Rainbow Vertex k-connection in Graphs

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Abstract

Let k be a positive integer and G be a k-connected graph. An edge-coloured path is *rainbow* if its edges have distinct colours. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum number of colours required to colour the edges of G so that any two vertices of G are connected by k internally vertex-disjoint rainbow paths. The function $rc_k(G)$ was first introduced by Chartrand, Johns, McKeon, and Zhang in 2009, and has since attracted considerable interest. In this paper, we consider a version of the function $rc_k(G)$ which involves vertex-colourings. A vertex-coloured path is *vertex-rainbow* if its internal vertices have distinct colours. The *rainbow vertex k-connection number* of G, denoted by $rvc_k(G)$, is the minimum number of colours required to colour the vertices of G so that any two vertices of G are connected by k internally vertex-disjoint vertex-rainbow paths. We shall study the function $rvc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. We also construct graphs G where $rc_k(G)$ is much larger than $rvc_k(G)$ and vice versa so that we cannot in general bound one of $rc_k(G)$ and $rvc_k(G)$ in terms of the other.

Keywords: Graph colouring, rainbow (vertex) connection number, k-connected

1 Introduction

In this paper, we consider graphs which are finite, simple, and undirected. For any undefined terms in graph theory, we refer the reader to the book by Bollobás [1].

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Throughout the paper, let k be a positive integer. For simplicity, a set of internally vertex-disjoint paths will be called *disjoint*. Recall that, by Menger's theorem [13], a graph is k-connected if and only if any two vertices are connected by k disjoint paths. An edge-coloured path is *rainbow* if its edges have distinct colours. An edge-colouring of a k-connected graph G, not necessarily proper, is *rainbow k-connected* if any two vertices of G are connected by k disjoint rainbow paths. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum integer t such that there exists a rainbow k-connected colouring of G, using t colours. For simplicity, we write $rc(G)$ for $rc_1(G)$. Note that, by Menger's theorem, $rc_k(G)$ is well defined if G is k-connected. The function $rc_k(G)$ was first introduced by Chartrand et al. ([2] for $k = 1$ (2008), and [3] for general k (2009)). Since then, a considerable amount of research has been carried out towards the study of $rc_k(G)$. The case for general k has been studied by Li and Sun [10, 11], and Fujita et al. [4], among others. For an overview of the rainbow connection subject, we refer the reader to the survey of Li et al. [9], and the book of Li and Sun [12].

Here, we consider a version of the function $rc_k(G)$ involving vertex-colourings. A vertex-coloured path is *vertex-rainbow* if the internal vertices have distinct colours. A vertex-colouring of a k-connected graph G, not necessarily proper and possibly with uncoloured vertices, is *rainbow vertex k-connected* if any two vertices of G are connected by k disjoint vertex-rainbow paths. The *rainbow vertex k-connection number* of G, denoted by $rvc_k(G)$, is the minimum integer t such that there exists a rainbow vertex k-connected colouring of G, using t colours. We write $rvc(G)$ for $rvc_1(G)$. Again by Menger's theorem, $rvc_k(G)$ is well defined if G is k-connected. The function $rvc(G)$ was first introduced by Krivelevich and Yuster [5], and has since been studied by Li and Shi [8], Li and Liu [6], and Li et al. [7].

Some initial observations can be made. If G is a connected graph on n vertices, then $rvc(G) = 0$ if and only if G is a clique. If $n \geq 2$ and q is the number of vertices of G with degree at least 2, then $rvc(G) \leq \min(n-2, q)$. Moreover, a result of Li et al. [7] implies that $rvc(G) = n-2$ if and only if G is a path. Furthermore, it is easy to prove that $rvc(G) = q$ if G is a tree. Also, if $\mathrm{diam}(G)$ denotes the *diameter* of G, then we have $rvc_k(G) \geq \mathrm{diam}(G)-1$, with equality if $k = 1$ and $\mathrm{diam}(G) = 1$ or 2. In fact, we have $rvc(G) = 1$ if and only if $\mathrm{diam}(G) = 2$. If $k \geq 2$ and G is a k-connected graph, then $rvc_k(G) \geq 1$, and equality holds if G is a clique on at least three vertices.

This paper is organised as follows. In Section 2, we determine the function $rvc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. In Section 3, we compare the functions $rc_k(G)$ and $rvc_k(G)$. We show that we cannot bound one of $rc_k(G)$ and $rvc_k(G)$ in terms of the other, by constructing examples of graphs G where $rc_k(G)$ is much larger than $rvc_k(G)$, and vice versa.

2 Rainbow Vertex k-connection Numbers of some Graphs

In this section, we shall determine the function $rvc_k(G)$ for some specific graphs G. Here, we will only consider vertex-colourings. For simplicity, a vertex-rainbow path will be called *rainbow*.

Let $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$ denote the *vertex-connectivity* of G. Note that $rvc_k(G)$ is defined for all $1 \leq k \leq \kappa(G)$. We begin with the case when G is a cycle. Let C_n denote the cycle of order n. The function $rvc(C_n)$ was determined by Li and Liu [6] as follows.

Theorem 1 (Li and Liu [6]) *For $3 \leq n \leq 15$, the values of $rvc(C_n)$ are given in the following table.*

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$rvc(C_n)$	0	1	1	2	3	3	3	4	5	5	6	7	7

For $n \geq 16$, we have $rvc(C_n) = \lceil \frac{n}{2} \rceil$.

Since $\kappa(C_n) = 2$, in addition to Theorem 1, we determine $rvc_2(C_n)$.

Theorem 2 *$rvc_2(C_3) = 1$, $rvc_2(C_4) = 2$, and $rvc_2(C_n) = n$ for $n \geq 5$.*

Proof. The assertion can be easily verified for C_3 and C_4. Now, let $n \geq 5$. Clearly, we have $rvc_2(C_n) \leq n$, by considering the colouring of C_n where the vertices are given distinct colours. If we have a vertex-colouring of C_n with at most $n - 1$ colours, then some two vertices u, v have the same colour. Since $n \geq 5$, we can take two vertices x, y which are internal vertices of one of the two $u - v$ paths. Then, we do not have two disjoint rainbow $x - y$ paths. Hence, $rvc_2(C_n) \geq n$. $\square$

A graph closely related to C_n is the *wheel* W_n. This is the graph obtained from C_n by joining a new vertex v to every vertex of C_n. The vertex v is the *centre* of W_n. Note that $\kappa(W_n) = 3$. We have the following.

Theorem 3

(a) $rvc(W_3) = 0$ and $rvc(W_n) = 1$ for $n \geq 4$.

(b) $rvc_2(W_3) = 1$ and $rvc_2(W_n) = rvc(C_n)$ for $n \geq 4$ (hence, $rvc_2(W_n)$ is determined and given by Theorem 1 for $n \geq 4$).

(c) $rvc_3(W_3) = 1$, $rvc_3(W_4) = 2$, and $rvc_3(W_n) = n$ for $n \geq 5$.

Proof. (a) This is clear, since $rvc(W_3) = rvc(K_4) = 0$ and $\mathrm{diam}(W_n) = 2$ for $n \geq 4$.

(b) The assertion $rvc_2(W_3) = 1$ is easily verified. Now, let $n \geq 4$. Clearly, $rvc_2(W_n) \leq rvc(C_n)$, since by taking a rainbow vertex connected colouring for the cycle C_n in W_n with $rvc(C_n)$ colours, and then colouring the centre with any used colour, we have a rainbow vertex 2-connected colouring for W_n. On the other hand, suppose that we have a vertex-colouring for W_n with fewer than $rvc(C_n)$ colours. Then, for some two vertices x, y in the cycle C_n of W_n, we do not have a rainbow $x - y$ path along the cycle. Hence, there is at most one rainbow $x - y$ path in W_n (using the centre of W_n). Therefore, $rvc_2(W_n) \geq rvc(C_n)$.

(c) This can be proved in a similar way as the proof of Theorem 2. $\square$

We now consider the function $rvc_k(G)$ when G is a complete multipartite graph. Let G have partite class-sizes $1 \leq n_1 \leq \cdots \leq n_t$ for some $t \geq 2$. We write $G = K_{n_1,\dots,n_t}$.

The analogous problem of the determination of $rc_k(K_{n_1,\dots,n_t})$ has only been solved completely for $k = 1$ by Chartrand et al. [2], as follows. For the bipartite case $rc(K_{m,n})$, where $1 \leq m \leq n$, we have

$$rc(K_{m,n}) = \begin{cases} n & \text{if } m = 1, \\ \min(\lceil \sqrt[m]{n} \rceil, 4) & \text{if } m \geq 2. \end{cases}$$

For the general multipartite case $rc(K_{n_1,\dots,n_t})$, where $t \geq 3$, $1 \leq n_1 \leq \cdots \leq n_t$, $m = \sum_{i=1}^{t-1} n_i$, and $n = n_t$, we have

$$rc(K_{n_1,\dots,n_t}) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_t \geq 2 \text{ and } m > n, \\ \min(\lceil \sqrt[m]{n} \rceil, 3) & \text{if } m \leq n. \end{cases}$$

The problem remains open for $k \geq 2$. In the case of the balanced complete bipartite graph $K_{n,n}$, Chartrand et al. [3] proved that $rc_k(K_{n,n}) = 3$ if $k \geq 2$ and $n = 2k\lceil \frac{k}{2} \rceil$. This result was later improved by Li and Sun [11], who proved that $rc_k(K_{n,n}) = 3$ if $k \geq 2$ and $n \geq 2k\lceil \frac{k}{2} \rceil$, and by Fujita et al. [4], who proved that $rc_k(K_{n,n}) = 3$ if k is sufficiently large and $n \geq 2k + o(k)$. As for balanced complete multipartite graphs, Fujita et al. also proved that $rc_k(K_{t\times n}) = 2$ if $t \geq 3$, k is sufficiently large, and $n \geq \frac{2k}{t-2} + o(k)$, where $K_{t\times n}$ denotes the complete t-partite graph with each class having n vertices. For general complete multipartite graphs, Fujita et al. asked the question of whether, for $k, t \geq 2$, there is a function $g(k,t)$ such that, if $n_1 \geq g(k,t)$, then $rc_k(K_{n_1,n_2}) = 3$ or 4, and $rc_k(K_{n_1,\dots,n_t}) = 2$ or 3 if $t \geq 3$. Moreover, they also asked the following: when do we have $rc_k(K_{n_1,n_2}) = 3$, and when do we have $rc_k(K_{n_1,\dots,n_t}) = 2$ if $t \geq 3$?

Here, we are able to completely determine $rvc_k(K_{n_1,\dots,n_t})$ for every complete multipartite graph $K_{n_1,\dots,n_t}$ and every $1 \leq k \leq \kappa(K_{n_1,\dots,n_t}) = m = \sum_{i=1}^{t-1} n_i$. Obviously, if $n_t = 1$, then $K_{n_1,\dots,n_t} = K_t$. Hence, $rvc(K_{n_1,\dots,n_t}) = 0$ and $rvc_k(K_{n_1,\dots,n_t}) = 1$ for $2 \leq k \leq m$. Now, let $n_t \geq 2$. The bipartite case of $t = 2$ can be easily obtained. We have $rvc(K_{n_1,n_2}) = 1$ and $rvc_k(K_{n_1,n_2}) = 2$ for $2 \leq k \leq m$. For the general multipartite case when $t \geq 3$, we have the following result.

Theorem 4 *Let $1 \leq n_1 \leq \cdots \leq n_t$, where $t \geq 3$, $n_t \geq 2$ and $m = \sum_{i=1}^{t-1} n_i$.*

(a) If $1 \leq k \leq m-2$, then we have the following.

- *(i) $rvc_k(K_{n_1,\dots,n_t}) = 1$ if $1 \leq k \leq m - n_{t-1} + 1$.*
- *(ii) $rvc_k(K_{n_1,\dots,n_t}) = 2$ if $m - n_{t-1} + 2 \leq k \leq m-2$.*

(b)

- *(i) $rvc_{m-1}(K_{n_1,\dots,n_t}) = 1$ if $n_{t-1} \leq 2$.*
- *(ii) $rvc_{m-1}(K_{n_1,\dots,n_t}) = 2$ if $n_{t-1} \geq 3$ and we do not have $n_t = n_{t-1} = n_{t-2}$ odd.*
- *(iii) $rvc_{m-1}(K_{n_1,\dots,n_t}) = 3$ if $n_t = n_{t-1} = n_{t-2} \geq 3$ are odd.*

(c)

- *(i) $rvc_m(K_{n_1,\dots,n_t}) = 1$ if $n_{t-1} = 1$.*
- *(ii) $rvc_m(K_{n_1,\dots,n_t}) = 2$ if $2 \leq n_{t-1} \leq n_t - 2$.*
- *(iii) $rvc_m(K_{n_1,\dots,n_t}) = 2$ if $n_{t-1} = n_t - 1 \geq 2$ and $n_{t-2} \leq 2$, or $n_{t-1} = n_t \geq 2$ and $n_{t-2} = 1$.*
- *(iv) $rvc_m(K_{n_1,\dots,n_t}) = 3$ if $n_{t-1} = n_t - 1$ and $n_{t-2} \geq 3$, or $n_{t-1} = n_t \geq 3$, $n_{t-2} \geq 2$, and we do not have $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$ and $t \geq 4$.*
- *(v) $rvc_m(K_{n_1,\dots,n_t}) = 4$ if $t \geq 4$ and $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$.*
- *(vi) $rvc_m(K_{n_1,\dots,n_t}) = s$ if $n_t = n_{t-1} = \cdots = n_{t-s+1} = 2$ and $n_{t-s} = n_{t-s-1} = \cdots = n_1 = 1$, for $1 \leq s \leq t$.*

Before we proceed to the proof of Theorem 4, we shall prove two auxiliary lemmas. Let H be a vertex-coloured complete bipartite graph with classes X and Y. We say that a matching in H is *vertex-rainbow* if, for every edge in the matching, the end-vertices have distinct colours. For $A \subset X$ and $B \subset Y$, let (A, B) denote the complete bipartite subgraph of H with classes A and B.

Lemma 5 *Let $1 \le p \le q$. Consider the complete bipartite graph $K_{p,q}$ with classes X and Y, where $|X| = p$ and $|Y| = q$. Suppose that $\lceil \frac{p}{2} \rceil$ vertices of X and b vertices of Y have colour 1, and $\lfloor \frac{p}{2} \rfloor$ vertices of X and a vertices of Y have colour 2, where $a + b = q$. Let $u \in X$ and $v \in Y$.*

(a) If $a = \lceil \frac{q}{2} \rceil$ and $b = \lfloor \frac{q}{2} \rfloor$, then there exist p disjoint rainbow $u - v$ paths if $q \ge p + 2$, and $p - 1$ disjoint rainbow $u - v$ paths if $q = p$ or $q = p + 1$.

(b) If $a = \lfloor \frac{q}{2} \rfloor$ and $b = \lceil \frac{q}{2} \rceil$, then there exist $p - 2$ disjoint rainbow $u - v$ paths if $p = q \ge 3$ are odd, and $p - 1$ disjoint rainbow $u - v$ paths otherwise.

Proof. The lemma holds for $p = 1$, so assume that $p \ge 2$. Clearly, for both (a) and (b), one rainbow $u - v$ path is the edge uv. To find the other rainbow $u - v$ paths, it is enough to find a sufficiently large vertex-rainbow matching in $K_{p,q} - \{u, v\}$. Such a matching with size h then gives h disjoint rainbow $u - v$ paths, where each path has the form $uyxv$ for some $x \in X \setminus \{u\}$ and $y \in Y \setminus \{v\}$, with xy an edge of the matching. Together with the edge uv, we have $h + 1$ disjoint rainbow $u - v$ paths. For $i = 1, 2$, let X_i and Y_i be the sets of vertices with colour i in $X \setminus \{u\}$ and $Y \setminus \{v\}$, respectively.

(a) If $q \ge p + 2$, then $|Y_2| \ge \lceil \frac{q}{2} \rceil - 1 \ge \lceil \frac{p}{2} \rceil \ge |X_1|$, and $|Y_1| \ge \lfloor \frac{q}{2} \rfloor - 1 \ge \lfloor \frac{p}{2} \rfloor \ge |X_2|$. Hence, we can find matchings in (X_1, Y_2) and (X_2, Y_1) of sizes $|X_1|$ and $|X_2|$, respectively. Thus, there is a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $|X_1| + |X_2| = p - 1$, and we have p disjoint rainbow $u - v$ paths. Now, let $q = p$ or $q = p + 1$. If v has colour 1, then $|Y_2| = \lceil \frac{q}{2} \rceil \ge \lceil \frac{p}{2} \rceil \ge |X_1|$ and $|Y_1| = \lfloor \frac{q}{2} \rfloor - 1 \ge \lfloor \frac{p}{2} \rfloor - 1 \ge |X_2| - 1$. If v has colour 2, then similarly we have $|Y_2| \ge |X_1| - 1$ and $|Y_1| \ge |X_2|$. In both cases, we obtain a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $|X_1| + |X_2| - 1 = p - 2$, and we have $p - 1$ disjoint rainbow $u - v$ paths.

(b) For $p = q \ge 3$ odd, $|Y_2| \ge \lfloor \frac{q}{2} \rfloor - 1 = \lceil \frac{p}{2} \rceil - 2 \ge |X_1| - 2$, and $|Y_1| \ge \lceil \frac{q}{2} \rceil - 1 = \lfloor \frac{p}{2} \rfloor \ge |X_2|$. As before, we have a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $|X_1| + |X_2| - 2 = p - 3$, which gives $p - 2$ disjoint rainbow $u - v$ paths. Now, suppose that we do not have $p = q \ge 3$ odd. If q is even, then, by (a), we have $p - 1$ disjoint rainbow $u - v$ paths. If q is odd, then $q \ge p + 1$. We delete a vertex of colour 1 from Y and apply (a) to the resulting $K_{p,q-1}$. This again gives $p - 1$ disjoint rainbow $u - v$ paths. $\square$

Lemma 6 *Let $1 \le p \le q$ with $(p, q) \ne (2, 2), (4, 4)$. Consider the complete bipartite graph $K_{p,q}$ with classes X and Y, where $|X| = p$ and $|Y| = q$. Suppose that $\lceil \frac{p}{3} \rceil$ vertices of X and $\lceil \frac{q}{3} \rceil$ vertices of Y have colour 1, $\lfloor \frac{p}{3} \rfloor$ vertices of X and $\lfloor \frac{q}{3} \rfloor$ vertices of Y have colour 3, and all the other vertices have colour 2. Then, for all $u \in X$ and $v \in Y$, there are p disjoint rainbow $u - v$ paths.*

Proof. Clearly, if the lemma holds for (p, q), then it holds for (p, q') for any $q' \ge q$. Hence, it suffices to prove the lemma for $(p, q) = (2, 3), (4, 5)$ and $(p, q) = (a, a)$ for $a \ne 2, 4$. As in Lemma 5, it suffices to find a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $p - 1$ (i.e., the matching is maximum, and perfect if $p = q$). For $1 \le i \le 3$, let X_i and Y_i be the sets of vertices with colour i in $X \setminus \{u\}$ and $Y \setminus \{v\}$, respectively, and $p_i = |X_i|$, $q_i = |Y_i|$. We obtain a suitable matching as follows.

Case 1. $p = q \equiv 0 \pmod 3$.

Without loss of generality, u has colour 1, and v has colour 1 or colour 2. If v has colour 2, then we take perfect matchings in (X_1, Y_2), (X_2, Y_3) and (X_3, Y_1). If v has colour 1, then let $y \in Y_2$, and take perfect matchings in $(X_1, Y_2 \setminus \{y\})$, (X_2, Y_3) and $(X_3, Y_1 \cup \{y\})$. In both cases, we have a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $p - 1$.

Case 2. $p \not\equiv 0 \pmod 3$.

The cases $(p, q) = (1, 1), (2, 3), (4, 5)$ can be verified easily. Now, let $p = q \geq 5$. Note that $q_2 - 1 \leq p_1 \leq q_2 + 2$, and since $p \not\equiv 0 \pmod 3$, we have $q_1 - 2 \leq p_3 \leq q_1$.

Subcase 2.1. $p_1 = q_2 - 1$.

Note that u has colour 1, and hence $X_3 \neq \emptyset$. Let $Z_1 \subset Y_1$ with $|Z_1| = p_3 - 1$ (note that $0 \leq p_3 - 1 < q_1$), and $y \in Y_2$. Take perfect matchings in $(X_1, Y_2 \setminus \{y\})$, $(X_2, (Y_1 \setminus Z_1) \cup Y_3)$ and $(X_3, Z_1 \cup \{y\})$. We have a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $p - 1$.

Subcase 2.2. $q_2 \leq p_1 \leq q_2 + 2$.

We have $p_1 \leq \lceil \frac{p}{3} \rceil \leq p - \lceil \frac{p}{3} \rceil - 1 \leq q_2 + q_3$ (since $p \geq 5$). Let $Z_1 \subset Y_1$ with $|Z_1| = p_3$ (note that $p_3 \leq q_1$), and $Z_3 \subset Y_3$ with $|Z_3| = p_1 - q_2$ (note that $0 \leq p_1 - q_2 \leq q_3$). Take perfect matchings in $(X_1, Y_2 \cup Z_3)$, $(X_2, (Y_1 \setminus Z_1) \cup (Y_3 \setminus Z_3))$ and (X_3, Z_1). Again, we have a vertex-rainbow matching in $K_{p,q} - \{u, v\}$ of size $p - 1$. □

Proof of Theorem 4. For $1 \leq i \leq t$, let V_i denote the class of $K_{n_1,\dots,n_t}$ with n_i vertices.

Since $n_t \geq 2$, clearly $rvc_k(K_{n_1,\dots,n_t}) \geq 1$ for $1 \leq k \leq m$. Observe that, given any vertex-colouring of $K_{n_1,\dots,n_t}$, any two vertices in the same class, say V_i, have $m + n_t - n_i \geq m$ disjoint rainbow paths of length 2 connecting them. Hence, to prove the theorem, it is enough to consider, in each case, pairs of vertices where the two vertices are in different classes.

First, let $1 \leq k \leq m - n_{t-1} + 1$. We colour all the vertices of $K_{n_1,\dots,n_t}$ with the same colour. If $u \in V_i$ and $v \in V_j$ for some $i \neq j$, then there are $1 + m + n_t - n_i - n_j \geq k$ disjoint rainbow $u - v$ paths, each with length at most 2. Hence, we have $rvc_k(K_{n_1,\dots,n_t}) \leq 1$, and this proves part (i) of (a), (b) and (c). Next, let $k \geq m - n_{t-1} + 2$. Suppose that all the vertices of $K_{n_1,\dots,n_t}$ are coloured with the same colour. Then, if $u \in V_{t-1}$ and $v \in V_t$, it is clear that the maximum number of disjoint $u - v$ paths of length at most 2 is $1 + m - n_{t-1} < k$. Hence, we cannot have k disjoint rainbow $u - v$ paths, and $rvc_k(K_{n_1,\dots,n_t}) \geq 2$. This proves that 2 is a lower bound for the remaining parts, except for part (c)(vi) when $s = 1$. We now prove the remaining assertions.

(a)(ii) We construct a colouring of $K_{n_1,\dots,n_t}$ with two colours, as follows. Assign colour 1 to $\lceil \frac{n_\ell}{2} \rceil$ vertices of V_ℓ for every $\ell < t$, and to $\lfloor \frac{n_t}{2} \rfloor$ vertices of V_t. Colour the remaining vertices with colour 2. Now, let $u \in V_i$ and $v \in V_j$ for some $i < j$. Using both (a) and (b) in Lemma 5, we have $n_i - 2$ disjoint rainbow $u - v$ paths, each using edges between V_i and V_j. With all the paths of $\{uwv : w \notin V_i \cup V_j\}$, we have $(n_i - 2) + (m + n_t - n_i - n_j) \geq m - 2$ disjoint rainbow $u - v$ paths. Hence, $rvc_k(K_{n_1,\dots,n_t}) \leq 2$.

(b)(ii) Consider the same colouring of $K_{n_1,\dots,n_t}$ with two colours as described in (a)(ii). Let $u \in V_i$ and $v \in V_j$ for some $i < j$.

- If $j = t$, then, by Lemma 5(a), we have $n_i - 1$ disjoint rainbow $u - v$ paths, each using edges between V_i and V_t. With all the paths of $\{uwv : w \notin V_i \cup V_t\}$, we have

$(n_i-1)+(m-n_i)=m-1$ disjoint rainbow $u-v$ paths.

- Let $j<t$. If $n_j\le n_t-1$, then, by Lemma 5(b), we have n_i-2 disjoint rainbow $u-v$ paths between V_i and V_j. As before, we have $(n_i-2)+(m+n_t-n_i-n_j)\ge m-1$ disjoint rainbow $u-v$ paths. Now, let $n_j=n_t$. Since we do not have $n_t=n_{t-1}=n_{t-2}$ odd, this means that we cannot have $n_i=n_j$ odd. By Lemma 5(b), we have n_i-1 disjoint rainbow $u-v$ paths between V_i and V_j, which again gives $(n_i-1)+(m+n_t-n_i-n_j)=m-1$ disjoint rainbow $u-v$ paths.

Hence, $rvc_{m-1}(K_{n_1,\dots,n_t})\le 2$.

(b)(iii) Suppose that we have a colouring of $K_{n_1,\dots,n_t}$ with two colours. Without loss of generality, there are sets $A\subset V_{t-1}$ and $B\subset V_t$ with $|A|=|B|=\frac{1}{2}(n_t+1)$, and all the vertices of $A\cup B$ have the same colour. Let $u\in V_{t-1}\setminus A$ and $v\in V_t\setminus B$. The maximum number of disjoint rainbow $u-v$ paths, using edges between V_{t-1} and V_t, is n_t-2. Hence, the maximum number of disjoint rainbow $u-v$ paths is $(n_t-2)+(m-n_{t-1})<m-1$. Therefore, $rvc_{m-1}(K_{n_1,\dots,n_t})\ge 3$.

The upper bound $rvc_{m-1}(K_{n_1,\dots,n_t})\le 3$ will follow immediately once we have proved the upper bound of part (c)(iv).

(c)(ii) Again, consider the same colouring of $K_{n_1,\dots,n_t}$ with two colours as described in (a)(ii). Let $u\in V_i$ and $v\in V_j$ for some $i<j$.

- If $j=t$, then, by Lemma 5(a), we have n_i disjoint rainbow $u-v$ paths, each using edges between V_i and V_t. With all the paths of $\{uwv: w\notin V_i\cup V_t\}$, we have $n_i+(m-n_i)=m$ disjoint rainbow $u-v$ paths.
- If $j<t$, then, by Lemma 5(b), we have n_i-2 disjoint rainbow $u-v$ paths between V_i and V_j. Then, as before, we have $(n_i-2)+(m+n_t-n_i-n_j)\ge m$ disjoint rainbow $u-v$ paths.

Hence, $rvc_m(K_{n_1,\dots,n_t})\le 2$.

(c)(iii) We colour $K_{n_1,\dots,n_t}$ with two colours, where all the vertices of V_t have colour 1 and all the other vertices have colour 2. Then, if $u\in V_i$ and $v\in V_j$ for some $i<j$, we can easily check that there are m disjoint rainbow $u-v$ paths. Hence, $rvc_m(K_{n_1,\dots,n_t})\le 2$.

(c)(iv) Suppose that there is a rainbow vertex m-connected colouring of $K_{n_1,\dots,n_t}$ with two colours, say colours 1 and 2. Then, for any $u\in V_{t-1}$ and $v\in V_t$, there must exist n_{t-1} disjoint rainbow $u-v$ paths, each using edges between V_{t-1} and V_t; otherwise, the maximum number of disjoint rainbow $u-v$ paths would be less than $n_{t-1}+(m-n_{t-1})=m$. It follows that all the vertices of V_t must have the same colour, and the same for V_{t-1}. Otherwise, if a and b vertices of V_{t-1} have colour 1 and colour 2, respectively, where $a+b=n_{t-1}$ and $a,b>0$, then $b+1$ and $a+1$ vertices of V_t have colour 1 and colour 2, respectively, contradicting $n_{t-1}=n_t-1$ or $n_{t-1}=n_t$. Assume that all the vertices of V_t have colour 1, and all the vertices of V_{t-1} have colour 2. Now, take a set $A\subset V_{t-2}$ such that $|A|=\lceil\frac{n_{t-2}}{2}\rceil$, with all the vertices of A having the same colour. Let $u\in V_{t-2}\setminus A$. If the vertices of A have colour 1 (respectively, colour 2), then let $v\in V_t$ (respectively, $v\in V_{t-1}$). There are at most $\lfloor\frac{n_{t-2}}{2}\rfloor$ disjoint rainbow $u-v$ paths, each using edges between V_{t-2} and V_t (respectively, V_{t-1}). Then, we can only have at most $\lfloor\frac{n_{t-2}}{2}\rfloor+m+n_t-n_{t-1}-n_{t-2}<m$ disjoint rainbow $u-v$ paths, a contradiction. Hence, $rvc_m(K_{n_1,\dots,n_t})\ge 3$.

Now, assume that, in addition, we have $(n_t,n_{t-1},n_{t-2},n_{t-3})\ne(4,4,4,4)$ and $t\ge 4$. We construct a colouring of $K_{n_1,\dots,n_t}$ with three colours as follows. If $(n_t,n_{t-1})\ne(4,4)$, then, for every V_i, we colour $\lceil\frac{n_i}{3}\rceil$ and $\lfloor\frac{n_i}{3}\rfloor$ vertices in V_i with colour 1 and colour 3, respectively, and colour the remaining vertices with colour 2. If $(n_t, n_{t-1}) = (4,4)$, we colour two vertices of V_t with colour 1 and the other two vertices with colours 2 and 3, and colour two vertices of V_{t-1} with colour 2 and the other two vertices with colours 3 and 1. If in addition $n_{t-2} = 4$, we colour two vertices of V_{t-2} with colour 3 and the other two vertices with colours 1 and 2. In both cases, colour each remaining V_i with three colours, as described in the case $(n_t, n_{t-1}) \neq (4,4)$. Let $u \in V_i$ and $v \in V_j$ for some $i < j$. By Lemma 6, if $(n_i, n_j) \neq (2,2), (4,4)$, then there are n_i disjoint rainbow $u-v$ paths, each using edges between V_i and V_j. With the paths of $\{uwv : w \notin V_i \cup V_j\}$, we have $n_i + (m + n_t - n_i - n_j) \geq m$ disjoint rainbow $u-v$ paths. If $(n_i, n_j) = (2,2)$, then similarly we have $1 + (m + n_t - n_i - n_j) \geq m$ disjoint rainbow $u-v$ paths. If $(n_i, n_j) = (4,4)$ and $n_j < n_t$, then we have $3 + (m + n_t - n_i - n_j) \geq m$ disjoint rainbow $u-v$ paths. If $(n_i, n_j) = (4,4)$ and $n_j = n_t$, then we have $4 + (m + n_t - n_i - n_j) = m$ disjoint rainbow $u-v$ paths. Hence, $rvc_m(K_{n_1,\dots,n_t}) \leq 3$ in all cases. This also completes the proof of part (b)(iii).

(c)(v) Suppose that we have a colouring of $K_{n_1,\dots,n_t}$ with at most three colours. Without loss of generality, we have $y, y' \in V_t$ and $x, x' \in V_{t-1}$, all having the same colour. Let $u \in V_{t-1} \setminus \{x, x'\}$ and $v \in V_t \setminus \{y, y'\}$. Then, the maximum number of disjoint rainbow $u-v$ paths is $3 + (m - n_{t-1}) < m$. Hence, $rvc_m(K_{n_1,\dots,n_t}) \geq 4$. Now, consider the colouring of $K_{n_1,\dots,n_t}$ with four colours, where, for every V_i, the vertices have colours $1, \dots, n_i$. Let $u \in V_i$ and $v \in V_j$ for some $i < j$. If $(n_i, n_j) \neq (2,2)$, then there are $n_i + (m + n_t - n_i - n_j) \geq m$ disjoint rainbow $u-v$ paths. If $(n_i, n_j) = (2,2)$, then there are $1 + (m + n_t - n_i - n_j) > m$ disjoint rainbow $u-v$ paths. Hence, $rvc_m(K_{n_1,\dots,n_t}) \leq 4$.

(c)(vi) The assertion holds for $s = 1$, so let $s \geq 2$. If we have a colouring of $K_{n_1,\dots,n_t}$ with at most $s-1$ colours, then, without loss of generality, there are $y \in V_t$ and $x \in V_{t-1}$ with the same colour. Let $u \in V_{t-1} \setminus \{x\}$ and $v \in V_t \setminus \{y\}$. Then, we can only have at most $m-1$ rainbow $u-v$ paths. Hence, $rvc_m(K_{n_1,\dots,n_t}) \geq s$. Now, the colouring where, for $1 \leq i \leq s$, both vertices of V_{t-i+1} have colour i, and all the other vertices have colour 1, is a rainbow vertex m-connected colouring. Hence, $rvc_m(K_{n_1,\dots,n_t}) \leq s$.

The completes the proof of Theorem 4. □

3 Comparing $rc_k(G)$ and $rvc_k(G)$

In [5], Krivelevich and Yuster compared the functions $rc(G)$ and $rvc(G)$. They observed that we cannot bound one of $rc(G)$ and $rvc(G)$ in terms of the other, by providing examples of graphs G where $rc(G)$ is much larger than $rvc(G)$, and vice versa. Their examples were as follows. By taking G to be the star $K_{1,s}$, we have $rc(G) = s$ and $rvc(G) = 1$. On the other hand, let G be constructed as follows. Take s vertex-disjoint triangles and, by designating a vertex from each triangle, add a complete graph on the designated vertices. Then $rc(G) \leq 4$ and $rvc(G) = s$.

Here, our goal is to compare the functions $rc_k(G)$ and $rvc_k(G)$. First, we construct graphs G where $rc_k(G)$ is larger than $rvc_k(G)$. Observe that we can extend a star to a *broom*. This is a graph formed by taking a path $xx_1 \cdots x_t$ of length t and adding a star with centre x_t and leaves $y_1, \dots, y_s$, for some $t, s \geq 1$. Let $B_{t,s}$ denote this broom graph; see Figure 1(a). Then, note that we have $rc(B_{t,s}) = t + s$ and $rvc(B_{t,s}) = t$. Hence, given any two integers $1 \leq b < a$, there exists a graph G' with $rc(G') = a$ and $rvc(G') = b$: we take $G' = B_{b,a-b}$. This fact has the following generalisation.

Theorem 7 *Given $1 \leq t < s$, there exists a graph G such that $rc_k(G) \geq s$ and $rvc_k(G) = t$.*

Proof. We take G to be a blow-up of the broom $B_{t,s}$, as follows. Take vertices $x, y_1, \dots, y_s$ and t copies of the clique K_k with vertex sets $X_1, \dots, X_t$. Add all the edges between x and X_1; X_i and X_{i+1} for all $1 \leq i \leq t-1$ (if $t \geq 2$); and y_j and X_t for all $1 \leq j \leq s$. See Figure 1(b). Then, $rc_k(G) \geq s$. Otherwise, if we have an edge-colouring of G with fewer than s colours, we do not have k disjoint rainbow $y_j - y_{j'}$ paths, for some $1 \leq j < j' \leq s$. Also, we have $rvc_k(G) \geq \operatorname{diam}(G) - 1 = t$. Finally, consider the vertex-colouring with t colours, where $x, y_1, \dots, y_s$ are given colour 1, and all the vertices of X_i are given colour i for $1 \leq i \leq t$. Then, we can easily check that this colouring is rainbow vertex k-connected. Hence, $rvc_k(G) \leq t$. $\square$

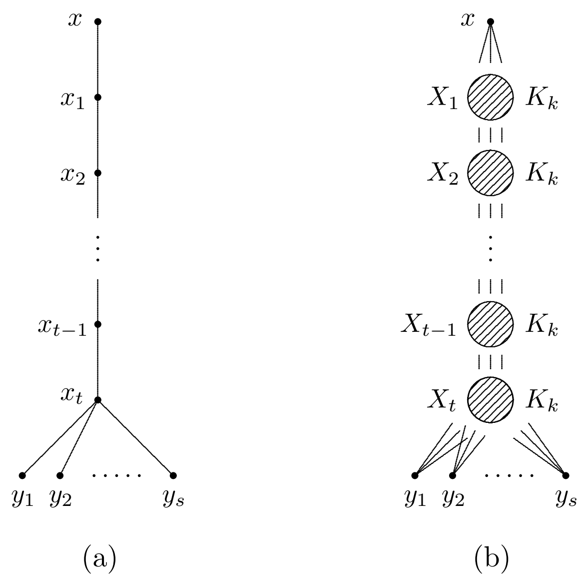

Figure 1. The broom $B_{t,s}$, and its blow-up.

Now, we proceed to construct graphs G where $rvc_k(G)$ is larger than $rc_k(G)$. We need the following result of Chartrand et al. [3].

Theorem 8 (Chartrand et al. [3]) *For $k \geq 2$ and $n \geq (k+1)^2$, we have $rc_k(K_n) = 2$.*

We have the following result.

Theorem 9 *Let $s \geq (k+1)^2$. Then, there exists a graph G such that $rc_k(G) \leq 9$ and $rvc_k(G) = s$.*

Proof. The case $k = 1$ follows from the construction of Krivelevich and Yuster with the disjoint triangles attached to the clique K_s, as described earlier. Now, let $k \geq 2$. We generalise the same construction by taking a blow-up, as follows. Take s disjoint k-sets of vertices $V_1, \dots, V_s$. Let $V_i = \{v_1^i, \dots, v_k^i\}$ for every $1 \leq i \leq s$. For every $1 \leq p \leq k$, we add a clique on $\{v_p^1, \dots, v_p^s\}$. This gives k disjoint copies of K_s. Let G_p be the copy of K_s on $\{v_p^1, \dots, v_p^s\}$ ($1 \leq p \leq k$). Take further disjoint sets $X_1, \dots, X_s$ and $Y_1, \dots, Y_s$, each with $(k+1)^2$ vertices. For each $1 \leq i \leq s$, add a clique with vertex set $X_i \cup Y_i$, and a complete bipartite graph with classes $X_i \cup Y_i$ and V_i. Let G be the resulting graph. We show that G is a suitable graph for the theorem. Let $x_1 \in X_1, \dots, x_s \in X_s$ and $y_1 \in Y_1, \dots, y_s \in Y_s$.

We first define an edge-colouring of G using nine colours. For every $1 \le i \le s$, colour all edges from x_i to V_i with colour 1, and those from y_i to V_i with colour 3. Colour all edges between $X_i \setminus \{x_i\}$ and V_i with colour 2; those between $Y_i \setminus \{y_i\}$ and V_i with colour 4; and those between X_i and Y_i with colour 5. By Theorem 8, we colour the edges of the copies of $K_{(k+1)^2}$ on X_i and Y_i with colours 6 and 7, and the edges of G_1 with colours 8 and 9, so that the edge-colouring within each clique is rainbow k-connected. Finally, colour the edges of $G_2, \dots, G_k$ identically as G_1. That is, for $1 \le i < j \le s$ and $2 \le p \le k$, the edge $v_p^i v_p^j$ has the same colour as the edge $v_1^i v_1^j$.

We claim that this is a rainbow k-connected colouring for G. Let $u, v \in V(G)$. It is easy to see that, if $u, v \in X_i \cup Y_i \cup V_i$ for some $1 \le i \le s$, or if $u \in X_i \cup Y_i$ and $v \in X_j \cup Y_j$ for some $1 \le i < j \le s$, then there are k disjoint rainbow $u - v$ paths. Note that, in the former, we use Theorem 8 when $u, v \in X_i$ or $u, v \in Y_i$. It remains to consider the case when $u \in V_i$ and $v \in X_j \cup Y_j \cup V_j$ for some $1 \le i < j \le s$. For simplicity, assume that $i = 1$, $j = 2$, and $u = v_1^1$. By Theorem 8, there are k disjoint rainbow $u - v_1^2$ paths in G_1, say $uv_1^2, uv_1^{\ell_1}v_1^2, \dots, uv_1^{\ell_{k-1}}v_1^2$ for some $3 \le \ell_1 < \dots < \ell_{k-1} \le s$. If $v \in V_2$ with $v = v_1^2$, then these are $u - v$ paths. Otherwise, if $v = v_p^2$ for some $1 < p \le k$, then we have k disjoint rainbow $u - v$ paths of the form

$$uv_1^2 x_2 x_2' v, uv_1^{\ell_1} x_{\ell_1} x_{\ell_1}' v_p^{\ell_1} v, \dots, uv_1^{\ell_{k-1}} x_{\ell_{k-1}} x_{\ell_{k-1}}' v_p^{\ell_{k-1}} v,$$

where $x_2' \in X_2, x_{\ell_1}' \in X_{\ell_1}, \dots, x_{\ell_{k-1}}' \in X_{\ell_{k-1}}$. If $v \in X_2$ then we have k disjoint rainbow $u - v$ paths, where one path is $uv_1^2 v$, and the other $k - 1$ paths are of the form

$$uv_1^{\ell_1} y_{\ell_1} y_{\ell_1}' v_2^{\ell_1} v_2^2 v, uv_1^{\ell_2} y_{\ell_2} y_{\ell_2}' v_3^{\ell_2} v_3^2 v, \dots, uv_1^{\ell_{k-1}} y_{\ell_{k-1}} y_{\ell_{k-1}}' v_k^{\ell_{k-1}} v_k^2 v,$$

where $y_{\ell_1}' \in Y_{\ell_1}, \dots, y_{\ell_{k-1}}' \in Y_{\ell_{k-1}}$. A similar argument holds for $v \in Y_2$. Hence, the colouring is rainbow k-connected, so $rc_k(G) \le 9$.

Next, suppose that we have a vertex-colouring of G with fewer than s colours. Then, without loss of generality, v_1^1 and v_1^2 have the same colour, and we cannot have k disjoint vertex-rainbow $u - v$ paths for any $u \in X_1 \cup Y_1$ and $v \in X_2 \cup Y_2$. Hence, $rvc_k(G) \ge s$.

Finally, consider the vertex-colouring of G with s colours, where for $1 \le i \le s$ and $1 \le p \le k$, the vertex v_p^i is given colour $i + p - 1$ (modulo s); the vertices of X_i are given colour $i + k$ (modulo s); and those of Y_i are given colour $i + k + 1$ (modulo s). We claim that this is a rainbow vertex k-connected colouring for G. Let $u, v \in V(G)$. Again, it is easy to check that, if $u, v \in V_i \cup X_i \cup Y_i$, or if $u \in X_i \cup Y_i$ and $v \in X_j \cup Y_j$, or if $u = v_p^i$ and $v = v_p^j$ for some $1 \le i \ne j \le s$ and $1 \le p \le k$, then there are k disjoint vertex-rainbow $u - v$ paths. Now, let $u \in V_i$ and $v \in V_j \cup X_j \cup Y_j$ for some $1 \le i \ne j \le s$. Let $u = v_p^i$ for some $1 \le p \le k$. If $v \in V_j$ with $v = v_q^j$ for some $1 \le q \ne p \le k$, then, for any $\ell_1, \dots, \ell_k \in \{1, \dots, s\} \setminus \{i, j\}$, the paths $uv_p^{\ell_1} x_{\ell_1} v_q^{\ell_1} v_q^j, \dots, uv_p^{\ell_k} x_{\ell_k} v_q^{\ell_k} v_q^j$ are disjoint vertex-rainbow $u - v$ paths. If $v \in X_j \cup Y_j$, then we obtain k disjoint vertex-rainbow $u - v$ paths, as follows. One path is $uv_p^j v$. To obtain the other $k - 1$ paths, perform the following procedure. For each $1 \le q \ne p \le k$, choose $h_q \in \{1, \dots, s\} \setminus \{i, j\}$ such that $v_p^{h_q}$ uses a different colour to that of v_q^j; then choose one of the paths $vv_p^{h_q} x_{h_q} v_q^{h_q} v_q^j v$ or $vv_p^{h_q} y_{h_q} v_q^{h_q} v_q^j v$, whichever one is vertex-rainbow. We choose h_q so that distinct q are assigned to distinct h_q, and this is possible since $s \ge (k + 1)^2$. Hence, the colouring is rainbow vertex k-connected, and $rvc_k(G) \le s$. $\square$

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