Total Rainbow k-connection in Graphs

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Abstract

Let k be a positive integer and G be a k-connected graph. In 2009, Chartrand, Johns, McKeon, and Zhang introduced the *rainbow k-connection number* $rc_k(G)$ of G. An edge-coloured path is *rainbow* if its edges have distinct colours. Then, $rc_k(G)$ is the minimum number of colours required to colour the edges of G so that any two vertices of G are connected by k internally vertex-disjoint rainbow paths. The function $rc_k(G)$ has since been studied by numerous researchers. An analogue of the function $rc_k(G)$ involving vertex colourings, the *rainbow vertex k-connection number* $rvc_k(G)$, was subsequently introduced. In this paper, we introduce a version which involves total colourings. A total-coloured path is *total-rainbow* if its edges and internal vertices have distinct colours. The *total rainbow k-connection number* of G, denoted by $trc_k(G)$, is the minimum number of colours required to colour the edges and vertices of G, so that any two vertices of G are connected by k internally vertex-disjoint total-rainbow paths. We study the function $trc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. We also compare the functions $rc_k(G)$, $rvc_k(G)$, and $trc_k(G)$, by considering how close and how far apart $trc_k(G)$ can be from $rc_k(G)$ and $rvc_k(G)$.

Keywords: Graph colouring, rainbow (vertex) connection number, k-connected

AMS subject classifications: 05C15, 05C40

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1 Introduction

In this paper, all graphs are finite, simple, and undirected. For undefined terms in graph theory, we refer the reader to the book by Bollobás [1]. A set of internally vertex-disjoint paths will be called *disjoint*. An edge-coloured path is *rainbow* if its edges have distinct colours. For a positive integer k, an edge-colouring of a k-connected graph G, not necessarily proper, is *rainbow k-connected* if any two vertices of G are connected by k disjoint rainbow paths. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum integer t such that there exists a rainbow k-connected colouring of G using t colours. By Menger's theorem [13], a graph is k-connected if and only if any two vertices are connected by k disjoint paths. Hence, $rc_k(G)$ is well-defined if and only if G is k-connected. The function $rc_k(G)$ was first introduced by Chartrand et al. ([2] for $k = 1$ in 2008, and [3] for general k in 2009) and has since been very well-studied. For an overview of the rainbow connection subject, we refer the reader to the survey by Li et al. [8], and the book by Li and Sun [11].

A vertex-coloured path is *vertex-rainbow* if its internal vertices have distinct colours. A vertex-colouring of a k-connected graph G, not necessarily proper, is *rainbow vertex k-connected* if any two vertices of G are connected by k disjoint vertex-rainbow paths. The *rainbow vertex k-connection number* of G, denoted by $rvc_k(G)$, is the minimum integer t such that there exists a rainbow vertex k-connected colouring of G using t colours. Again, $rvc_k(G)$ is well-defined if and only if G is k-connected. The function $rvc_k(G)$ was introduced by Krivelevich and Yuster [7] (for $k = 1$ in 2010), and by the authors [12] (for general k in 2012).

Here, we consider an analogous function using total colourings. A total-coloured path is *total-rainbow* if its edges and internal vertices have distinct colours. A total colouring of a k-connected graph G, not necessarily proper, is *total-rainbow k-connected* if any two vertices of G are connected by k disjoint total-rainbow paths. The *total rainbow k-connection number* of G, denoted by $trc_k(G)$, is the minimum integer t such that, there exists a total-rainbow k-connected colouring of G using t colours. We have $trc_k(G)$ is well-defined if and only if G is k-connected. We write $trc(G)$ for $trc_1(G)$, and similarly for $rc(G)$ and $rvc(G)$.

For a non-trivial connected graph G, many observations about the function $trc_k(G)$ can be made. We have $trc(G) = 1$ if and only if G is a complete graph, and $trc(G) \geq 3$ if G is not complete. A simple upper bound is $trc(G) \leq n - 1 + q$, where $|V(G)| = n$ and q is the number of vertices of G with degree at least 2, and equality holds if and only if G is a tree (see Proposition 1). If G is k-connected, then $trc_k(G) \geq 3$ if $k \geq 2$, and $trc_k(G) \geq 2\,\mathrm{diam}(G) - 1$ for $k \geq 1$, where $\mathrm{diam}(G)$ denotes the *diameter* of G. In relation to $rc_k(G)$ and $rvc_k(G)$, we have $trc_k(G) \geq \max(rc_k(G), rvc_k(G))$. Also, if $rc_k(G) = 2$, then $trc_k(G) = 3$. If $rvc_k(G) \geq 2$, then $trc_k(G) \geq 5$.

In the rest of this paper, we will study the function $trc_k(G)$ when G is a cycle, a wheel, a complete bipartite graph, and a complete multipartite graph. We will also compare the functions $trc_k(G)$, $rc_k(G)$, and $rvc_k(G)$, by considering how close and how far apart $trc_k(G)$ can be from each of $rc_k(G)$, $rvc_k(G)$, and $\max(rc_k(G), rvc_k(G))$.

2 Total Rainbow k-connection Numbers of some Graphs

In this section, we first derive the upper bound on $trc(G)$ as mentioned in Introduction. Then, we study the function $trc_k(G)$ for some specific graphs G.

Proposition 1. *Let G be a connected graph on n vertices, with q vertices having degree at least 2. Then, $trc(G) \leq n-1+q$, with equality if and only if G is a tree.*

Proof. The proposition is trivial for $n = 1, 2$. Now, let $n \geq 3$.

First, we show that $trc(G) \leq n-1+q$. Take a spanning tree T of G, and note that T has at most q non-leaves. Then, take a total colouring of G with at most $n-1+q$ colours such that the edges and non-leaves of T have distinct colours. We have a total-rainbow connected colouring for G.

Now, let G be a tree. Suppose that we have a total colouring for G with fewer than $n-1+q$ colours. Then, either there are two edges, or two non-leaves, or an edge and a non-leaf, with the same colour. In each case, we can find two vertices $u, v \in V(G)$ such that the unique $u-v$ path in G is not total-rainbow. Hence, $trc(G) \geq n-1+q$.

Conversely, if G is not a tree, then G contains a spanning unicyclic subgraph G'. Let C be the unique cycle of G'. We recall from Chartrand et al. [2, Proposition 2.1] that $rc(C_3) = 1$ and $rc(C_p) = \lceil \frac{p}{2} \rceil$ for $p \geq 4$, where C_p is the cycle of order p. Now, consider a total colouring of G' where C is given a rainbow connected colouring with $rc(C)$ colours, and the vertices of degree at least 2 in G' and the edges of $E(G') \setminus E(C)$ are given distinct colours. Then, we have a total-rainbow connected colouring for G', and hence for G, with at most $e(G') - e(C) + rc(C) + q \leq n-2+q$ colours. Therefore, $trc(G) < n-1+q$. □

Now, we study the function $trc_k(G)$ for some specific graphs G. Let $\kappa(G) = \max\{k : G$ is k-connected$\}$ denote the *vertex connectivity* of G. Note that $trc_k(G)$ is well-defined if and only if $1 \leq k \leq \kappa(G)$. We first consider the case when G is a cycle. Let C_n denote the cycle of order n, and note that $\kappa(C_n) = 2$.

Theorem 2.

(a) *For $3 \leq n \leq 12$, the values of $trc(C_n)$ are given in the following table.*

n	3	4	5	6	7	8	9	10	11	12
$trc(C_n)$	1	3	3	5	6	7	8	9	11	11

For $n \geq 13$, we have $trc(C_n) = n$.

(b) *$trc_2(C_3) = 3$, $trc_2(C_4) = 6$, and $trc_2(C_n) = 2n$ for $n \geq 5$.*

Proof. Throughout, let the vertices of C_n be $v_0, \dots, v_{n-1}$, with the indices taken cyclically modulo n.

(a) One can easily verify that $trc(C_3) = 1$ and $trc(C_4) = trc(C_5) = 3$. Now, let $n \geq 6$. We first prove the upper bounds. Define a total colouring c_n of C_n, using colours $0, 1, 2, \dots$, as follows. For $6 \leq n \leq 10$ or $n = 12$, let $c_n(v_0) = c_n(v_{\lfloor n/3 \rfloor}) = c_n(v_{\lfloor 2n/3 \rfloor}) = 0$, and $c_n(v_1) = c_n(v_{\lfloor n/3 \rfloor + 1}) = c_n(v_{\lfloor 2n/3 \rfloor + 1}) = 1$. Then, colour $v_0v_1, v_1v_2, v_2, v_2v_3, v_3, \dots, v_{n-1}, v_{n-1}v_0$ with the colours $2, 3, \dots, n-2, 2, 3, \dots, n-2$, omitting $v_0, v_1, v_{\lfloor n/3 \rfloor}, v_{\lfloor n/3 \rfloor + 1}, v_{\lfloor 2n/3 \rfloor}$ and $v_{\lfloor 2n/3 \rfloor + 1}$. Note that each of the colours $2, 3, \dots, n-2$ appear exactly twice. See Figure 1. Then, every path of length at most $\lceil \frac{n}{2} \rceil - 1$ is total-rainbow, and when n is even, any two opposite vertices $v_i, v_{i+n/2}$ $(0 \leq i \leq n-1)$ are connected by a total-rainbow path. Hence, c_n is total-rainbow connected, and $trc(C_n) \leq n-1$.

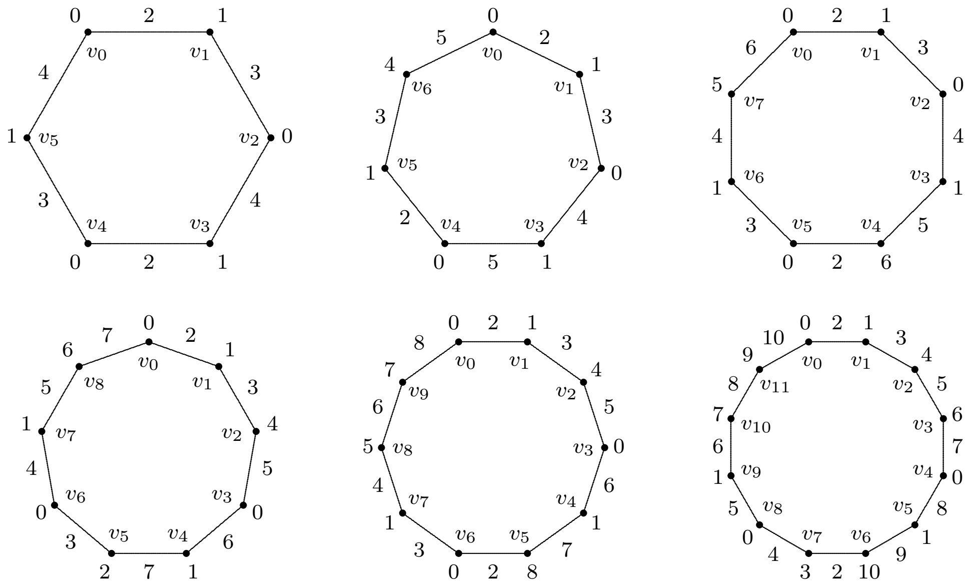

Figure 1. Total-rainbow connected colourings of C_n, for $6 \le n \le 10$ and $n = 12$.

For $n = 11$ or $n \ge 13$, and $0 \le i \le n-1$, let $c_n(v_iv_{i+1}) = i$, and $c_n(v_i) = i + \lfloor \frac{n}{2} \rfloor$ (mod n). Then, for $0 \le i \le n-1$, consider the path $Q_i = v_iv_{i+1}\cdots v_{i+\lceil n/2 \rceil}$. The edges of Q_i have colours $i, i+1, \dots, i+\lceil \frac{n}{2} \rceil - 1$ (mod n), and the internal vertices have colours $i+\lfloor \frac{n}{2} \rfloor + 1, \dots, i+n-1$ (mod n). Hence, Q_i is a total-rainbow path. Since any two vertices v_j and v_ℓ are contained in some Q_i, there exists a total-rainbow $v_j - v_\ell$ path. Hence, c_n is total-rainbow connected, and $trc(C_n) \le n$.

Now we prove the lower bounds. We have $trc(C_n) \ge 2\,\mathrm{diam}(C_n) - 1 = 2\lfloor \frac{n}{2} \rfloor - 1$, and this gives the matching lower bound for $n = 6, 8, 10, 12$. To see that $trc(C_7) \ge 6$ (resp. $trc(C_9) \ge 8$), one can check that in any total colouring of C_7 (resp. C_9) with at most five (resp. seven) colours, there exists a path of length 3 (resp. 4) which is not total-rainbow, so that the end-vertices of such a path are not connected by a total-rainbow path. Finally, for $n = 11$ or $n \ge 13$, take a total colouring of C_n with fewer than n colours. Then for some $m \in \{0, 1, 2, 3\}$, we have m edges and $3 - m$ vertices with the same colour. Without loss of generality, for some $1 < i \le \lfloor \frac{n-m}{3} \rfloor + 1$, either v_1 and v_i, or v_1 and v_iv_{i+1}, or v_0v_1 and v_iv_{i+1}, have the same colour. Then, the path $v_0v_1\cdots v_{i+1}$ is not total-rainbow, and the path $v_{i+1}v_{i+2}\cdots v_{n-1}v_0$ has at least $2n - 2\lfloor \frac{n-m}{3} \rfloor - 5 > n - 1$ edges and internal vertices, and hence is also not total-rainbow. Therefore, $trc(C_n) \ge n$.

(b) One can easily verify that $trc_2(C_3) = 3$ and $trc_2(C_4) = 6$. For $n \ge 5$, we clearly have $trc_2(C_n) \le 2n$. If we have a total colouring of C_n with fewer than $2n$ colours, then some $a, b \in E(C_n) \cup V(C_n)$ have the same colour. There exist vertices v_i and v_j such that one of the two $v_i - v_j$ paths, say P, satisfies $a, b \in E(P) \cup V(P - \{v_i, v_j\})$, so that P is not total-rainbow. Hence, $trc(C_n) \ge 2n$. $\square$

A graph closely related to the cycle C_n is the *wheel* W_n. This is the graph obtained from C_n by joining a new vertex v to every vertex of C_n. The vertex v is the *centre* of W_n. We now determine $trc_k(W_n)$. Observe that $\kappa(W_n) = 3$.

Theorem 3.

(a) $trc(W_3) = 1$, $trc(W_n) = 3$ *for* $4 \le n \le 6$, $trc(W_n) = 4$ *for* $7 \le n \le 9$, *and* $trc(W_n) = 5$ *for* $n \ge 10$.

(b) $trc_2(W_3) = 3$, $trc_2(W_4) = 3$, $trc_2(W_5) = 5$, *and* $trc_2(W_n) = trc(C_n)$ *for* $n \ge 6$ *(hence,* $trc_2(W_n)$ *is determined and given by Theorem 2(a) for* $n \ge 6$*).*

(c) $trc_3(W_3) = 4$, $trc_3(W_4) = 6$, *and* $trc_3(W_n) = 2n$ *for* $n \ge 5$.

Proof. Let v be the centre of W_n, and the vertices of the cycle C_n be $v_0, \dots, v_{n-1}$. Throughout this proof, indices of the vertices are taken cyclically modulo n.

(a) Clearly, we have $trc(W_3) = 1$. Now, let $n \ge 4$. We first prove the upper bounds. Define a total colouring f_n of W_n, using colours $0, 1, 2, \dots$, as follows. For $4 \le n \le 9$ and $0 \le i \le n-1$, let $f_n(vv_i) = \lfloor \frac{i}{3} \rfloor$, $f_n(v_iv_{i+1}) = i \pmod 3$; and $f_n(v) = f_n(v_i) = 2$ if $4 \le n \le 6$; and $f_n(v) = f_n(v_i) = 3$ if $7 \le n \le 9$. For $n \ge 10$ and $0 \le i \le n-1$, let $f_n(vv_i) = i \pmod 2$, $f_n(v_iv_{i+1}) = 2$, $f_n(v_i) = 3$, and $f_n(v) = 4$. See Figure 2 for the cases $n = 5, 7, 10$. Then, f_n is total-rainbow connected. Hence, $trc(W_n) \le 3$ for $4 \le n \le 6$, $trc(W_n) \le 4$ for $7 \le n \le 9$, and $trc(W_n) \le 5$ for $n \ge 10$.

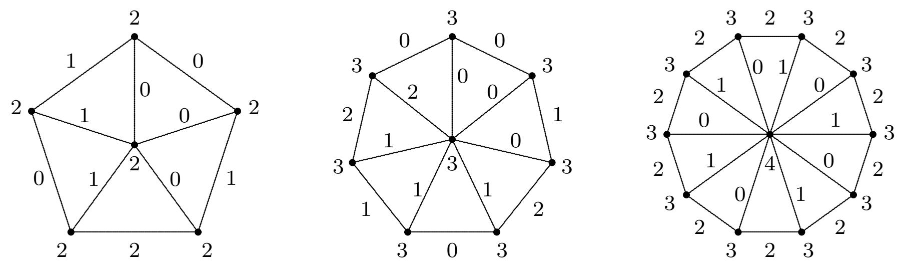

Figure 2. Total-rainbow connected colourings of W_5, W_7 and W_{10}.

Now, we prove the lower bounds. Clearly, we have $trc(W_n) \ge 3$ for $n \ge 4$. Suppose that we have a total-rainbow connected colouring f'_n of W_n with colours $0, 1, \dots, b-1$, where $b = 3$ for $7 \le n \le 9$, and $b \in \{3, 4\}$ for $n \ge 10$. Assume that $f'_n(v) = 0$. If $f'_n(vv_i) = 0$ for some $0 \le i \le n-1$, then there is no total-rainbow $v_i - v_{i+3}$ path. Hence, some four of the edges $vv_0, \dots, vv_{n-1}$ have the same colour. Assume that for some $3 \le i \le n-3$, we have $f'_n(vv_0) = f'_n(vv_i) = 1$. But then, we do not have a total-rainbow $v_0 - v_i$ path, a contradiction. Hence, $trc(W_n) \ge 4$ for $7 \le n \le 9$, and $trc(W_n) \ge 5$ for $n \ge 10$.

(b) We first prove the upper bounds. For $n \ge 3$, we define a total colouring g_n of W_n, using colours $0, 1, 2, \dots$, as follows. For g_3, g_4 and g_5, we take the total-rainbow 2-connected colourings as shown in Figure 3. Hence, $trc_2(W_3) \le 3$, $trc_2(W_4) \le 3$ and $trc_2(W_5) \le 5$.

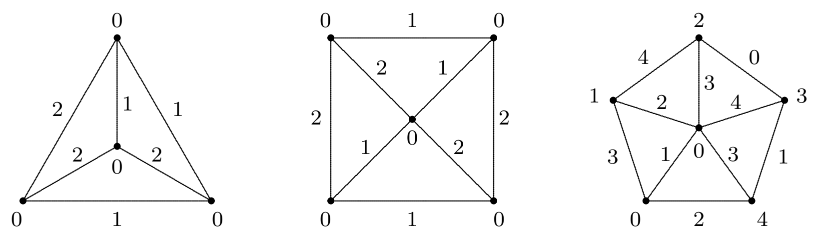

Figure 3. Total-rainbow 2-connected colourings of W_3, W_4 and W_5.

For $6 \le n \le 10$ and $n = 12$, let $g_n \equiv c_n$ when we restrict g_n to the cycle C_n, where c_n is the total colouring on C_n as defined in Theorem 2(a). Let $g_n(v) = 0$, and colour the edges vv_i, for $0 \le i \le n-1$, as in Figure 4. Note that g_n uses $trc(C_n) = n-1$ colours. In each case, g_n is total-rainbow 2-connected. Indeed, for v and a vertex v_i, we have vv_i and $vv_{i-1}v_i$ are total-rainbow. For two vertices v_i and v_j, assume that $v_iv_{i+1}\cdots v_j$ is total-rainbow (by the choice of $g_n \equiv c_n$ on C_n). If v_ivv_j is total-rainbow, then this is the second suitable $v_i - v_j$ path. Otherwise, we can take the second path to be $v_jv_{j+1}\cdots v_i$ if $n = 4, 6, 8$, or $v_iv_{i-1}vv_j$ if $n = 7, 9, 10, 12$. Hence, we have $trc_2(W_n) \le trc(C_n) = n-1$.

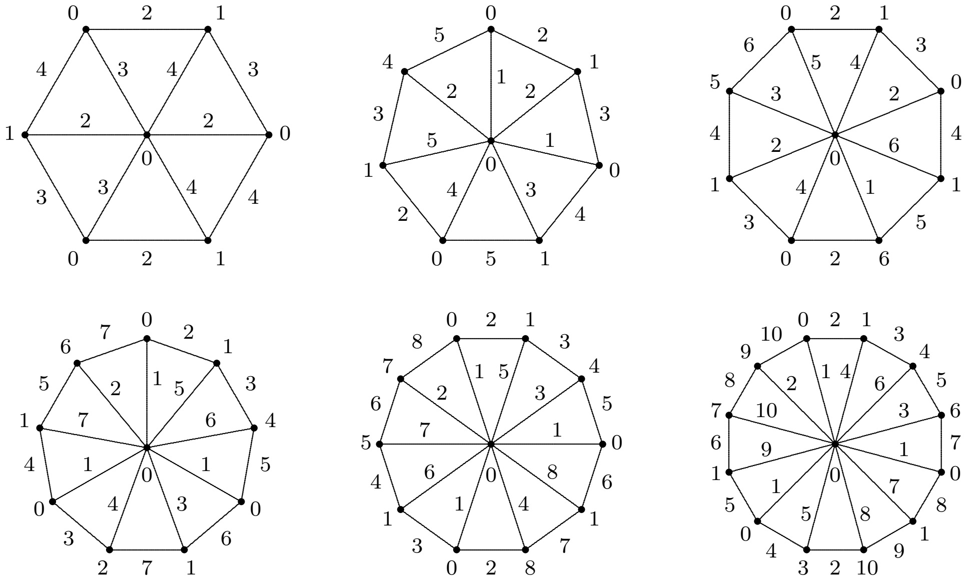

Figure 4. Total-rainbow 2-connected colourings of W_n, for $6 \le n \le 10$ and $n = 12$.

For $n = 11$ or $n \ge 13$, let $g_n \equiv c_n$ on the cycle C_n as before. Let $g_n(vv_i) = i$ for $0 \le i \le n-2$, $g_n(vv_{n-1}) = \lceil \frac{n}{2} \rceil - 1$ and $g_n(v) = n-1$. Note that g_n uses $trc(C_n) = n$ colours. We claim that g_n is total-rainbow 2-connected. For two vertices v_i and v_j, two of $v_iv_{i+1}\cdots v_j$, $v_jv_{j+1}\cdots v_i$ and v_ivv_j are total-rainbow. For v and a vertex v_i, we have vv_i and $vv_{i+1}v_i$ are total-rainbow, except for $i = n-2$ and n even, in which case we take vv_{n-2} and $vv_0v_{n-1}v_{n-2}$. Hence, we have $trc_2(W_n) \le trc(C_n) = n$.

Now, we prove the lower bounds. Clearly, we have $trc_2(W_3) \ge 3$ and $trc_2(W_4) \ge 3$. If we have a total colouring of W_5 with fewer than five colours, then without loss of generality, vv_0 and vv_i have the same colour for $i = 1$ or $i = 2$, and we do not have two disjoint total-rainbow $v_0 - v_i$ paths. Hence, $trc_2(W_5) \ge 5$. Now let $n \ge 6$. If we have a total colouring of W_n with fewer than $trc(C_n)$ colours, then there exist v_i and v_j where both of the $v_i - v_j$ paths in the C_n, say P and P', are not total-rainbow. Since any two disjoint $v_i - v_j$ paths in W_n must include at least one of P and P', we cannot have two disjoint total-rainbow $v_i - v_j$ paths in W_n. Hence, $trc_2(W_n) \ge trc(C_n)$.

(c) We first prove the upper bounds. For $n \ge 3$, we define a total colouring h_n of W_n, using colours $0, 1, 2, \ldots$, as follows. For h_3 and h_4, we take the total-rainbow 3-connected colourings as shown in Figure 5. Hence, $trc_3(W_3) \le 4$ and $trc_3(W_4) \le 6$.

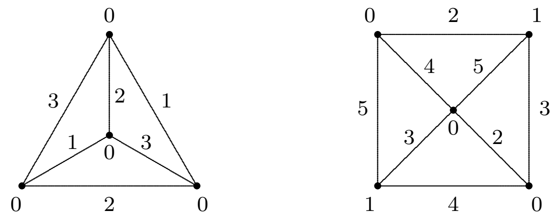

Figure 5. Total-rainbow 3-connected colourings of W_3 and W_4.

For $n \geq 5$, let $h_n(v_i) = i$, $h_n(v_iv_{i+1}) = i + n$ and $h_n(vv_i) = i + 1$ for every $0 \leq i \leq n-1$, and $h_n(v) = 0$. Then h_n is total-rainbow 3-connected, and $trc_3(W_n) \leq 2n$.

Now, we prove the lower bounds. Clearly, for a total-rainbow 3-connected colouring of W_3, we need v_0vv_1, v_1vv_2 and v_2vv_0 to be total-rainbow. This implies that $trc_3(W_3) \geq 4$. For $n \geq 4$, suppose that we have a total colouring of W_n, using fewer than $trc_2(C_n)$ colours. Then, there exist v_i and v_j such that one of the $v_i - v_j$ paths along the C_n is not total-rainbow. Hence, we cannot have three disjoint total-rainbow $v_i - v_j$ paths in W_n, and by Theorem 2(b), $trc_3(W_4) \geq 6$ and $trc_3(W_n) \geq 2n$ for $n \geq 5$. □

Next, we consider the function $trc_k(G)$ when G is a complete bipartite graph $K_{m,n}$, where $1 \leq m \leq n$. Note that $\kappa(K_{m,n}) = m$. We first determine $trc(K_{m,n})$ exactly. Clearly, $trc(K_{1,1}) = 1$ and $trc(K_{1,n}) = n + 1$ if $n \geq 2$. For $m \geq 2$, we have the following result.

Theorem 4. *For $2 \leq m \leq n$, we have $trc(K_{m,n}) = \min(\lceil \sqrt[m]{n}\, \rceil + 1, 7)$.*

Proof. Let the classes of $K_{m,n}$ be U and V, where $U = \{u_1, \dots, u_m\}$ and $|V| = n$. Let $b = \lceil \sqrt[m]{n}\, \rceil \geq 2$, and note that $m \leq n \leq 6^m$ if and only if $3 \leq b + 1 \leq 7$. Hence, we need to prove that $trc(K_{m,n}) = b + 1$ if $m \leq n \leq 6^m$, and $trc(K_{m,n}) = 7$ if $n > 6^m$.

We first prove the upper bound. Note that whenever we have constructed a total colouring of $K_{m,n}$ and want to show that it is total-rainbow connected, it suffices to verify that any two vertices in the same class of $K_{m,n}$ are connected by a total-rainbow path, since any two vertices in different classes are adjacent.

For $m \leq n \leq 6^m$, we define a total colouring f of $K_{m,n}$ with $b + 1$ colours as follows. Assign to the vertices of V distinct vectors of length m, with entries from $\{1, \dots, b\}$, such that the vectors $(2, 1, \dots, 1), (1, 2, 1, \dots, 1), \dots, (1, \dots, 1, 2)$ are all present. For $v \in V$, let v' denote the vector assigned to v. For $u_i \in U$ and $v \in V$, let $f(u_iv) = v'_i$ (where v'_i denotes the ith coordinate of v'). Let $f(w) = b + 1$ for all $w \in V(K_{m,n})$. Now for $u_i, u_j \in U$, the path u_izu_j is total-rainbow, where $z \in V$ is assigned the vector $z' = (1, \dots, 1, 2, 1, \dots, 1)$, with the 2 in the ith position. For $x, y \in V$, there exists $1 \leq i \leq m$ such that $x'_i \neq y'_i$, and the path xu_iy is total-rainbow. Hence, f is total-rainbow connected, and $trc(K_{m,n}) \leq b+1$.

For $n > 6^m$, we define a total colouring f' of $K_{m,n}$ with seven colours as follows. Let $V = V' \,\dot\cup\, V''$, where $|V'| = 3^m$. Assign the 3^m distinct vectors of length m with entries from $\{1, 2, 3\}$ to the vertices of V', and the vector $(4, 3, \dots, 3)$ (with length m) to every vertex of V''. Again for $v \in V$, let v' be the vector assigned to v. For $u_i \in U$ and $v \in V$, let $f'(u_iv) = v'_i$. Let $f'(u_1) = 5$, $f'(u_i) = 6$ for $2 \leq i \leq m$, and $f'(w) = 7$ for all $w \in V$. Then as before, any two vertices $u_i, u_j \in U$, and any two vertices $x, y \in V'$, are connected by a total-rainbow path. Now, for $x \in V'$ and $y \in V''$, the path xu_1y is total-rainbow. For $x, y \in V''$, the path xu_1zu_2y is total-rainbow, where $z \in V$ is assigned the vector $z' = (2, 1, \dots, 1)$. Hence, f' is total-rainbow connected, and $trc(K_{m,n}) \leq 7$.

Now, we prove the lower bound. If $m \le n \le 2^m$, then $trc(K_{m,n}) \ge 3 = b+1$. Next, let $2^m < n \le 6^m$. Then $(b-1)^m < n \le b^m$, with $b \in \{3,4,5,6\}$. Let g be a total colouring of $K_{m,n}$, using colours from $\{1,\dots,b\}$. For $v \in V$, assign v with the vector v' of length m, where $v'_i = g(u_i v)$ for $1 \le i \le m$. For two partitions $\mathcal{P}$ and $\mathcal{P}'$ of V, we say that $\mathcal{P}$ *refines* $\mathcal{P}'$, written as $\mathcal{P}' \prec \mathcal{P}$, if for all $A \in \mathcal{P}$, we have $A \subseteq B$ for some $B \in \mathcal{P}'$. In other words, $\mathcal{P}$ can be obtained from $\mathcal{P}'$ by partitioning some of the sets of $\mathcal{P}'$. We define a sequence of refining partitions $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \cdots \prec \mathcal{P}_m$ of V, with $|\mathcal{P}_i| \le (b-1)^i$ for $0 \le i \le m$, as follows. Initially, set $\mathcal{P}_0 = \{V\}$. Now, for $1 \le i \le m$, suppose that we have defined $\mathcal{P}_{i-1}$ with $|\mathcal{P}_{i-1}| \le (b-1)^{i-1}$. Let $\mathcal{P}_{i-1} = \{A_1,\dots,A_\ell\}$, where $\ell \le (b-1)^{i-1}$. Define $\mathcal{P}_i$ as follows. For $1 \le q \le \ell$ and $A_q \in \mathcal{P}_{i-1}$, let

$$\begin{aligned} B_1^q &= \{v \in A_q : v'_i = g(u_i) \text{ or } g(u_i)+1 \pmod b\}, \\ B_r^q &= \{v \in A_q : v'_i = g(u_i)+r \pmod b\}, \text{ for } 2 \le r \le b-1. \end{aligned}$$

Let $\mathcal{P}_i = \{B_r^q : 1 \le q \le \ell,\ 1 \le r \le b-1 \text{ and } B_r^q \ne \emptyset\}$, so that $\mathcal{P}_i$ is a partition of V with $|\mathcal{P}_i| \le (b-1)^i$ and $\mathcal{P}_{i-1} \prec \mathcal{P}_i$. Proceeding inductively, we obtain the partitions $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \cdots \prec \mathcal{P}_m$ of V, with $|\mathcal{P}_i| \le (b-1)^i$ for $0 \le i \le m$. Now, observe that for every $1 \le i \le m$, and any two vertices y and z in the same set in $\mathcal{P}_i$, the path yu_iz is not total-rainbow, since $g(yu_i) = y'_i$ and $g(zu_i) = z'_i$ are either in $\{g(u_i), g(u_i)+1\} \pmod b$, or they are both $g(u_i)+r \pmod b$ for some $2 \le r \le b-1$. Since $n > (b-1)^m \ge |\mathcal{P}_m|$, there exists a set in $\mathcal{P}_m$ with at least two vertices w and x, and since $\mathcal{P}_1 \prec \cdots \prec \mathcal{P}_m$, this means that w and x are in the same set in $\mathcal{P}_i$ for every $1 \le i \le m$. Therefore, wu_ix is not a total-rainbow path for every $1 \le i \le m$. Since any other $w-x$ path has length at least 4, and g uses only at most $b \le 6$ colours, we cannot have a total-rainbow $w-x$ path. Hence, g is not total-rainbow connected, and $trc(K_{m,n}) \ge b+1$.

Finally, let $n > 6^m$. Let g' be a total colouring of $K_{m,n}$, using colours from $\{1,\dots,6\}$. For $v \in V$, assign to v the vector v' of length m, where $v'_i = g'(u_i v)$ for $1 \le i \le m$. Then, there exist $x, y \in V$ with $x' = y'$, and hence the path xu_iy is not total-rainbow for any $1 \le i \le m$. Since any other $x-y$ path has length at least 4, and g' uses only at most six colours, we cannot have a total-rainbow $x-y$ path. Hence, g' is not total-rainbow connected, and $trc(K_{m,n}) \ge 7$. □

Now, we consider $trc_k(K_{m,n})$ for $2 \le k \le m \le n$. We may consider the related problems of finding $rc_k(K_{m,n})$ and $rvc_k(K_{m,n})$ for $1 \le k \le m \le n$. It is easy to see that $rvc(K_{1,1}) = 0$, $rvc(K_{m,n}) = 1$ if $n \ge 2$, and $rvc_k(K_{m,n}) = 2$ for $k \ge 2$. Also, we have $rc(K_{1,n}) = n$, and Chartrand et al. [2] proved that $rc(K_{m,n}) = \min(\lceil \sqrt[m]{n}\,\rceil, 4)$ if $2 \le m \le n$. For $k \ge 2$, the determination of $rc_k(K_{m,n})$ has been well-studied and remains an open problem. Partial solutions for the balanced case $rc_k(K_{n,n})$ have been obtained by Chartrand et al. [3], Li and Sun [10], and Fujita et al. [5]. The result of Fujita et al. [5, Theorem 1.6] says that if $0 < \varepsilon < \frac{1}{2}$ and $k \ge \frac{1}{2}(\theta-1)(1-2\varepsilon)+2$, where $\theta = \theta(\varepsilon)$ is the largest solution of $2x^2 e^{-\varepsilon^2(x-2)} = 1$, then we have $rc_k(K_{n,n}) = 3$ for $n \ge \frac{2k-4}{1-2\varepsilon}+1$. From this result, we have the following corollary.

Corollary 5. *Let $0 < \varepsilon < \frac{1}{2}$ and $k \ge \frac{1}{2}(\theta-1)(1-2\varepsilon)+2$, where $\theta = \theta(\varepsilon)$ is the largest solution of $2x^2 e^{-\varepsilon^2(x-2)} = 1$. If $n \ge \frac{2k-4}{1-2\varepsilon}+1$, then $trc_k(K_{n,n}) = 5$.*

Proof. By the result of Fujita et al., we have $rc_k(K_{n,n}) = 3$. We can take a rainbow k-connected colouring of $K_{n,n}$ with three colours, and colour the vertices with two further colours, with the vertices within each class having the same colour. This gives $trc_k(K_{n,n}) \le$

5. Also, we have $trc_k(K_{n,n}) \geq 5$, since to have at least two disjoint total-rainbow paths connecting two vertices in different classes, at least one path must have length at least 3. $\square$

For example, if we set $\varepsilon = \frac{1}{6}$ in Corollary 5, then for $k \geq 159$ and $n \geq 3k - 5$, we have $trc_k(K_{n,n}) = 5$.

Fujita et al. [5, Problem 5.3] also asked the following question: "For $2 \leq k \leq m \leq n$, is it true that $rc_k(K_{m,n}) \in \{3, 4\}$ for sufficiently large m?" Here, we may ask a similar question.

Problem 6. *For $2 \leq k \leq m \leq n$, is it true that $trc_k(K_{m,n}) \in \{5, 6, 7\}$ for sufficiently large m? Also, for which values of m and n does $trc_k(K_{m,n})$ take a particular value in $\{5, 6, 7\}$?*

Next, we consider the more general problem of finding $trc_k(G)$ when G is a complete multipartite graph. For $t \geq 3$, let $K_{n_1,\dots,n_t}$ be the complete multipartite graph with class-sizes $1 \leq n_1 \leq \cdots \leq n_t$. Let $m = \sum_{i=1}^{t-1} n_i$ and $n = n_t$. Observe that $\kappa(K_{n_1,\dots,n_t}) = m$. We first determine $trc(K_{n_1,\dots,n_t})$.

Theorem 7. *Let $t \geq 3$, $1 \leq n_1 \leq \cdots \leq n_t$, $m = \sum_{i=1}^{t-1} n_i$ and $n = n_t$. Then,*

$$trc(K_{n_1,\dots,n_t}) = \begin{cases} 1 & \text{if } n = 1, \\ 3 & \text{if } n \geq 2 \text{ and } m > n, \\ \min(\lceil \sqrt[m]{n}\, \rceil + 1, 5) & \text{if } m \leq n. \end{cases}$$

Proof. Write G for $K_{n_1,\dots,n_t}$, and let V_i be the ith class (with n_i vertices) for $1 \leq i \leq t$. If $n = 1$, then $G = K_t$ and $trc(G) = 1$. For the case $n \geq 2$ and $m > n$, Chartrand et al. [2, Theorem 2.7] proved that $rc(G) = 2$. This implies that $trc(G) = 3$.

Now, let $m \leq n$. Let $b = \lceil \sqrt[m]{n}\, \rceil \geq 2$, and note that $m \leq n \leq 4^m$ if and only if $3 \leq b+1 \leq 5$. Hence, we need to prove that $trc(G) = b + 1$ if $m \leq n \leq 4^m$, and $trc(G) = 5$ if $n > 4^m$.

We first prove the upper bound. For the case $m \leq n \leq 4^m$, observe that $K_{m,n}$ is a spanning subgraph of G with classes $V_1 \cup \cdots \cup V_{t-1}$ and V_t. Hence, by Theorem 4 we have $trc(G) \leq trc(K_{m,n}) = b + 1$.

For the case $n > 4^m$, we define a total colouring f of G with five colours as follows. Let $V_t = V \,\dot\cup\, V'$, where $|V| = 3^m - 1$. Assign the $3^m - 1$ distinct vectors of length m with entries from $\{1, 2, 3\}$, except for $(1, 2, \dots, 2)$, to the vertices of V. Assign the vector $(1, 2, \dots, 2)$ to all vertices of V'. For $v \in V_t$, let v' denote the vector assigned to v. Let $V_1 \cup \cdots \cup V_{t-1} = \{u_1, \dots, u_m\}$, where $u_1 \in V_1$ and $u_2 \in V_2$. For $1 \leq i \leq m$ and $v \in V_t$, let $f(u_i v) = v'_i$. Let $f(e) = 3$ for every edge e inside $V_1 \cup \cdots \cup V_{t-1}$ (note that such edges exist, since $t \geq 3$), $f(u_1) = 4$, and $f(w) = 5$ for all $w \in V(G) \setminus \{u_1\}$. As in the proof of Theorem 4, we only need to verify that any two vertices in the same class of G are connected by a total-rainbow path. Any two vertices in $V(G) \setminus V'$, and one vertex in each of $V(G) \setminus V'$ and V', are connected by a total-rainbow path. If $x, y \in V'$, then the path $x u_1 u_2 y$ is total-rainbow. Therefore, f is total-rainbow connected, and $trc(G) \leq 5$.

Now, we prove the lower bound. If $m \leq n \leq 2^m$, then $trc(G) \geq 3 = b + 1$. Next, let $2^m < n \leq 4^m$. Then $(b-1)^m < n \leq b^m$, with $b \in \{3, 4\}$. Suppose that we have a total colouring of G, using at most b colours. Again, $K_{m,n}$ is a spanning subgraph of G with classes $V_1 \cup \cdots \cup V_{t-1}$ and V_t. Apply the same argument involving the refining partitions as in Theorem 4. We have vertices $u, v \in V_t$ which are not connected by a total-rainbow path of length 2. Since we have used at most $b \leq 4$ colours, we do not have a total-rainbow $u - v$ path of length at least 3 in G. Therefore, $trc(G) \geq b+1$. Finally for $n > 4^m$, consider a total colouring of G with at most four colours. We obtain $K_{m,n}$ as before and apply the same argument as in the end of Theorem 4. Then there are $u, v \in V_t$ which are not connected by a total-rainbow path of length 2. Since we have used only at most four colours, again we cannot have a total-rainbow $u - v$ path in G. Hence, $trc(G) \geq 5$. $\square$

Now, we consider $trc_k(K_{n_1,\dots,n_t})$ for $2 \leq k \leq m$. Again, we have the related problems of finding $rc_k(K_{n_1,\dots,n_t})$ and $rvc_k(K_{n_1,\dots,n_t})$ for $1 \leq k \leq m$, and these have been well-studied. The function $rvc_k(K_{n_1,\dots,n_t})$ has been completely determined by Liu et al. [12]. Roughly speaking, for $1 \leq s \leq t$, if $n_i = 1$ for $i \leq t - s$ and $n_i = 2$ for $i > t - s$, then we have $rvc_k(K_{n_1,\dots,n_t}) = s$. Otherwise, we have $rvc_k(K_{n_1,\dots,n_t}) \in \{1, 2, 3, 4\}$. Also, Chartrand et al. [2] proved that

$$rc(K_{n_1,\dots,n_t}) = \begin{cases} 1 & \textit{if } n = 1, \\ 2 & \textit{if } n \geq 2 \textit{ and } m > n, \\ \min(\lceil \sqrt[m]{n}\, \rceil, 3) & \textit{if } m \leq n. \end{cases}$$

The problem of determining $rc_k(K_{n_1,\dots,n_t})$ remains open for $k \geq 2$. For the balanced case, let $K_{t\times n}$ denote the graph $K_{n,\dots,n}$ with t classes. Fujita et al. [5, Theorem 1.8] proved that if $0 < \varepsilon < \frac{1}{2}$ and $k \geq \frac{1}{2}\theta(t-2)(1-2\varepsilon) + 1$, where $\theta = \theta(\varepsilon, t)$ is the largest solution of $\frac{1}{2}t^2x^2e^{-(t-2)\varepsilon^2 x} = 1$, then we have $rc_k(K_{t\times n}) = 2$ for $n \geq \frac{2k-2}{(t-2)(1-2\varepsilon)}$. From this, we instantly have the following corollary.

Corollary 8. *Let $t \geq 3$, $0 < \varepsilon < \frac{1}{2}$, and $k \geq \frac{1}{2}\theta(t-2)(1-2\varepsilon) + 1$, where $\theta = \theta(\varepsilon, t)$ is the largest solution of $\frac{1}{2}t^2x^2e^{-(t-2)\varepsilon^2 x} = 1$. If $n \geq \frac{2k-2}{(t-2)(1-2\varepsilon)}$, then $trc_k(K_{t\times n}) = 3$.* $\square$

For example, if we set $t = 3$ and $\varepsilon = \frac{1}{6}$ in Corollary 8, then for $k \geq 169$ and $n \geq 3k - 3$, we have $trc_k(K_{3\times n}) = 3$.

For general complete multipartite graphs, Fujita et al. [5, Problem 5.3] asked the following question for $rc_k(K_{n_1,\dots,n_t})$: "For $t \geq 3$, $k \geq 2$ and sufficiently large n_1, is it true that $rc_k(K_{n_1,\dots,n_t}) \in \{2, 3\}$?" Here, we ask a similar question for $trc_k(K_{n_1,\dots,n_t})$.

Problem 9. *For $t \geq 3$, $1 \leq n_1 \leq \cdots \leq n_t$ and $2 \leq k \leq m$, where $m = \sum_{i=1}^{t-1} n_i$, is it true that $trc_k(K_{n_1,\dots,n_t}) \in \{3, 4, 5\}$ for sufficiently large n_1? Also, for which values of $n_1, \dots, n_t$ does $trc_k(K_{n_1,\dots,n_t})$ take a particular value in $\{3, 4, 5\}$?*

To end this section, we mention the analogous situation for complete graphs. Obviously, we have $rc(K_n) = 1$, $rvc(K_n) = 0$, and $rvc_k(K_n) = 1$ if $k \geq 2$. The problem of determining $rc_k(K_n)$ for $k \geq 2$ has also been well-studied. Chartrand et al. [3] proved that if $k \geq 2$ and $n \geq (k+1)^2$, then we have $rc_k(K_n) = 2$. The bound $n \geq (k+1)^2$ was subsequently improved to $n \geq ck^{3/2} + o(k^{3/2})$ (for some constant c) by Li and Sun [9], and then to $n \geq (2+o(1))k$ by Dellamonica et al. [4, Theorem 2]. This latter bound for n is asymptotically best possible, and we have the following corollary.

Corollary 10. *For $k \geq 2$, we have $trc_k(K_n) = 3$ for $n \geq (2 + o(1))k$.* $\square$

3 Comparing $rc_k(G)$, $rvc_k(G)$, and $trc_k(G)$

In [7], Krivelevich and Yuster observed that we cannot upper-bound one of the functions $rc(G)$ and $rvc(G)$ in terms of the other, by providing examples of graphs G where $rc(G)$ is much larger than $rvc(G)$, and vice versa. By taking G to be the star $K_{1,s}$, we have $rc(G) = s$ and $rvc(G) = 1$. On the other hand, let G be constructed as follows. Take s vertex-disjoint triangles and, by designating a vertex from each triangle, add a complete graph on the designated vertices. Then $rc(G) \leq 4$ and $rvc(G) = s$. Subsequently, Liu et al. [12] compared the functions $rc_k(G)$ and $rvc_k(G)$ by extending these examples of Krivelevich and Yuster, and they obtained similar results.

Here, we similarly compare $trc_k(G)$ with $rc_k(G)$ and $rvc_k(G)$. Recall that for any k-connected graph G, we have

$$trc_k(G) \geq \max(rc_k(G), rvc_k(G)). \tag{1}$$

The first question we may ask is, how tight are the inequalities $trc_k(G) \geq rc_k(G)$ and $trc_k(G) \geq rvc_k(G)$? Theorem 11 below shows that the second inequality is the best possible in the sense that, for every sufficiently large s, there exists an example of a graph G where $trc_k(G) = rvc_k(G) = s$. This is somewhat surprising since in total colourings, we colour edges and vertices of graphs, as opposed to only vertices for vertex-colourings, so we could possibly expect that $trc_k(G) > rvc_k(G)$ if $rvc_k(G)$ is sufficiently large.

Theorem 11. *For every $s \geq 11k+1470$, there exists a graph G with $trc_k(G) = rvc_k(G) = s$.*

To prove Theorem 11, we need the result of Chartrand et al. [3] which was mentioned at the end of Section 2, as well as an auxiliary lemma.

Theorem 12 (Chartrand et al. [3, Theorem 2.4])**.** *For $k \geq 2$ and $n \geq (k+1)^2$, we have $rc_k(K_n) = 2$.*

Lemma 13. *Let $k \geq 2$, $s \geq 11k+1470$, and $u_1, \dots, u_s$ be the vertices of the complete graph K_s. For $1 \leq i \leq s$, let $L(u_i) \subset \{1, \dots, s\}$, where $L(u_i) = \{i, i+1, \dots, i+k+4\}$ modulo s. Then, there exists an edge-colouring of K_s, using at most $11k+1470$ colours from $\{1, \dots, s\}$ such that, for every $y, z \in V(K_s)$, the following property holds.*

- *(P) There exist $3k+10$ disjoint rainbow $y-z$ paths of length 2, say $yv_1z, \dots, yv_{3k+10}z$ for some $v_1, \dots, v_{3k+10} \in V(K_s)$, such that for every $1 \leq j \leq 3k+10$, the path yv_jz does not use the colours from $L(y) \cup L(z) \cup L(v_j)$.*

Proof. We take a random edge-colouring of K_s with $11k+1470$ colours. For $y, z \in V(K_s)$, let $E_{y,z}$ be the event that the property (P) does not hold. For fixed y and z, we can choose a set $L \subset \{1, \dots, s\}$ such that $L(y) \cup L(z) \subset L$ and $|L| = 2k+10$, and then a set $S \subset V(K_s) \setminus \{y, z\}$ such that $|S| = s-4k-18$, and $L(v) \cap L = \emptyset$ for all $v \in S$. For $v \in S$, we say that the path yvz is *good* if it is rainbow, and does not use the colours of $L(v) \cup L$. The probability that the path yvz is good is $p = \frac{(8k+1455)^2-(8k+1455)}{(11k+1470)^2}$, and for different v in S, these probabilities are independent. Let X be the number of good $y-z$ paths. Then, $X \sim \mathrm{Bi}(s-4k-18, p)$. By the Chernoff bound (see for example, [6, Theorem 2.1]), we have

$$\begin{aligned} \mathbb{P}(E_{y,z}) &\leq \mathbb{P}(X \leq 3k+9) \leq \mathbb{P}(X \leq (1-\varepsilon)(s-4k-18)p) \\ &\leq \exp\Big(-\frac{1}{2}\varepsilon^2(s-4k-18)p\Big), \end{aligned}$$

where ε is a constant such that $0 < \varepsilon < 1 - \frac{3k+9}{(s-4k-18)p}$. Note that we have $p > \frac{64}{121}$ so that

$$1 - \frac{3k+9}{(s-4k-18)p} > 1 - \frac{3}{7} \cdot \frac{121}{64} = \frac{85}{448}, \tag{2}$$

and we may take $\varepsilon = \frac{85}{448}$. Applying the union bound, we have

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{y,z\in V(K_s)} E_{y,z}\Big) &\le \binom{s}{2}\exp\Big(-\frac{1}{2}\varepsilon^2(s-4k-18)p\Big)\\ &< \binom{s}{2}\exp(-0.0095(s-4k-18)) < 1,\end{aligned}$$

if $s \ge 11k+1470$ and $k \ge 2$. Hence, there exists an edge-colouring of K_s, using at most $11k+1470$ colours, such that the property (P) holds for all $y, z \in V(K_s)$. $\square$

We remark that in Lemma 13, the constant 11 can be replaced by any constant greater than θ, where $\theta \approx 10.08$ is the largest solution of the cubic equation $(x-3)^2(x-4)-3x^2 = 0$. Using any such constant in place of 11 will ensure that the term corresponding to ε can be found, as in the calculation in (2). The constant 1470 will then have to be replaced by another constant.

Proof of Theorem 11. We shall focus mainly on the case when $k \ge 2$, and then briefly consider the case $k = 1$ at the end.

We generalise the construction of Krivelevich and Yuster with the disjoint triangles attached to a clique, as described earlier. Take s disjoint k-sets of vertices $V_1, \dots, V_s$. Let $V_i = \{v_1^i, \dots, v_k^i\}$ for every $1 \le i \le s$. For every $1 \le p \le k$, we add a clique on $\{v_p^1, \dots, v_p^s\}$. This gives k disjoint copies of K_s. Let G_p be the copy of K_s on $\{v_p^1, \dots, v_p^s\}$ ($1 \le p \le k$). Take further disjoint sets $X_1, \dots, X_s$ and $Y_1, \dots, Y_s$, each with $(k+1)^2$ vertices. For each $1 \le i \le s$, add a clique with vertex set $X_i \cup Y_i$, and a complete bipartite graph with classes $X_i \cup Y_i$ and V_i. Let G be the resulting graph. We claim that $rvc_k(G) \ge s$, and $trc_k(G) \le s$.

First, suppose that we have a vertex-colouring of G, using fewer than s colours. We may assume that $v_1^1, v_1^2 \in V(G_1)$ have the same colour. Let $u \in X_1$ and $v \in X_2$. Then, in any set of k disjoint $u - v$ paths in G, one path must use both v_1^1 and v_1^2; note that the edges and vertices between v_1^1 and v_1^2 in such a path are in G_1. Hence, $rvc_k(G) \ge s$.

Now, we construct a total colouring c of G, using the colours $1, \dots, s$, as follows. Let c' be an edge-colouring on K_s as given by Lemma 13, where the vertices of the K_s are $u_1, \dots, u_s$ and the sets $L(u_i) \subset \{1, \dots, s\}$ are as defined. For $w \in V_i$ and $w' \in V_j$, for some $i \ne j$, if $ww' \in E(G)$, we let $c(ww') = c'(u_iu_j)$. For $1 \le i \le s$ and $1 \le p \le k$, let $c(v_p^i) = i+p-1$, $c(e) = i+k$ for $e \in E(V_i, X_i)$, $c(x) = i+k+1$ for $x \in X_i$, $c(e') = i+k+2$ for $e' \in E(X_i, Y_i)$, $c(y) = i+k+3$ for $y \in Y_i$, and $c(e'') = i+k+4$ for $e'' \in E(Y_i, V_i)$, all modulo s. Here, $E(A, B)$ denotes the set of edges in G with one end-vertex in A and the other in B, for disjoint sets of vertices $A, B \subset V(G)$. Note that for $1 \le i \le s$, the set of colours used within $V_i \cup X_i \cup Y_i$ is $L(u_i) = \{i, i+1, \dots, i+k+4\} \pmod{s}$. By Theorem 12, we can give rainbow k-connected edge-colourings to the copies of $K_{(k+1)^2}$ induced by X_i and Y_i with two colours not in $L(u_i)$, for $1 \le i \le s$.

We claim that c is total-rainbow k-connected for G. Let $u, v \in V(G)$. We show that there exist k disjoint total-rainbow $u - v$ paths. It is easy to see that such paths exist if $u, v \in V_i \cup X_i \cup Y_i$ for some i. Note that we apply Theorem 12 if $u, v \in X_i$ or $u, v \in Y_i$.

Let $u \in V_i$ and $v \in V_j$ for some $i \ne j$, with $u = v_p^i$ and $v = v_q^j$ for some p and q. If $p = q$, then $u, v \in V(G_p)$. Applying Lemma 13 on G_p, we can find k suitable $u - v$ paths of length 2, all within G_p. If $p \ne q$, then applying Lemma 13 on G_p and G_q, we can find the k suitable $u - v$ paths, where each path has the form $uv_p^\ell xyv_q^\ell v$ for some $\ell \notin \{i, j\}$, $x \in X_\ell$ and $y \in Y_\ell$, with distinct paths having distinct values of ℓ.

Let $u \in X_i \cup Y_i$ and $v \in V_j$ for some $i \neq j$, with $v = v_q^j$ for some q. By Lemma 13, in the edge-colouring c' on K_s, there are $3k+10$ disjoint rainbow $u_i - u_j$ paths of length 2, where each such path u_iwu_j does not use the colours of $L(u_i) \cup L(u_j) \cup L(w)$. We may choose k of these paths, say $u_iu_{\ell_1}u_j, \ldots, u_iu_{\ell_k}u_j$, such that $L(u_{\ell_r}) \cap L(u_i) = \emptyset$ for $1 \leq r \leq k$. Then, we have the k suitable $u - v$ paths, where one path is $uv_q^iv_q^{\ell_1}v$, and the other $k-1$ paths have the form $uv_p^iv_p^{\ell}xyv_q^{\ell}v$, for some $\ell \in \{\ell_2, \ldots, \ell_k\}$, $p \neq q$, $x \in X_\ell$ and $y \in Y_\ell$, with distinct paths having distinct values of p and ℓ.

Let $u \in X_i$ and $v \in X_j$ for some $i \neq j$. For $1 \leq p \leq k$, one of the following holds.

- $c(uv_p^i) = i+k$, $c(v_p^i) = i+p-1$, $c(vv_p^j) = j+k$ and $c(v_p^j) = j+p-1 \pmod s$ are pairwise distinct.

- $c(uy) = i+k+2$, $c(y) = i+k+3$, $c(yv_p^i) = i+k+4$, $c(v_p^i) = i+p-1$, $c(vv_p^j) = j+k$ and $c(v_p^j) = j+p-1 \pmod s$ are pairwise distinct, where $y \in Y_i$.

- $c(uv_p^i) = i+k$, $c(v_p^i) = i+p-1$, $c(vy') = j+k+2$, $c(y') = j+k+3$, $c(y'v_p^j) = j+k+4$ and $c(v_p^j) = j+p-1 \pmod s$ are pairwise distinct, where $y' \in Y_j$.

Hence, there is a $u - v_p^i$ path Q_p^i and a $v - v_p^j$ path Q_p^j such that, the colours of the elements of $V(Q_p^i - u) \cup E(Q_p^i) \cup V(Q_p^j - v) \cup E(Q_p^j)$ are distinct. We may take the paths Q_p^i and Q_p^j, for $1 \leq p \leq k$, such that the only common vertex of the paths Q_p^i is u, and the only common vertex of the paths Q_p^j is v. Now, by Lemma 13, in the edge-colouring c' on K_s, there are $3k+10$ disjoint rainbow $u_i - u_j$ paths of length 2, where each such path u_iwu_j does not use the colours of $L(u_i) \cup L(u_j) \cup L(w)$. Hence, we may choose k of these paths, say $u_iu_{\ell_1}u_j, \ldots, u_iu_{\ell_k}u_j$, such that $c(v_p^{\ell_p}) \notin L(u_i) \cup L(u_j)$ for $1 \leq p \leq k$. Therefore, we have the k suitable $u-v$ paths $uQ_p^iv_p^iv_p^{\ell_p}v_p^jQ_p^jv$, for $1 \leq p \leq k$. The cases when $u \in Y_i$ and $v \in Y_j$, and when $u \in X_i$ and $v \in Y_j$ for some $i \neq j$, can be considered similarly.

Therefore, c is total-rainbow k-connected, and $trc_k(G) \leq s$.

Finally, for the case $k = 1$, we may simply use the construction of Krivelevich and Yuster. Let G' be the graph in their construction, where the clique K_s has $s \geq 13$. Let $v_1, \ldots, v_s$ be the vertices of the K_s. We have already seen that $rvc(G') = s$. Now, consider a total colouring c'' on G' with s colours as follows. For the vertices and edges other than the edges of the K_s, we colour these as in the total colouring c on G. For $i \neq j$, let $c''(v_iv_j)$ be a colour not present in the triangles at v_i and v_j. This can be done since $s \geq 13$. Then, c'' is total-rainbow connected for G', and $trc(G') \leq s$. □

Next, we consider the tightness of the inequality $trc_k(G) \geq rc_k(G)$. For $s \geq k+1$, let G be the complete bipartite graph $K_{k,s}$. Then, we have $rc_k(G) = s$ and $trc_k(G) = s+1$. Indeed, the lower bounds $rc_k(G) \geq s$ and $trc_k(G) \geq s+1$ are fairly trivial. To see that $rc_k(G) \leq s$, we take an edge-colouring of G with colours $1, \ldots, s$ as follows. Let the classes of G be $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_s\}$, and colour x_iy_j with colour $i+j-1 \pmod s$. Then, this is a rainbow k-connected colouring for G. To see that $trc_k(G) \leq s+1$, we take a total colouring of G with colours $1, \ldots, s+1$ by taking the above edge-colouring and in addition, colour every x_i with colour $s+1$, and y_j with colour $j+k \pmod s$. This is a total-rainbow k-connected colouring for G.

Hence, we see that $rc_k(G)$ and $trc_k(G)$ can differ by 1, and attaining all sufficiently large values. However, we have not been able to improve this to obtain an analogue of Theorem 11, and we pose the following problem.

Problem 14. *Let $k \geq 1$. Does there exist an integer $N = N(k)$ such that for all $s \geq N$, there exists a graph G with $trc_k(G) = rc_k(G) = s$?*

There exist some graphs G such that $trc_k(G) = rc_k(G)$. Let $\chi'(H)$ denote the *chromatic index* of the graph H. Chartrand et al. [3, Proposition 2.1] proved that

$$rc_k(K_{k+1}) = \chi'(K_{k+1}) = \begin{cases} k+1 & \text{if } k \text{ is even,} \\ k & \text{if } k \text{ is odd.} \end{cases}$$

Hence, if k is even, we have a total-rainbow k-connected colouring for K_{k+1} with $k+1$ colours as follows. Take a proper edge-colouring of K_{k+1} with $k+1$ colours, and then for every vertex $v \in V(K_{k+1})$, colour v with the colour not used by the edges at v. It follows that $trc_k(K_{k+1}) = rc_k(K_{k+1}) = k+1$. Note that this argument does not hold if k is odd.

Also, we have $trc(C_5) = rc(C_5) = 3$. Moreover, there are infinitely many graphs G such that $trc(G) = rc(G) = 3$. We may take G to be any graph formed by taking C_5, then replacing a vertex v by a clique and joining all edges from the clique to the two neighbours of v in C_5. For $k \geq 3$ odd, we have not been able to find a graph G such that $trc_k(G) = rc_k(G)$.

Now, the second question we can ask is, how far from equality can the inequality (1) be? As we have mentioned at the beginning of this section, Liu et al. [12] showed that the functions $rvc_k(G)$ and $rc_k(G)$ can be arbitrarily far apart. They proved that, for $1 \leq t < s$, there exists a graph G such that $rc_k(G) \geq s$ and $rvc_k(G) = t$ [12, Theorem 7]. They also proved that for $s \geq (k+1)^2$, there exists a graph G such that $rvc_k(G) = s$ and $rc_k(G) \leq 9$ [12, Theorem 9]. Hence, we instantly have the following corollaries.

Corollary 15. *Given $1 \leq t < s$, there exists a graph G such that $trc_k(G) \geq s$ and $rvc_k(G) = t$.* □

Corollary 16. *Let $s \geq (k+1)^2$. Then, there exists a graph G such that $trc_k(G) \geq s$ and $rc_k(G) \leq 9$.* □

The constructions given by Liu et al. were as follows. For Corollary 15, we take the star $K_{1,s}$ and identify the centre with one end-vertex of the path of length t. This graph is a *broom*. We take a blow-up of this broom by replacing each non-leaf vertex with a clique K_k, and joining an edge between two vertices if the corresponding two vertices in the broom are neighbours. For Corollary 16, we take the same construction as that of Theorem 11, but with the condition $s \geq (k+1)^2$ instead of $s \geq 11k + 1470$.

We observe that the difference between $trc_k(G)$ and $\max(rc_k(G), rvc_k(G))$ can be arbitrarily large. Take a path $x_0x_1 \cdots x_s$ of length $s \geq 4$. Then, take a blow-up by replacing each vertex x_i $(1 \leq i \leq s-1)$ with a clique $K_{(k+1)^2}$ on vertex set X_i, and joining all edges between x_0 and X_1; x_s and X_{s-1}; and X_i and X_{i+1} for $1 \leq i \leq s-2$. Let G be the resulting graph. Clearly, $trc_k(G) \geq 2s-1$ and $rvc_k(G) = s-1$. Using $s \geq 4$ and Theorem 12, we have $rc_k(G) = s$, and hence $trc_k(G) - \max(rc_k(G), rvc_k(G)) \geq s-1$ can be arbitrarily large.

However, in this simple construction, we see that $\max(rc_k(G), rvc_k(G)) = s$ is unbounded as s increases. We may ask the following question.

($*$) *For fixed k, does there exist an infinite family $\mathcal{F}$ of k-connected graphs such that $\max(rc_k(G), rvc_k(G))$ is bounded on $\mathcal{F}$, but $trc_k(G)$ is not?*

We see that the constructions of Corollaries 15 and 16 do not answer this question. In the first construction, we have $rc_k(G) \geq s$. Similarly in the second construction, using

Theorem 11, we have $rvc_k(G) = trc_k(G) = s$ if we also have $s \geq 11k + 1470$. In both cases, $\max(rc_k(G), rvc_k(G))$ is unbounded as s increases.

We make the following conjecture.

Conjecture 17. *For every $k \geq 1$, there exists a function $f_k : \mathbb{N} \to \mathbb{N}$ such that if G is a k-connected graph and $\max(rc_k(G), rvc_k(G)) = c$, then $trc_k(G) \leq f_k(c)$.*

A positive solution to Conjecture 17 would imply that the answer to the question ($*$) is "no".

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