < k < \ell\}\) and \(\{j < k < \ell\}\) are restrictions of \(\sigma_1\), \(\sigma_2\) and \(\sigma_3\), respectively. If \(\sigma_1 \neq \sigma_2\) and \(\sigma_1 \neq \sigma_3\) then, since \(\sigma_1\) and \(\sigma_2\) coincide in \(\{i < k\}\) and \(\sigma_1\) and \(\sigma_3\) coincide in \(\{j < k\}\), we have \(n = 2(k - j)\) and \(n = 2(j - i)\), whence \(i = j\), which is a contradiction. Thus \(\sigma_1 = \sigma_2\) or \(\sigma_1 = \sigma_3\) and so \(\alpha\) is a restriction of \(\sigma_1\).

Finally, suppose that \(|\operatorname{Dom}(\alpha)| = m \geq 5\) and admit, by induction hypothesis, that any partial injective transformation \(\beta\) of \(X_n\) such that \(|\operatorname{Dom}(\beta)| = m - 1\) and \(|(j)\beta - (i)\beta| \in \{j - i, n - (j - i)\}\), for all \(i, j \in \operatorname{Dom}(\beta)\) with \(i < j\), is a restriction of a permutation of \(D_{2n}\). Let \(\sigma_1, \sigma_2 \in D_{2n}\) be such that the restrictions of \(\alpha\) to \(\operatorname{Dom}(\alpha) \setminus \{\min(\operatorname{Dom}(\alpha))\}\) and \(\operatorname{Dom}(\alpha) \setminus \{\max(\operatorname{Dom}(\alpha))\}\) are restrictions of \(\sigma_1\) and \(\sigma_2\), respectively. Then \(\sigma_1\) and \(\sigma_2\) coincide in \(\operatorname{Dom}(\alpha) \setminus \{\min(\operatorname{Dom}(\alpha)), \max(\operatorname{Dom}(\alpha))\}\), a set with \(m - 2 \geq 3\) elements, whence \(\sigma_1 = \sigma_2\). Thus \(\alpha\) is a restriction of a permutation of \(D_{2n}\), as required. \(\square\)

Let \(x \in X_n\). We denote by \(c_x\) the constant transformation of \(T_n\) with image \(\{x\}\). Observe that, given \(\alpha \in T_n\), we have
\[
\epsilon_x \alpha = \epsilon_{x \alpha}
\]
(2) and
\[
\alpha \epsilon_x = \epsilon_x
\]
(3)
These immediate equalities allow us to easily deduce the following properties.

**Lemma 1.2** Let \(Y\) and \(Z\) be nonempty subsets of \(X_n\) and let \(\Theta : \mathcal{OP}_n(Y) \rightarrow \mathcal{OP}_n(Z)\) be an isomorphism. Then:

1. For all \(y \in Y\) there exists (a unique) \(z \in Z\) such that \(\epsilon_y \Theta = \epsilon_z\);
2. \(\Theta\) induces a bijection \(\theta : Y \rightarrow Z\) defined by \(\epsilon_y \Theta = \epsilon_{y \theta}\), for all \(y \in Y\);
3. \((y \theta)(\alpha \Theta) = (y \alpha \theta)\), for all \(y \in Y\) and \(\alpha \in \mathcal{OP}_n(Y)\);
4. \(\operatorname{Im}(\alpha \Theta) = (\operatorname{Im}(\alpha))\theta\), for any idempotent \(\alpha \in \mathcal{OP}_n(Y)\).

**Proof** Let \(y \in Y\). Then, by (3) we have \(\alpha \epsilon_y = \epsilon_y\), for all \(\alpha \in \mathcal{OP}_n(Y)\), whence \((\alpha \Theta)(\epsilon_y \Theta) = \alpha \epsilon_y \Theta = \epsilon_{y \theta} \Theta\), for all \(\alpha \in \mathcal{OP}_n(Y)\). Since \(\Theta\) is surjective, it follows that \(\beta(\epsilon_y \Theta) = \epsilon_y \Theta\), for all \(\beta \in \mathcal{OP}_n(Z)\). In particular, given (any) \(x \in Z\), we obtain \(\epsilon_y \Theta = \epsilon_x(\epsilon_y \Theta) = \epsilon_{x \theta}(\epsilon_y \Theta)\), using also equality (2). Thus \(\epsilon_y \Theta = \epsilon_z\), with \(z = (x)(\epsilon_y \Theta) \in Z\) (which does not depend of the taken \(x \in Z\), since \(\Theta\) is a function).

Therefore, we proved property 1 and so we have a well defined function \(\theta : Y \rightarrow Z\) satisfying the equality \(\epsilon_y \Theta = \epsilon_{y \theta}\), for all \(y \in Y\). A similar reasoning applied to the inverse isomorphism \(\Theta^{-1} : \mathcal{OP}_n(Z) \rightarrow \mathcal{OP}_n(Y)\) allows us to deduce the existence of a function \(\theta' : Z \rightarrow Y\) satisfying the equality \(\epsilon_z \Theta^{-1} = \epsilon_{z \theta'}\), for all \(z \in Z\). Moreover, we have \(\epsilon_y\) = \(\epsilon_y \Theta \Theta^{-1} = \epsilon_{y \theta} \Theta^{-1} = \epsilon_{y \theta' \theta}\), for all \(y \in Y\), and similarly \(\epsilon_z = \epsilon_{z \theta' \theta}\), for all \(z \in Z\), which shows that \(\theta\) and \(\theta'\) are mutually inverse bijections. Thus, we just proved 2.

Next, we prove property 3. Let \(y \in Y\) and \(\alpha \in \mathcal{OP}_n(Y)\). Then
\[
\epsilon_{(y \theta)(\alpha \Theta)} = \epsilon_{y \theta}(\alpha \Theta) = (\epsilon_y \Theta)(\alpha \Theta) = (\epsilon_y \alpha \Theta) = \epsilon_{(y \alpha) \theta},
\]
by using equality (2), the definition of \(\theta\) and the fact that \(\Theta\) is a homomorphism, whence \((y \theta)(\alpha \Theta) = (y \alpha \theta)\).

It remains to prove 4. Let \(\alpha\) be an idempotent of \(\mathcal{OP}_n(Y)\). Then \(\alpha \Theta\) is an idempotent of \(\mathcal{OP}_n(Z)\) and so \(\operatorname{Im}(\alpha \Theta) = \operatorname{Fix}(\alpha \Theta)\). Let \(z \in \operatorname{Im}(\alpha \Theta) \subseteq Z\). Then \(z = z(\alpha \Theta)\) and, on the other hand, \(z = y \theta\), for some \(y \in Y\). Hence, by property 3, \(z = z(\alpha \Theta) = (y \theta)(\alpha \Theta) = (y \alpha) \theta \in (\operatorname{Im}(\alpha) \theta)\). Conversely, let \(z \in (\operatorname{Im}(\alpha) \theta)\). Then \(z = y \theta\), for some \(y \in \operatorname{Im}(\alpha) \subseteq Y\). As \(\alpha\) is an idempotent (hence \(\operatorname{Im}(\alpha) = \operatorname{Fix}(\alpha)\)), we have \(y = y \alpha\) and so, by using property 3, we obtain \(z = y \theta = (y \alpha) \theta = (y \theta)(\alpha \Theta) = z(\alpha \Theta) \in \operatorname{Im}(\alpha \Theta)\). Thus \(\operatorname{Im}(\alpha \Theta) = (\operatorname{Im}(\alpha)) \theta\), as required. \(\square\)
Theorem 1.3 Let $Y$ and $Z$ be nonempty subsets of $X_n$. Then $\mathcal{OP}_n(Y)$ and $\mathcal{OP}_n(Z)$ are isomorphic if and only if there exists $\sigma \in D_{2n}$ such that $Y\sigma = Z$.

Proof If there exists $\sigma \in D_{2n}$ such that $Y\sigma = Z$, then it is easy to show that the mapping $\mathcal{OP}_n(Y) \to \mathcal{OP}_n(Z)$, $\alpha \mapsto \sigma^{-1}\alpha\sigma$, is an isomorphism. Conversely, suppose there exists an isomorphism $\Theta : \mathcal{OP}_n(Y) \to \mathcal{OP}_n(Z)$ and let $\theta : Y \to Z$ be the bijection induced by $\Theta$ given by Lemma 1.2. Let $i, j \in Y$ be such that $i < j$. Take

$$A(i, j) = \{\alpha \in \mathcal{OP}_n(Y) \mid \alpha = \alpha^2 \text{ and } \text{Im(}\alpha ) = \{i < j\} \}. $$

Then, by Lemma 1.2, we have

$$A(i, j)\Theta = \{\beta \in \mathcal{OP}_n(Z) \mid \beta = \beta^2 \text{ and } \text{Im(}\beta ) = \{i\theta, j\theta\} \}. $$

Moreover, by enumerating their elements, it is not difficult to conclude that $|A(i, j)| = (j - i)(n - (j - i))$ and $|A(i, j)\Theta| = |j\theta - i\theta|(n - |j\theta - i\theta|)$. As $\Theta$ is an isomorphism, we get $|A(i, j)\Theta| = |A(i, j)|$, i.e. $(j - i)(n - (j - i)) = |j\theta - i\theta|(n - |j\theta - i\theta|)$, whence $|j\theta - i\theta| = j - i$ or $|j\theta - i\theta| = n - (j - i)$. Therefore, by Proposition 1.1, it follows that $\theta$ is a restriction of some permutation $\sigma$ of $D_{2n}$ and so $Z = Y\theta = Y\sigma$, as required. \qed

Now, observing that we can replace $\mathcal{OP}_n$ by $\mathcal{OR}_n$ in the proof of both Lemma 1.2 and Theorem 1.3 (also observe that all the idempotents of $\mathcal{OR}_n$ belong to $\mathcal{OP}_n$), we obtain an analogous characterization:

Theorem 1.4 Let $Y$ and $Z$ be nonempty subsets of $X_n$. Then $\mathcal{OR}_n(Y)$ and $\mathcal{OR}_n(Z)$ are isomorphic if and only if there exists $\sigma \in D_{2n}$ such that $Y\sigma = Z$.

We finish this section by determining the cardinality of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, for each nonempty subset $Y$ of $X_n$.

Recall that Catarino and Higgins showed in [4] that $|\mathcal{OP}_n| = n(2n - n - 1)$ and $|\mathcal{OR}_n| = n(2n - n - 1) + n$. One of the methods used by Catarino and Higgins to find this formula for $|\mathcal{OP}_n|$ consisted of counting the number of transformations of rank $k$, for $1 \leq k \leq n$. In fact, they showed that, for $2 \leq k \leq n$, the number of transformations with a fixed image of size $k$ is $k\binom{n}{k}$. In addition, $\mathcal{OP}_n$ has $n$ constant transformations.

Now, let $Y$ be a subset of $X_n$, with $1 \leq |Y| = r \leq n$. Since the number of distinct images contained in $Y$ of size $k$ is $\binom{n}{k}$, we get

$$|\{\alpha \in \mathcal{OP}_n(Y) \mid |\text{Im(}\alpha )| = k\}| = k\binom{n}{k}\binom{r}{k}, $$

for $2 \leq k \leq r$. As $\mathcal{OP}_n(Y)$ also contains $r$ constant transformations, it follows that

$$|\mathcal{OP}_n(Y)| = r + \sum_{k=2}^{r} k\binom{r}{k}\binom{n}{k} = \sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} - r(n - 1). $$

Noticing that $\sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} = r\binom{n+r-1}{r}$ (Combinatorial Identity (3.30) of [13, page 25]), we obtain:

Theorem 1.5 Let $Y$ be a subset of $X_n$ of size $r$, for $1 \leq r \leq n$. Then $|\mathcal{OP}_n(Y)| = r\binom{n+r-1}{r} - r(n - 1)$.

Regarding $\mathcal{OR}_n$, Catarino and Higgins [4] proved that, for $3 \leq k \leq n$, the number of transformations with a fixed image of size $k$ is $2k\binom{n}{k}$. Moreover, they also observed that $\{\alpha \in \mathcal{OR}_n \mid |\text{Im(}\alpha )| \leq 2\} = \{\alpha \in \mathcal{OP}_n \mid |\text{Im(}\alpha )| \leq 2\}$. Therefore

$$|\mathcal{OR}_n(Y)| = r + 2\binom{r}{2}\binom{n}{2} + \sum_{k=3}^{r} 2k\binom{r}{k}\binom{n}{k} = 2\sum_{k=1}^{r} k\binom{r}{k}\binom{n}{k} - 2\binom{r}{2}\binom{n}{2} - 2rn + r $$

and so, by using again the above combinatorial identity, we obtain:

Theorem 1.6 Let $Y$ be a subset of $X_n$ of size $r$, for $1 \leq r \leq n$. Then $|\mathcal{OR}_n(Y)| = 2r\binom{n+r-1}{r} - \frac{rn}{2}(rn - r - n + 5) + r$.

Notice that the sizes of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$ just depend of the size of $Y$. 

4
2 Regularity and Green’s relations

Recall that Catarino and Higgins showed in [4] that both $\mathcal{OP}_n$ and $\mathcal{OR}_n$ are regular semigroups. Thus, Green’s relations $\mathcal{L}$ and $\mathcal{R}$ in $\mathcal{OP}_n$ and in $\mathcal{OR}_n$ are just restrictions of the correspondent relations in $\mathcal{T}_n$. This is also the case for Green’s relation $\mathcal{D}$, as proved by Catarino and Higgins in [4]. Therefore, if $\alpha, \beta \in \mathcal{OP}_n$ [respectively, $\alpha, \beta \in \mathcal{OR}_n$], we have

1. $\alpha \mathcal{L} \beta$ if and only if $\operatorname{Im}(\alpha) = \operatorname{Im}(\beta)$,
2. $\alpha \mathcal{R} \beta$ if and only if $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\beta)$ and
3. $\alpha \mathcal{D} \beta$ if and only if $|\operatorname{Im}(\alpha)| = |\operatorname{Im}(\beta)|$,

in $\mathcal{OP}_n$ [respectively, $\mathcal{OR}_n$]. Regarding Green’s relation $\mathcal{H}$, if $\alpha$ is an element of $\mathcal{OP}_n$ of rank $k$, for $1 \leq k \leq n$, then the $\mathcal{H}$-class in $\mathcal{OP}_n$ of $\alpha$ has $k$ elements (all with the same domain and image): in particular, the $\mathcal{H}$-class in $\mathcal{OP}_n$ of an idempotent of rank $k$, for $1 \leq k \leq n$, is a cycle group of order $k$; if $\alpha$ is an element of $\mathcal{OR}_n$ of rank $k$, for $3 \leq k \leq n$, then the $\mathcal{H}$-class in $\mathcal{OR}_n$ of $\alpha$ has $2k$ elements (all with the same domain and image): in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n$ of an idempotent of rank $k$, for $3 \leq k \leq n$, is a dihedral group of order $2k$; if $\alpha$ is an element of $\mathcal{OR}_n$ of rank 1 or 2, then the $\mathcal{H}$-class in $\mathcal{OR}_n$ of $\alpha$ has 1 or 2 elements (all with the same domain and image), respectively; in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n$ of an idempotent of rank 1 or 2 is a cycle group of order 1 or 2, respectively. These facts were proved in [4]. See also [16].

Let $Y$ be a nonempty subset of $X_n$ of size $r$ ($1 \leq r \leq n$). In this section we discuss regularity and give descriptions for Green’s relations of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$.

Let

$$\mathcal{FOP}_n(Y) = \{ \alpha \in \mathcal{OP}_n(Y) \mid \operatorname{Im}(\alpha) = Y \alpha \}$$

and

$$\mathcal{FOR}_n(Y) = \{ \alpha \in \mathcal{OR}_n(Y) \mid \operatorname{Im}(\alpha) = Y \alpha \}.$$  

Clearly $\mathcal{FOP}_n(Y)$ and $\mathcal{FOR}_n(Y)$ are right ideals of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, respectively. In fact, we will show that these subsets determine relevant aspects of the structure of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. Regarding the regularity, we have:

**Theorem 2.1** Let $Y$ be a nonempty subset of $X_n$. Then $\mathcal{FOP}_n(Y)$ [respectively, $\mathcal{FOR}_n(Y)$] is the set of all regular elements of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$]. Furthermore:

1. $\mathcal{FOP}_n(Y)$ [respectively, $\mathcal{FOR}_n(Y)$] is the largest regular subsemigroup of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$];
2. $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$] is regular if and only if $|Y| = 1$ or $Y = X_n$.

**Proof** If $\alpha$ is a regular element of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$], then $\alpha = \alpha \beta \alpha$, for some $\beta \in \mathcal{OP}_n(Y)$ [respectively, $\beta \in \mathcal{OR}_n(Y)$], whence $X \alpha = (X \alpha) \beta \alpha \subseteq Y \alpha$ and so $X \alpha = Y \alpha$, i.e. $\alpha \in \mathcal{FOP}_n(Y)$ [respectively, $\alpha \in \mathcal{FOR}_n(Y)$].

Conversely, take $\alpha \in \mathcal{FOP}_n(Y)$ [respectively, $\alpha \in \mathcal{FOR}_n(Y)$] and suppose that $\operatorname{Im}(\alpha) = \{ a_1 < \cdots < a_k \}$, for some $a_1, \ldots, a_k \in Y$, with $1 \leq k \leq |Y|$. Then there exist $b_1, \ldots, b_k \in Y$ such that $b_i \alpha = a_i$, for $1 \leq i \leq k$. Let $\beta$ be the transformation of $X_n$ defined by

$$x \beta = \begin{cases} b_k & \text{if } 1 \leq x < a_1 \text{ or } a_k \leq x \leq n, \\ b_j & \text{if } a_j \leq x < a_{j+1} \text{ and } 1 \leq j \leq k - 1. \end{cases}$$

It is easy to check that $\beta \in \mathcal{OP}_n(Y)$ [respectively, $\beta \in \mathcal{OR}_n(Y)$] and $\alpha = \alpha \beta \alpha$. Thus $\alpha$ is a regular element of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$].

Hence, we proved that $\mathcal{FOP}_n(Y)$ [respectively, $\mathcal{FOR}_n(Y)$] is the set of all regular elements of $\mathcal{OP}_n(Y)$ [respectively, $\mathcal{OR}_n(Y)$].

Statement 1 is obvious, since $\mathcal{FOP}_n(Y)$ and $\mathcal{FOR}_n(Y)$ are subsemigroups of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$, respectively.

Regarding statement 2, if $Y$ is a proper subset of $X_n$ such that $|Y| \geq 2$ then we may consider two distinct elements $y, y' \in Y$ and an element $z \in X_n \setminus Y$. Thus, we define a transformation $\alpha$ on $X_n$ by $x \alpha = y$, if $x = z$, and $x \alpha = y'$, if $x \in X_n \setminus \{ z \}$. Clearly, $\alpha \in \mathcal{OP}_n(Y)$ and, since $\operatorname{Im}(\alpha) = \{ y, y' \}$ and $Y \alpha = \{ y' \}$, $\alpha$ is not regular both in $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. For the converse, we already recalled that $\mathcal{OP}_n = \mathcal{OP}_n(X_n)$ and $\mathcal{OR}_n = \mathcal{OR}_n(X_n)$ are regular semigroups and, on the other hand, if $|Y| = 1$ then $\mathcal{OP}_n(Y) = \mathcal{OR}_n(Y)$ is trivial (it is just formed by the constant mapping with range $Y$), whence a regular semigroup, as required. □

We finish this section with the following characterization of Green’s relations in $\mathcal{OP}_n(Y)$ and in $\mathcal{OR}_n(Y)$:
Theorem 2.2 Let $Y$ be a nonempty subset of $X_n$ of size $r$. Let $S = \mathcal{OP}_n(Y)$ or $S = \mathcal{OR}_n(Y)$. Let $\alpha, \beta \in S$. Then:

1. $\alpha \mathcal{L} \beta$ in $S$ if and only if either $\alpha = \beta$ or both $\alpha$ and $\beta$ are regular and $\text{Im}(\alpha) = \text{Im}(\beta)$;
2. $\alpha \mathcal{R} \beta$ in $S$ if and only if $\ker(\alpha) = \ker(\beta)$;
3. $\alpha \mathcal{D} \beta$ in $S$ if and only if either (i) both $\alpha$ and $\beta$ are regular and $|\text{Im}(\alpha)| = |\text{Im}(\beta)|$ or (ii) both $\alpha$ and $\beta$ are not regular and $\ker(\alpha) = \ker(\beta)$;
4. If $\alpha$ is a non regular element of $\mathcal{OP}_n(Y)$ then the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of $\alpha$ is trivial; if $\alpha$ is a regular element of $\mathcal{OP}_n(Y)$ of rank $k$, for $1 \leq k \leq r$, then the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of $\alpha$ has $k$ elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OP}_n(Y)$ of an idempotent of rank $k$, for $1 \leq k \leq r$, is a cycle group of order $k$;
5. If $\alpha$ is a non regular element of $\mathcal{OR}_n(Y)$ then the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of $\alpha$ is trivial; if $\alpha$ is a regular element of $\mathcal{OR}_n(Y)$ of rank $k$, for $3 \leq k \leq r$, then the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of $\alpha$ has $2k$ elements (all with the same domain and image); in particular, the $\mathcal{H}$-class in $\mathcal{OR}_n(Y)$ of an idempotent of rank $1$ or $2$ is a cycle group of order $1$ or $2$, respectively.

Proof 1. First, suppose that $\alpha \mathcal{L} \beta$ in $S$. Then $\alpha = \gamma \beta$ and $\beta = \lambda \alpha$, for some $\gamma, \lambda \in S^1$. If $\alpha \neq \beta$ then $\gamma \neq 1$ and $\lambda \neq 1$, whence $X \alpha = X \gamma \beta \subseteq Y \beta \subseteq X \beta = X \lambda \alpha \subseteq Y \alpha \subseteq X \alpha$ and so $\text{Im}(\alpha) = Y \alpha = Y \beta = \text{Im}(\beta)$, i.e. $\alpha$ and $\beta$ are regular and $\text{Im}(\alpha) = \text{Im}(\beta)$.

The converse is obvious, since $\text{Im}(\alpha) = \text{Im}(\beta)$ implies $\alpha \mathcal{L} \beta$ in $T_n$ and so, being either $\alpha = \beta$ or $\alpha$ and $\beta$ are regular elements in $S$, it follows that $\alpha \mathcal{L} \beta$ in $S$ (in fact, no matter who is the subsemigroup $S$ of $T_n$).

2. If $\alpha \mathcal{R} \beta$ in $S$ then $\alpha \mathcal{R} \beta$ in $T_n$ and so $\ker(\alpha) = \ker(\beta)$.

In order to prove the converse, it is useful to introduce the following notation. Let $I = \{i_1 < \cdots < i_k\}$ be a nonempty subset of $X_n$ ($1 \leq k \leq n$). Define the transformation $\iota_I$ of $T_n$ by

$$\iota_I = \begin{cases} i_k & \text{if } 1 \leq x < i_1 \text{ or } i_k \leq x \leq n \\ i_j & \text{if } i_j \leq x < i_{j+1} \text{ and } 1 \leq j \leq k-1. \end{cases}$$

Clearly, $\iota_I$ is an idempotent of $\mathcal{OP}_n$, with image (fixed points) $I$. In particular, if $I \subseteq Y$ then $\iota_I \in \mathcal{OP}_n(Y)$.

Now, take $T = \mathcal{OP}_n$ for $S = \mathcal{OP}_n(Y)$ and $T = \mathcal{OR}_n$ for $S = \mathcal{OR}_n(Y)$. Suppose that $\ker(\alpha) = \ker(\beta)$. Then $\alpha \mathcal{R} \beta$ in $T$ and so $\alpha = \beta \gamma$ and $\beta = \alpha \lambda$, for some $\gamma, \lambda \in T$. Thus, it is clear that we also have $\alpha = \beta \iota_{\text{Im}(\beta)} \gamma$ and $\beta = \alpha \iota_{\text{Im}(\alpha)} \lambda$. Moreover, since $\iota_{\text{Im}(\beta)} \gamma$ and $\iota_{\text{Im}(\alpha)} \lambda$ have the same rank of $\alpha$ (and $\beta$), it follows that $\text{Im}(\iota_{\text{Im}(\beta)} \gamma) = \text{Im}(\alpha)$ and $\text{Im}(\iota_{\text{Im}(\alpha)} \lambda) = \text{Im}(\beta)$, whence $\iota_{\text{Im}(\beta)} \gamma, \iota_{\text{Im}(\alpha)} \lambda \in S$ and so $\alpha \mathcal{R} \beta$ in $S$.

3. If $\alpha \mathcal{D} \beta$ in $S$ then there exists $\gamma \in S$ such that $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$ in $S$. On the other hand, we also have $\alpha \mathcal{D} \beta$ in $T_n$ and so $|\text{Im}(\alpha)| = |\text{Im}(\beta)|$. Moreover, as in any semigroup, either both $\alpha$ and $\beta$ are regular or both are non regular. Hence, in case both $\alpha$ and $\beta$ are non regular, it follows that $\gamma$ is also non regular and thus $\alpha = \gamma$ and $\ker(\alpha) = \ker(\gamma) = \ker(\beta)$.

Conversely, if (both $\alpha$ and $\beta$ are non regular) and $\ker(\alpha) = \ker(\beta)$ then $\alpha \mathcal{R} \beta$ in $S$ and so $\alpha \mathcal{D} \beta$ in $S$. On the other hand, suppose that both $\alpha$ and $\beta$ are regular and $|\text{Im}(\alpha)| = |\text{Im}(\beta)|$. Then $\alpha \mathcal{D} \beta$ in $T$ (with $T$ as defined above) and so $\alpha = \gamma_1 \beta_2$ and $\beta = \lambda_1 \alpha \lambda_2$, for some $\gamma_1, \beta_2, \lambda_1, \lambda_2 \in T$. Let $\beta' \in S$ be an inverse of $\beta$. Then $\alpha = (\gamma_1 \beta_2 \beta') \beta' \beta_2 \gamma_2$. Clearly, $\gamma_1 \beta_2 \beta' \gamma_2 \in S$. On the other hand, since $\beta' \beta_2 \gamma_2$ has the same rank as $\alpha$, it should have the same image as $\alpha$, whence also $\beta' \beta_2 \gamma_2 \in S$. Similarly, if $\alpha' \in S$ is an inverse of $\alpha$ then $\beta = (\lambda_1 \alpha \alpha') \alpha (\alpha' \lambda_2)$ and $\lambda_1 \alpha \alpha', \alpha' \lambda_2 \in S$. Thus, also in this case, $\alpha \mathcal{D} \beta$ in $S$.

Properties 4 and 5 are immediate, since $\mathcal{L}$-classes in $S$ of non regular elements are trivial and it is clear that $\mathcal{H}$-classes in $T$ (with $T$ as defined above) of regular elements of $S$ must coincide with the respective $\mathcal{H}$-classes in $S$. □

3 Ranks

Let $Y$ be a nonempty subset of $X_n$. In this section we determine the ranks of the semigroups $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$. Surprisingly, as opposed to the case of $O_n(Y)$ [8], we will show that the ranks of $\mathcal{OP}_n(Y)$ and $\mathcal{OR}_n(Y)$ only depend of the size of $Y$.

It is well known that $\mathcal{OP}_n$ and $\mathcal{OR}_n$ have, respectively, ranks 2 and 3 (see [2, 4]). Therefore, in what follows, we suppose that $Y$ is a proper subset of $X_n$. Let $r = |Y|$. We begin by showing that $\mathcal{OP}_n(Y)$ is generated by its elements of rank $r$.
Lemma 3.1 Any transformation of \( \mathcal{OP}_n(Y) \) of rank \( k \) is a product of two elements of \( \mathcal{OP}_n(Y) \) of rank \( k+1 \), for \( 1 \leq k < r \).

Proof Let \( \alpha \) be an element of \( \mathcal{OP}_n(Y) \) of rank \( k \). By [4, Theorem 2.6], there exist \( \beta \in \mathcal{O}_n \) and \( 0 \leq t < n \) such that \( \alpha = g^t \beta \), where \( g \) is the permutation of \( X_n \), defined in Section 1. Since \( g^t \) is a permutation, then \( \text{Im}(\beta) = \text{Im}(\alpha) \) and so, in addition, \( \beta \in \mathcal{O}_n(Y) \). Now, by [8, Lemma 3.5], there exist transformations \( \beta_1, \beta_2 \in \mathcal{O}_n(Y) \) of rank \( k+1 \) such that \( \beta = \beta_1 \beta_2 \). Let \( \alpha_1 = g^t \beta_1 \). Then \( \beta_1 \in \mathcal{OP}_n \). Again, since \( g^t \) is a permutation, we have \( \text{Im}(\alpha_1) = \text{Im}(\beta_1) \), whence \( \alpha_1 \) is an element of \( \mathcal{OP}_n(Y) \) with rank \( k+1 \). Thus, \( \alpha = \alpha_1 \beta_2 \), with both \( \alpha_1 \) and \( \beta_2 \) elements of \( \mathcal{OP}_n(Y) \) of rank \( k+1 \), as required. \( \Box \)

From this lemma, it follows immediately that \( \mathcal{OP}_n(Y) \) is generated by its elements of rank \( r \).

Now, suppose that \( Y = \{y_1 < y_2 < \cdots < y_r\} \). Let

\[
\tilde{g}_Y = \begin{pmatrix}
A_1 & A_2 & \cdots & A_{r-1} & A_r \\
y_2 & y_3 & \cdots & y_r & y_1
\end{pmatrix} \in \mathcal{OP}_n(Y),
\]

where \( A_j = \{y_j, \ldots, y_{j+1} - 1\}, 1 \leq j \leq r - 1, \) and \( A_r = \{y_r, \ldots, n, 1, \ldots, y_1 - 1\} \).

Lemma 3.2 Let \( \alpha, \beta \in \mathcal{OP}_n(Y) \) be two elements of rank \( r \) such that \( \text{Ker}(\beta) = \text{Ker}(\alpha) \). Then \( \beta = \alpha \tilde{g}_Y^k \), for some \( k \in \{0, \ldots, r-1\} \).

Proof Suppose that \( I_1, I_2, \ldots, I_k \) are the kernel classes of \( \alpha \) (and \( \beta \)) in order \( \max I_i < \max I_{i+1} \), for \( i = 1, \ldots, r - 1 \). Then

\[
\alpha = \begin{pmatrix}
I_1 & I_2 & \cdots & I_{r-i} & I_{r-i+1} & I_{r-i+2} & \cdots & I_r \\
y_{i+1} & y_{i+2} & \cdots & y_r & y_1 & y_2 & \cdots & y_i
\end{pmatrix}
\]

and

\[
\beta = \begin{pmatrix}
I_1 & I_2 & \cdots & I_{r-j} & I_{r-j+1} & I_{r-j+2} & \cdots & I_r \\
y_{j+1} & y_{j+2} & \cdots & y_r & y_1 & y_2 & \cdots & y_j
\end{pmatrix},
\]

for some \( 1 \leq i, j \leq r \). Take \( k = j - i \), if \( i \leq j \), and \( k = r - i + j \), otherwise. Hence, it is a routine matter to prove that \( \beta = \alpha \tilde{g}_Y^k \), as required. \( \Box \)

Now, notice that any generating set of \( \mathcal{OP}_n(Y) \) (and of \( \mathcal{OR}_n(Y) \)) must contain at least one element from each distinct kernel of transformations of rank \( r \). On the other hand, the number of distinct kernels of transformations of \( \mathcal{OP}_n(Y) \) (and of \( \mathcal{OR}_n(Y) \)) of rank \( r \) coincides with the number of distinct kernels of transformations of \( \mathcal{OP}_n \) of rank \( r \), which is precisely \( \binom{n}{r} \) (see [4]). These observations, together with the previous two lemmas, prove the following result.

Theorem 3.3 The semigroup \( \mathcal{OP}_n(Y) \) is generated by any subset of transformations of rank \( r \) containing \( \tilde{g}_Y \) and at least one element from each distinct kernel. Furthermore, \( \mathcal{OP}_n(Y) \) has rank equal to \( \binom{n}{r} \).

Next, let

\[
\tilde{h}_Y = \begin{pmatrix}
B_1 & B_2 & \cdots & B_{r-1} & B_r \\
y_r & y_{r-1} & \cdots & y_2 & y_1
\end{pmatrix} \in \mathcal{OR}_n(Y),
\]

where \( B_1 = \{1, \ldots, y_1, y_r + 1, \ldots, n\} \) and \( B_j = \{y_j - 1 + 1, \ldots, y_j\}, 2 \leq j \leq r \).

Notice that

\[
\text{Ker}(\tilde{h}_Y) = \text{Ker}(\tilde{g}_Y) \iff B_j = A_j, 1 \leq j \leq r \iff B_j = A_j = \{y_j\}, 1 \leq j \leq r \iff r = n,
\]

whence \( \tilde{h}_Y \) and \( \tilde{g}_Y \) have distinct kernels. On the other hand, clearly

\[
\tilde{h}_Y^2 = \begin{pmatrix}
B_1 & B_2 & \cdots & B_{r-1} & B_r \\
y_1 & y_2 & \cdots & y_{r-1} & y_r
\end{pmatrix} \in \mathcal{OP}_n(Y)
\]

is a right identity of \( \mathcal{OR}_n(Y) \). Thus, if \( \alpha \in \mathcal{OR}_n(Y) \setminus \mathcal{OP}_n(Y) \) then \( \alpha = (\alpha \tilde{h}_Y) \tilde{h}_Y \), with \( \text{Ker}(\alpha) = \text{Ker}(\alpha \tilde{h}_Y) \) and \( \alpha \tilde{h}_Y \in \mathcal{OP}_n(Y) \). Therefore, it is easy to conclude:

Theorem 3.4 The semigroup \( \mathcal{OR}_n(Y) \) is generated by any subset of transformations of rank \( r \) containing both \( \tilde{g}_Y \) and \( \tilde{h}_Y \) and at least one element from each distinct kernel. Furthermore, \( \mathcal{OR}_n(Y) \) has rank equal to \( \binom{n}{r} \).
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VÍTOR H. FERNANDES, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal; also: Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal; e-mail: vhf@fct.unl.pt

PREEYANUCH HONYAM, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; e-mail: preeyanuch_h@hotmail.com

TERESA M. QUINTEIRO, Instituto Superior de Engenharia de Lisboa, Rua Conselheiro Emídio Navarro 1, 1950-062 Lisboa, Portugal; also: Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal; e-mail: tmelo@adm.isel.pt

BOORAPA SINGHA, Department of Mathematics and Statistics, Faculty of Science and Technology, Chiang Mai Rajabhat University, Chiang Mai 50300, Thailand; e-mail: boorapas@yahoo.com