Rainbow Connection for some Families of Hypergraphs

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1 February, 2014

Abstract

An edge-coloured path in a graph is *rainbow* if its edges have distinct colours. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the minimum number of colours required to colour the edges of G so that any two vertices of G are connected by a rainbow path. The function $rc(G)$ was first introduced by Chartrand et al. [*Math. Bohem.*, 133 (2008), pp. 85-98], and has since attracted considerable interest. In this paper, we introduce two extensions of the rainbow connection number to hypergraphs. We study these two extensions of the rainbow connection number in minimally connected hypergraphs, hypergraph cycles and complete multipartite hypergraphs.

Keywords: Graph colouring, hypergraph colouring, rainbow connection number

1 Introduction

In this paper, we shall consider hypergraphs which are finite, undirected and without multiple edges. For any undefined terms we refer to [1]. Also, for basic terminology for graphs we refer to [2].

The concept of *rainbow connection* in graphs was first introduced by Chartrand et al. [5] in 2008. An edge-coloured path is *rainbow* if the colours of its edges are distinct. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is the minimum

*This work was supported by FCT - Fundação para a Ciência e a Tecnologia through Project PEst-OE/MAT/UI0297/2011 (CMA).

†This work was supported by FCT - Fundação para a Ciência e a Tecnologia through Project PTDC/MAT/113207/2009.

integer t for which there exists a colouring of the edges of G with t colours such that, any two vertices of G are connected by a rainbow path. In their original paper, Chartrand et al. [5] studied the function $rc(G)$ for many graphs G, including when G is a tree, a cycle, a wheel, and a complete multipartite graph. Since then, the rainbow connection subject has attracted considerable interest. Many results about $rc(G)$ have been proved when G satisfies some property, such as a minimum degree condition, a diameter condition, a connectivity condition, and when G is a regular graph or a random graph. Several related functions have also been introduced and studied. These include the *rainbow k-connection number* $rc_k(G)$ and the *rainbow vertex connection number* $rvc(G)$. See for example, Caro et al. [3], Chandran et al. [4], Chartrand et al. [6], Fujita et al. [7], Krivelevich and Yuster [9], and Li et al. [10], among others. A survey by Li et al. [11] and a book by Li and Sun [12] summarising the rainbow connection subject have also appeared recently.

Here, our aim is to extend the notion of rainbow connection to hypergraphs. Such an extension depends on the definition of a path in a hypergraph. To clarify this, we will actually consider two types of paths. For $\ell \geq 1$, a *Berge path*, or simply a *path*, is a hypergraph $\mathcal{P}$ consisting of a sequence $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$, where $v_1, \dots, v_{\ell+1}$ are distinct vertices, $e_1, \dots, e_\ell$ are distinct edges, and $v_i, v_{i+1} \in e_i$ for every $1 \leq i \leq \ell$. The *length* of a path is the number of its edges. If $\mathcal{H}$ is a connected hypergraph, then for $x, y \in V(\mathcal{H})$, an *$x - y$ path* is a path with a sequence $v_1, e_1, \dots, v_\ell, e_\ell, v_{\ell+1}$, where $x = v_1$ and $y = v_{\ell+1}$. The *distance* from x to y, denoted by $d(x, y)$, is the minimum possible length of an $x - y$ path in $\mathcal{H}$. The *diameter* of $\mathcal{H}$ is $\mathrm{diam}(\mathcal{H}) = \max_{x,y \in V(\mathcal{H})} d(x, y)$.

For $\ell \geq 1$ and $1 \leq s < r$, an *(r, s)-path* is an r-uniform hypergraph $\mathcal{P}'$ with vertex set $V(\mathcal{P}') = \{v_1, \dots, v_{(\ell-1)(r-s)+r}\}$ and edge set

$$E(\mathcal{P}') = \{v_1 \cdots v_r, v_{r-s+1} \cdots v_{r-s+r}, v_{2(r-s)+1} \cdots v_{2(r-s)+r}, \dots, v_{(\ell-1)(r-s)+1} \cdots v_{(\ell-1)(r-s)+r}\}.$$

In other words, $\mathcal{P}'$ is an interval hypergraph where all the intervals have size r, and they can be linearly ordered so that every two consecutive intervals intersect in exactly s vertices. For a hypergraph $\mathcal{H}$ and $x, y \in V(\mathcal{H})$, an *$x - y$ (r, s)-path* is an (r, s)-path as described above, with $x = v_1$ and $y = v_{(\ell-1)(r-s)+r}$, if such an (r, s)-path exists in $\mathcal{H}$. Let $\mathcal{F}_{r,s}$ be the family of the hypergraphs $\mathcal{H}$ such that, for every $x, y \in V(\mathcal{H})$, there exists an $x - y$ (r, s)-path. Note that every member of $\mathcal{F}_{r,s}$ is connected. For $\mathcal{H} \in \mathcal{F}_{r,s}$ and $x, y \in V(\mathcal{H})$, the *(r, s)-distance* from x to y, denoted by $d_{r,s}(x, y)$, is the minimum possible length of an $x - y$ (r, s)-path in $\mathcal{H}$. The *(r, s)-diameter* of $\mathcal{H}$ is $\mathrm{diam}_{r,s}(\mathcal{H}) = \max_{x,y \in V(\mathcal{H})} d_{r,s}(x, y)$. If an (r, s)-path has edges $e_1, \dots, e_\ell$, then we will often write the (r, s)-path as $\{e_1, \dots, e_\ell\}$.

The definition of Berge paths was introduced by Berge in the 1970's. The introduction of (r, s)-paths appeared more recently. Notably, in 1999, Katona and Kierstead [8] studied (r, s)-paths when they posed a problem concerning a generalisation of Dirac's theorem to hypergraphs, and since then, such paths have been well-studied.

An edge-coloured path or (r, s)-path (for $1 \leq s < r$) is *rainbow* if its edges have distinct colours. For a connected hypergraph $\mathcal{H}$, an edge-colouring of $\mathcal{H}$ is *rainbow connected* if for any two vertices $x, y \in V(\mathcal{H})$, there exists a rainbow $x - y$ path. The *rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H})$, is the minimum integer t for which there exists a rainbow connected edge-colouring of $\mathcal{H}$ with t colours. Clearly, we have $rc(\mathcal{H}) \geq \mathrm{diam}(\mathcal{H})$. Similarly, for $\mathcal{H} \in \mathcal{F}_{r,s}$, an edge-colouring of $\mathcal{H}$ is *(r, s)-rainbow connected* if for any two vertices $x, y \in V(\mathcal{H})$, there exists a rainbow $x - y$ (r, s)-path. The *(r, s)-rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H}, r, s)$, is the minimum integer t for which there exists an (r, s)-rainbow connected edge-colouring of $\mathcal{H}$ with t colours. Again, we have $rc(\mathcal{H}, r, s) \geq \text{diam}_{r,s}(\mathcal{H})$. Also, note that for $n \geq r \geq 2$, we have $rc(\mathcal{K}_n^r) = rc(\mathcal{K}_n^r, r, s) = 1$, where $\mathcal{K}_n^r$ is the complete r-uniform hypergraph on n vertices.

Hence, we have two generalisations of the rainbow connection number from graphs to hypergraphs. There are good reasons to consider both generalisations. We consider the version with Berge paths because this covers the situation for a larger class of hypergraphs, namely, all connected hypergraphs, rather than just the class $\mathcal{F}_{r,s}$ for the (r, s)-paths version. On the other hand, for many hypergraphs, the version with the (r, s)-paths is more interesting than the one with the Berge paths, in the sense that $rc(\mathcal{H}, r, s)$ is much more difficult to determine than $rc(\mathcal{H})$.

This paper will be organised as follows. In Section 2, we shall give a characterisation of those hypergraphs $\mathcal{H}$ with $rc(\mathcal{H}) = e(\mathcal{H})$ and study $rc(\mathcal{H})$ and $rc(\mathcal{H}, r, s)$ for some specific hypergraphs, namely, cycles and complete multipartite hypergraphs. In Section 3, we will show that the functions $rc(\mathcal{H})$ and $rc(\mathcal{H}, r, s)$ (for $1 \leq s < r$ and $r \geq 3$) are separated in the following sense: there is an infinite family of hypergraphs $\mathcal{G} \subset \mathcal{F}_{r,s}$ such that, $rc(\mathcal{H})$ is bounded on $\mathcal{G}$ by an absolute constant – we will in fact show that $rc(\mathcal{H}) = 2$ on $\mathcal{G}$; and $rc(\mathcal{H}, r, s)$ is unbounded. Note that we have $rc(\mathcal{H}, r, s) \geq rc(\mathcal{H})$ for all $\mathcal{H} \in \mathcal{F}_{r,s}$. Similarly, we will show that the functions $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ (for $1 \leq s \neq s' < r$ and $r \geq 3$) are separated, by proving that $rc(\mathcal{H}, r, s) = 2$ and $rc(\mathcal{H}, r, s')$ is unbounded on an infinite family of hypergraphs $\mathcal{G} \subset \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$. Hence, a bound for one of $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ does not in general imply a bound for the other.

2 Rainbow Connection of some Hypergraphs

In [5], Proposition 1.1, Chartrand et al. proved that for a connected graph G, we have $rc(G) = e(G)$ if and only if G is a tree. We would like to say something similar for hypergraphs. That is, what is a necessary and sufficient condition for a hypergraph $\mathcal{H}$ to have $rc(\mathcal{H}) = e(\mathcal{H})$?

Recall that a hypergraph $\mathcal{T}$ is a *hypertree* if $\mathcal{T}$ is connected, and there exists a simple tree T with $V(T) = V(\mathcal{T})$, with the vertex set of every edge of $\mathcal{T}$ inducing a subtree of T. Unfortunately, in the hypergraphs setting, a necessary and sufficient condition on $\mathcal{H}$ for $rc(\mathcal{H}) = e(\mathcal{H})$ is *not* that $\mathcal{H}$ is a hypertree. There are infinitely many hypertrees $\mathcal{T}$ where $rc(\mathcal{T}) < e(\mathcal{T})$. For example, consider the hypertree $\mathcal{T}$ which is the $(3, 2)$-path of length ℓ, where $\ell \geq 3$. Let $e_1, \ldots, e_\ell$ be the consecutive edges of $\mathcal{T}$. By assigning distinct colours to the edges

$$\begin{cases} e_1, e_3, e_5 \ldots, e_\ell & \text{if } \ell \text{ is odd,} \\ e_1, e_3, e_5 \ldots, e_{\ell-1}, e_\ell & \text{if } \ell \text{ is even,} \end{cases}$$

and then arbitrary (used) colours to the remaining edges, we have a rainbow connected edge-colouring for $\mathcal{T}$, and $rc(\mathcal{T}) \leq \lfloor \frac{\ell}{2} \rfloor + 1$. In fact, we have $rc(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1$, since $rc(\mathcal{T}) \geq \text{diam}(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1$. Hence, we have $rc(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1 < \ell = e(\mathcal{T})$.

Nevertheless, we can still find such a necessary and sufficient condition, which will be a connectivity property. Recall that a graph G with $e(G) \geq 1$ is *minimally connected* if G is connected, and for every $e \in E(G)$ the graph $(V(G), E(G) \setminus \{e\})$ is disconnected. It is well-known that if $e(G) \geq 1$, then G is minimally connected if and only if G is a tree. Hence, Chartrand et al.'s result can be restated as follows: "For a connected graph G with

$e(G) \geq 1$, we have $rc(G) = e(G)$ if and only if G is minimally connected". In this direction, we *do* have the analogous situation for hypergraphs.

We say that a hypergraph $\mathcal{H}$ with $e(\mathcal{H}) \geq 1$ is *minimally connected* if $\mathcal{H}$ is connected, and for every $e \in E(\mathcal{H})$, the hypergraph $(V(\mathcal{H}), E(\mathcal{H}) \setminus \{e\})$ is disconnected. Note that, unlike in the graphs setting, hypertrees and minimally connected hypergraphs are two rather different families. Indeed, any $(3,2)$-path of length at least 3 is a hypertree which is not minimally connected. On the other hand, any 3-uniform hypergraph $\mathcal{C}$, where $V(\mathcal{C}) = \{v_0, \dots, v_{2\ell-1}\}$ and $E(\mathcal{C}) = \{v_0v_1v_2, v_2v_3v_4, \dots, v_{2\ell-4}v_{2\ell-3}v_{2\ell-2}, v_{2\ell-2}v_{2\ell-1}v_0\}$, for some $\ell \geq 2$, is an example of a minimally connected hypergraph which is not a hypertree.

Theorem 1. *Let $\mathcal{H}$ be a connected hypergraph with $e(\mathcal{H}) \geq 1$. Then, $rc(\mathcal{H}) = e(\mathcal{H})$ if and only if $\mathcal{H}$ is minimally connected.*

Proof. Firstly, suppose that $rc(\mathcal{H}) = e(\mathcal{H})$. If $\mathcal{H}$ is not minimally connected, then there exists $e \in E(\mathcal{H})$ such that $\mathcal{H}' = (V(\mathcal{H}), E(\mathcal{H}) \setminus \{e\})$ is connected. The colouring of $\mathcal{H}'$ where every edge is given a distinct colour is rainbow connected for $\mathcal{H}'$, and uses $e(\mathcal{H}) - 1$ colours. Since $V(\mathcal{H}) = V(\mathcal{H}')$, we have $rc(\mathcal{H}) \leq e(\mathcal{H}) - 1$, a contradiction.

Conversely, suppose that $\mathcal{H}$ is minimally connected. Clearly, $rc(\mathcal{H}) \leq e(\mathcal{H})$, and $rc(\mathcal{H}) = e(\mathcal{H})$ if $e(\mathcal{H}) = 1$. Now, assume that $e(\mathcal{H}) \geq 2$. Suppose that we have a colouring for $\mathcal{H}$ with fewer than $e(\mathcal{H})$ colours. Then, there are two edges $e_1, e_2 \in E(\mathcal{H})$ with the same colour. Let $\mathcal{H}' = (V(\mathcal{H}), E(\mathcal{H}) \setminus \{e_1\})$, so that $\mathcal{H}'$ is disconnected. There are two components $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{H}'$ such that $e_2 \notin E(\mathcal{C}_1)$ and $e_2 \in E(\mathcal{C}_2)$. Let $\mathcal{C}_2' = (V(\mathcal{C}_2), E(\mathcal{C}_2) \setminus \{e_2\})$. If each component of $\mathcal{C}_2'$ has a vertex in e_1, then $(V(\mathcal{H}), E(\mathcal{H}) \setminus \{e_2\})$ would be connected, contradicting that $\mathcal{H}$ is minimally connected. Hence, there exists a component $\mathcal{C}_3$ of $\mathcal{C}_2'$ which does not have a vertex in e_1. Now, taking $x \in V(\mathcal{C}_1)$ and $y \in V(\mathcal{C}_3)$, any $x - y$ path in $\mathcal{H}$ must use both e_1 and e_2, and so is not rainbow. Hence, we have $rc(\mathcal{H}) \geq e(\mathcal{H})$. □

Our next aim is to study rainbow connection for hypergraph cycles. For $n > r \geq 2$, the *(n,r)-cycle* $\mathcal{C}_n^r$ is the r-uniform hypergraph on n vertices, say $V(\mathcal{C}_n^r) = \{v_0, \dots, v_{n-1}\}$, with the edge set $E(\mathcal{C}_n^r) = \{e_i = v_iv_{i+1}\dots v_{i+r-1} : i = 0, \dots, n-1\}$, where throughout this subsection concerning cycles, indices of vertices and edges are always taken cyclically modulo n. It is easy to see that $\mathcal{C}_n^r \in \mathcal{F}_{r,s}$ for every $1 \leq s < r$, and hence we can consider $rc(\mathcal{C}_n^r, r, s)$ and $rc(\mathcal{C}_n^r)$. In the case for simple cycles, Chartrand et al. ([5], Prop. 2.1) proved that $rc(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 4$, where C_n denotes the cycle on n vertices. Here, we shall extend this result to the hypergraph cycles $\mathcal{C}_n^r$, as follows.

Theorem 2. *Let $n > r \geq 2$ and $1 \leq s \leq r-2$. Then for sufficiently large n, we have the following.*

(a) $rc(\mathcal{C}_n^r) = rc(\mathcal{C}_n^r, r, 1) = \lceil \frac{n}{2(r-1)} \rceil$.

(b) $rc(\mathcal{C}_n^r, r, r-1) = \lceil \frac{n}{2} \rceil$.

(c) $rc(\mathcal{C}_n^r, r, s) \in \{d, d+1\}$, *where* $d = \mathrm{diam}_{r,s}(\mathcal{C}_n^r) = \lceil \frac{n+1-2s}{2(r-s)} \rceil$.

Before we prove Theorem 2, we prove two lemmas. Firstly, we determine $\mathrm{diam}_{r,s}(\mathcal{C}_n^r)$ for $1 \leq s < r$, and $\mathrm{diam}(\mathcal{C}_n^r)$.

Lemma 3. *For $1 \leq s < r < n$, we have* $\mathrm{diam}_{r,s}(\mathcal{C}_n^r) = \lceil \frac{n+1-2s}{2(r-s)} \rceil$ *and* $\mathrm{diam}(\mathcal{C}_n^r) = \lceil \frac{n-1}{2(r-1)} \rceil$.

Proof. Let $d = \mathrm{diam}_{r,s}(\mathcal{C}_n^r)$. The (r,s)-path with length ℓ has $(\ell-1)(r-s)+r$ vertices, so for any $x \in V(\mathcal{C}_n^r)$, the number of vertices $y \in V(\mathcal{C}_n^r)$ with $d_{r,s}(y,x) \le \ell$ is $2((\ell-1)(r-s)+r)-1$, if $2((\ell-1)(r-s)+r)-1 \le n$. Hence, d is the minimum ℓ for which $2((\ell-1)(r-s)+r)-1 \ge n$. That is, $2((d-2)(r-s)+r)-1 < n \le 2((d-1)(r-s)+r)-1$, which rearranges to

$$\frac{n+1-2s}{2(r-s)} \le d < \frac{n+1-2s}{2(r-s)} + 1.$$

Therefore, $d = \lceil \frac{n+1-2s}{2(r-s)} \rceil$.

Similarly, let $d' = \mathrm{diam}(\mathcal{C}_n^r)$. For any $x \in V(\mathcal{C}_n^r)$, the number of vertices $y \in V(\mathcal{C}_n^r)$ with $d(y,x) \le \ell$ is $2((\ell-1)(r-1)+r)-1$, if $2((\ell-1)(r-1)+r)-1 \le n$. Therefore, d' is the minimum ℓ for which $2((\ell-1)(r-1)+r)-1 \ge n$, and as before, we have $d' = \lceil \frac{n-1}{2(r-1)} \rceil$. $\square$

Secondly, we prove an auxiliary upper bound for $rc(\mathcal{C}_n^r, r, s)$.

Lemma 4. *For $1 \le s < r < n$, we have $rc(\mathcal{C}_n^r, r, s) \le \lceil \frac{n}{2(r-s)} \rceil$.*

Proof. Throughout this proof, we let $c = \lceil \frac{n}{2(r-s)} \rceil$. We divide into two cases.

Case 1. $r-s \mid n$.

We colour the edges of $\mathcal{C}_n^r$ by giving the edge e_k colour $\lfloor \frac{k}{r-s} \rfloor \pmod c$, for $0 \le k \le n-1$. Note that we have the following.

- If $2(r-s) \mid n$, then the colours $0, 1, \dots, c-1$ each occur exactly $2(r-s)$ times, and every (r,s)-path in $\mathcal{C}_n^r$ of length at most c is rainbow.
- If $2(r-s) \nmid n$, then the colours $0, 1, \dots, c-2$ each occur exactly $2(r-s)$ times, and the colour $c-1$ occurs exactly $r-s$ times. Also, every (r,s)-path in $\mathcal{C}_n^r$ of length at most c, whose edges have increasing indices, is rainbow; and every (r,s)-path in $\mathcal{C}_n^r$ of length at most $c-1$ is also rainbow.

Let $v_i, v_j \in V(\mathcal{C}_n^r)$, where $0 \le i < j \le n-1$. Suppose first that $j-i \le c(r-s)$. Consider the (r,s)-path $\mathcal{P} = \{e_i, e_{i+r-s}, e_{i+2(r-s)}, \dots, e_{i+(c-1)(r-s)}\}$. If $i+(c-1)(r-s) < n$, then $j-i \le c(r-s)$ implies that $i+(c-1)(r-s)+r-1 \ge j$. Otherwise, if $i+(c-1)(r-s) \ge n$, then we have $i+c'(r-s)+r-1 \ge j$, where c' is such that $i+c'(r-s) < n \le i+(c'+1)(r-s)$. Therefore, by neglecting the edges of $\mathcal{P}$ with indices at least n (if such edges exist), it follows that $\mathcal{P}$ contains a rainbow $v_i - v_j$ (r,s)-path.

Now, consider the case when $j-i > c(r-s)$, so that we have $n+i-j \le n-c(r-s)-1$. Assume first that $2(r-s) \mid n$, and consider the rainbow (r,s)-path $\mathcal{P}' = \{e_j, e_{j+r-s}, e_{j+2(r-s)}, \dots, e_{j+(c-1)(r-s)}\}$. Easy calculations show that $j+(c-1)(r-s)+r-1 \ge n+i$, which implies that $\mathcal{P}'$ contains a rainbow $v_i - v_j$ (r,s)-path. To conclude this case, suppose that $2(r-s) \nmid n$. Since $r-s \mid n$, we have $n = (2q+1)(r-s)$ for some $q \ge 1$. Consider the rainbow (r,s)-path $\mathcal{P}'' = \{e_j, e_{j+r-s}, e_{j+2(r-s)}, \dots, e_{j+(c-2)(r-s)}\}$. For $\mathcal{P}''$ to contain a rainbow $v_i - v_j$ (r,s)-path, it suffices to have $j+(c-2)(r-s)+r-1 \ge n+i$. This inequality holds if $(c-2)(r-s)+r-1 \ge n-c(r-s)-1$, or equivalently, $2q(r-s)+r \ge n$, which is clearly true since $n = (2q+1)(r-s)$.

Case 2. $r-s \nmid n$.

Let $g = \gcd(r-s, n)$. Consider the subgroup generated by the element $r-s$ in the cyclic group $\mathbb{Z}_n$. The elements of the subgroup are $\{0, r-s, 2(r-s), \dots, (\frac{n}{g}-1)(r-s)\}$, so that the subgroup is isomorphic to $\mathbb{Z}_{n/g}$. The same subgroup is also generated by g, and when the elements of $\{0, r-s, 2(r-s), \dots, (\frac{n}{g}-1)(r-s)\}$ are reduced modulo n and rearranged in ascending order, we get the arithmetic progression $\{0, g, 2g, \dots, (\frac{n}{g}-1)g\}$. We colour the edges as follows. For $0 \le k \le \frac{n}{g}-1$, we colour $e_{k(r-s)}$ with colour k (mod c). This colours the edges whose indices lie in $\mathbb{Z}_{n/g}$. Now, for any coset $\mathbb{Z}_{n/g}+a$, where $0 < a < g$, we colour the edges with indices lying in $\mathbb{Z}_{n/g}+a$ by giving $e_{a+k(r-s)}$ colour k (mod c), the same colour that $e_{k(r-s)}$ received, for $0 \le k \le \frac{n}{g}-1$. Observe that any (r,s)-path consisting of at most c edges, where the indices of the edges are congruent to consecutive members of some coset $\{a, a+r-s, a+2(r-s), \dots, a+(\frac{n}{g}-1)(r-s)\}$ (in this order, where $0 \le a < g$) modulo n, is rainbow.

Let $v_i, v_j \in V(\mathcal{C}_n^r)$, where $0 \le i < j \le n-1$. Suppose first that $j-i \le c(r-s)$. As in Case 1, by considering the (r,s)-path $\mathcal{P} = \{e_i, e_{i+r-s}, e_{i+2(r-s)}, \dots, e_{i+(c-1)(r-s)}\}$, it follows that $\mathcal{P}$ contains a rainbow $v_i - v_j$ (r,s)-path.

To complete the proof, suppose that $j-i > c(r-s)$, so that we have $n+i-j \le n-c(r-s)-1$. Consider the (r,s)-paths $\mathcal{P}' = \{e_j, e_{j+r-s}, e_{j+2(r-s)}, \dots, e_{j+(c-1)(r-s)}\}$ and $\mathcal{P}'' = \{e_{j-g}, e_{j-g+r-s}, e_{j-g+2(r-s)}, \dots, e_{j-g+(c-1)(r-s)}\}$. We first prove that at least one of $\mathcal{P}', \mathcal{P}''$ is a rainbow (r,s)-path. To prove this, we claim that the indices of the edges of at least one of $\mathcal{P}', \mathcal{P}''$ are congruent to c consecutive elements of the coset $\mathbb{Z}_{n/g}+\bar{j} = \{\bar{j}, \bar{j}+r-s, \bar{j}+2(r-s), \dots, \bar{j}+(\frac{n}{g}-1)(r-s)\}$ (in this order) modulo n, where $\bar{j} \equiv j$ (mod g) and $0 \le \bar{j} < g$. By hypothesis, $r-s \nmid n$, so that $g < r-s$. Note that the final $c-1$ members of the coset, when reduced modulo n, are $\bar{j}+n-(c-1)(r-s) < \bar{j}+n-(c-2)(r-s) < \cdots < \bar{j}+n-2(r-s) < \bar{j}+n-(r-s)$ (in this order). Indeed, we have $\bar{j}+n-(c-1)(r-s) \ge \frac{n}{2} > 0$, since $c-1 \le \frac{n}{2(r-s)}$, and $\bar{j}+n-(r-s) < g+n-(r-s) \le n-1$. Also, note that $j-g > c(r-s)-(r-s) \ge 0$, so that $j-g$ is already reduced modulo n. If the claim is false, then we have

$$j = \bar{j}+n-p(r-s) \quad \text{and} \quad j-g = \bar{j}+n-q(r-s) \tag{1}$$

for some $1 \le p \ne q \le c-1$. But (1) is impossible, since on one hand, j and $j-g$ differ by g, but on the other hand, $\bar{j}+n-p(r-s)$ and $\bar{j}+n-q(r-s)$ differ by $|p-q|(r-s) > g$. This proves the claim.

Now, if

$$j-g+(c-1)(r-s)+r-1 \ge n+i, \tag{2}$$

then this would imply that either $\mathcal{P}'$ or $\mathcal{P}''$ contains a rainbow $v_i - v_j$ (r,s)-path (whichever one of $\mathcal{P}', \mathcal{P}''$ is rainbow). Therefore, it suffices to prove the inequality (2). Since we have $n+i-j \le n-c(r-s)-1$, it is enough to show that $-g+(c-1)(r-s)+r-1 \ge n-c(r-s)-1$, which rearranges to

$$2(r-s)\Big\lceil \frac{n}{2(r-s)} \Big\rceil - n + s \ge g. \tag{3}$$

Let $2(r-s) \mid n+b$, where $1 \le b \le 2(r-s)-1$. Then $g \mid b$ since $g = \gcd(r-s, n)$, which implies that $b \ge g$. Therefore, $2(r-s)\lceil \frac{n}{2(r-s)} \rceil - n + s = b + s > g$. Inequality (3) holds, and the proof is complete. $\square$

We are now able to prove Theorem 2.

Proof of Theorem 2. (a) Since $rc(\mathcal{C}_n^r) \le rc(\mathcal{C}_n^r, r, 1) \le \lceil \frac{n}{2(r-1)} \rceil$, where we have used Lemma 4 for the second inequality, it suffices to show that $rc(\mathcal{C}_n^r) \ge \lceil \frac{n}{2(r-1)} \rceil$. By Lemma 3, $rc(\mathcal{C}_n^r) \ge$

$\operatorname{diam}(\mathcal{C}_n^r) = \lceil \frac{n-1}{2(r-1)} \rceil$. We have $\lceil \frac{n-1}{2(r-1)} \rceil = \lceil \frac{n}{2(r-1)} \rceil$ if and only if $n \not\equiv 1 \pmod{2(r-1)}$. Now, let $n \equiv 1 \pmod{2(r-1)}$, with $n = 2k(r-1)+1$. We have to prove that any rainbow connected colouring requires at least $k+1$ colours. Suppose that there exists a rainbow connected colouring using at most k colours. Observe that the $(r,1)$-path $\mathcal{P} = \{e_0, e_{r-1}, e_{2(r-1)}, \ldots, e_{(k-1)(r-1)}\}$ is the unique path of minimum length from v_0 to $v_{k(r-1)}$, and has length k. Hence, we must use exactly k colours, and $\mathcal{P}$ is rainbow. Let $e_{i(r-1)}$ have colour i for $0 \le i \le k-1$. Similarly, the $(r,1)$-path $\{e_{r-1}, e_{2(r-1)}, \ldots, e_{k(r-1)}\}$ is rainbow, and hence e_0 and $e_{k(r-1)}$ must both have colour 0. Repeating the same argument, we find that the edges appear successively as $e_0, e_{r-1}, e_{2(r-1)}, \ldots, e_{(n-1)(r-1)}$, and since n and $r-1$ are coprime, these are exactly all the edges of $\mathcal{C}_n^r$. Also, $e_{i(r-1)}$ has colour $i \pmod{k}$ for $0 \le i \le n-1$. Now, the unique path of length k from $v_{(n-k+1)(r-1)}$ to v_{r-1} is the $(r,1)$-path $\{e_{(n-k+1)(r-1)}, \ldots, e_{(n-1)(r-1)}, e_0\}$, and so must be rainbow. But since $n-1 \equiv 0 \pmod{k}$, this means that $e_{(n-1)(r-1)}$ and e_0 both have colour 0, a contradiction.

(b) Again by Lemma 4, we have $rc(\mathcal{C}_n^r, r, r-1) \le \lceil \frac{n}{2} \rceil$. Now, we prove that $rc(\mathcal{C}_n^r, r, r-1) \ge \lceil \frac{n}{2} \rceil$. Suppose that the edges of $\mathcal{C}_n^r$ are coloured with fewer than $\lceil \frac{n}{2} \rceil$ colours. Then, there are three edges with the same colour. Without loss of generality, for some $1 < i \le \frac{n}{3}$, the edges e_1 and e_i have the same colour. Now, there are exactly two $v_1 - v_{i+r-1}$ $(r,r-1)$-paths. One uses e_1 and e_i, which is not rainbow. The other has length $n-i-2(r-2) > \lceil \frac{n}{2} \rceil$ for n sufficiently large, and hence is also not rainbow.

(c) By Lemma 3 we have $d = \operatorname{diam}_{r,s}(\mathcal{C}_n^r) = \lceil \frac{n+1-2s}{2(r-s)} \rceil$. Therefore, $rc(\mathcal{C}_n^r, r, s) \ge d$. Now, we show that $rc(\mathcal{C}_n^r, r, s) \le d+1$. It suffices to colour only some of the edges with $d+1$ colours, and to show that any two vertices are connected by an (r,s)-path using only the coloured edges.

Suppose firstly that $r-s \nmid n$. In this case, for $0 \le k \le p$, where $p = \lfloor \frac{2n}{r-s} \rfloor$, we colour the edge $e_{k(r-s)}$ with colour $k \pmod{d+1}$. Note that $r-s \nmid n$ implies that $e_0, e_{r-s}, e_{2(r-s)}, \ldots, e_{p(r-s)}$ are distinct edges, and that any (r,s)-path formed by using at most $d+1$ consecutive members of $e_0, e_{r-s}, e_{2(r-s)}, \ldots, e_{p(r-s)}$ is rainbow. Now let $v_i, v_j \in V(\mathcal{C}_n^r)$, with $0 \le i < j \le n-1$. If $j-i \le \lceil \frac{n}{2} \rceil$, then consider the (r,s)-path $\mathcal{P} = \{e_{q(r-s)}, e_{(q+1)(r-s)}, \ldots, e_{(q+d)(r-s)}\}$, where $q = \lfloor \frac{i}{r-s} \rfloor$. Note that $q+d \le \frac{i}{r-s} + \frac{n+1-2s}{2(r-s)} + 1 \le p$ for n sufficiently large, so that $\mathcal{P}$ consists of $d+1$ consecutive members of $e_0, e_{r-s}, e_{2(r-s)}, \ldots, e_{p(r-s)}$. Therefore, $\mathcal{P}$ is rainbow. Also, $q(r-s) \ge i-r+s+1$ and $d(r-s) \ge \frac{n+1}{2} - s$, so that

$$(q+d)(r-s)+r-1 \ge (i-r+s+1) + \left(\frac{n+1}{2} - s\right) + r - 1 = i + \frac{n+1}{2} \ge j.$$

Thus, $\mathcal{P}$ contains a rainbow $v_i - v_j$ (r,s)-path. If $j-i > \lceil \frac{n}{2} \rceil$, then $n+i-j < \lceil \frac{n}{2} \rceil$. In this case, we can obtain a rainbow $v_i - v_j$ (r,s)-path with the same argument, by replacing i and j with j and $n+i$ respectively.

Now, suppose that $r-s \mid n$. For $0 \le k \le \frac{n}{r-s} - 1$, we colour the edge $e_{k(r-s)}$ with colour $k \pmod{d+1}$. Then, let $a \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}$ where $r-s \nmid a$, and note that a exists since $s \le r-2$. For $0 \le k \le \frac{n}{r-s} - 1$, we colour the edge $e_{a+k(r-s)}$ with colour $k \pmod{d+1}$. Now, let $v_i, v_j \in V(\mathcal{C}_n^r)$, with $0 \le i < j \le n-1$. If $j-i \le \lceil \frac{n}{2} \rceil$, then consider the (r,s)-path $\mathcal{P} = \{e_{q(r-s)}, e_{(q+1)(r-s)}, \ldots, e_{q'(r-s)}\}$, where $q = \lfloor \frac{i}{r-s} \rfloor$ and $q' = \min(q+d, \frac{n}{r-s} - 1)$. It is easy to check that $q'(r-s)+r-1 \ge j$, and hence the same argument as before shows that $\mathcal{P}$ contains the required rainbow $v_i - v_j$ (r,s)-path. If $j-i > \lceil \frac{n}{2} \rceil$, then $n+i-j < \lceil \frac{n}{2} \rceil$, $a \le j \le n-1$ and $n \le n+i < \lfloor \frac{3n}{2} \rfloor - 1$. We consider the (r,s)-path $\mathcal{P}' = \{e_{a+q(r-s)}, e_{a+(q+1)(r-s)}, \dots, e_{a+q'(r-s)}\}$, where $q = \lfloor \frac{j-a}{r-s} \rfloor$ and $q' = \min(q+d, \frac{n}{r-s} - 1)$. Again, it is easy to check that $a + q'(r-s) + r - 1 > n + i$, and hence $\mathcal{P}'$ contains the required rainbow $v_i - v_j$ (r,s)-path. The proof is now complete. $\square$

Our final task in this section is to study rainbow connection for complete multipartite hypergraphs. For $t \geq r \geq 2$ and $1 \leq n_1 \leq \cdots \leq n_t$, the r-uniform hypergraph $\mathcal{K}^r_{n_1,\dots,n_t}$ has vertex set consisting of t disjoint sets of vertices with sizes $n_1, \dots, n_t$, say $V(\mathcal{K}^r_{n_1,\dots,n_t}) = V_1 \dot\cup \cdots \dot\cup V_t$, where $|V_i| = n_i$ for all $1 \leq i \leq t$, and edge set $E(\mathcal{K}^r_{n_1,\dots,n_t})$ consisting of all possible r-edges which meet each V_i in at most one vertex. Such a hypergraph $\mathcal{K}^r_{n_1,\dots,n_t}$ is a *complete multipartite hypergraph*, and the V_i are the *(partite) classes* of $\mathcal{K}^r_{n_1,\dots,n_t}$.

For the case of simple graphs (i.e., $r = 2$), Chartrand et al. ([5], Proposition 1.1; Theorems 2.6 and 2.7) determined $rc(\mathcal{K}^2_{n_1,\dots,n_t}) = rc(\mathcal{K}^2_{n_1,\dots,n_t}, 2, 1)$ exactly, as follows. If $m = \sum_{i=1}^{t-1} n_i$ and $n = n_t$, then

$$rc(\mathcal{K}^2_{n_1,\dots,n_t}) = rc(\mathcal{K}^2_{n_1,\dots,n_t}, 2, 1) = \begin{cases} n & \text{if } t = 2 \text{ and } n_1 = 1, \\ \min(\lceil \sqrt[m]{n}\, \rceil, 4) & \text{if } t = 2 \text{ and } 2 \leq n_1 \leq n_2, \\ 1 & \text{if } t \geq 3 \text{ and } n = 1, \\ 2 & \text{if } t \geq 3,\ n \geq 2 \text{ and } m > n, \\ \min(\lceil \sqrt[m]{n}\, \rceil, 3) & \text{if } t \geq 3 \text{ and } m \leq n. \end{cases}$$

Here, we extend their result to complete multipartite hypergraphs. Firstly, we consider $rc(\mathcal{K}^r_{n_1,\dots,n_t})$.

Theorem 5. *Let $t \geq r \geq 3$ and $1 \leq n_1 \leq \cdots \leq n_t$. Then,*

$$rc(\mathcal{K}^r_{n_1,\dots,n_t}) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_{t-1} \geq 2, \text{ or } t > r,\ n_{t-1} = 1 \text{ and } n_t \geq 2, \\ n_t & \text{if } t = r \text{ and } n_{t-1} = 1. \end{cases}$$

Proof. Write $\mathcal{H}$ for $\mathcal{K}^r_{n_1,\dots,n_t}$. Clearly, $rc(\mathcal{H}) = 1$ if $n_t = 1$, since $\mathcal{H} \cong \mathcal{K}^r_t$.

Next, let $n_{t-1} \geq 2$. Then $rc(\mathcal{H}) \geq 2$, since $d(x,y) = 2$ for $x, y \in V_i$, for some $1 \leq i \leq t$. Now, we colour the edges of $\mathcal{H}$ as follows. Assign 0 to one vertex in each V_i, and 1 to all other vertices. For $e \in E(\mathcal{H})$, we colour e with colour 1 if the sum of the vertices of e is odd, and with colour 2 if the sum is even. We claim that this colouring is rainbow connected. Any two vertices in different classes are connected by an edge. Now, let $x, y \in V_i$ for some $1 \leq i \leq t$. If x is assigned with 0 and y is assigned with 1, take $r-1$ vertices $u_1, \dots, u_{r-1} \in V(\mathcal{H}) \setminus V_i$ with no two in the same class. Then, $x, xu_1 \cdots u_{r-1}, u_1, u_1 \cdots u_{r-1}y, y$ is a rainbow $x - y$ path. If x and y are both assigned with 1, take r vertices $v_1, \dots, v_r \in V(\mathcal{H}) \setminus V_i$, where v_j and $v_{j'}$ are in the same class only for $\{j, j'\} = \{1, 2\}$, and v_1 is assigned with 0. Then (since $r \geq 3$), $x, xv_1v_3 \cdots v_r, v_3, v_2v_3 \cdots v_ry, y$ is a rainbow $x - y$ path. Hence, $rc(\mathcal{H}) \leq 2$.

Now, let $t > r$, $n_{t-1} = 1$ and $n_t \geq 2$. Again, we have $rc(\mathcal{H}) \geq 2$. Since $t \geq 3$, we can consider the subhypergraph $\mathcal{H}' \subset \mathcal{H}$, where $V(\mathcal{H}') = V(\mathcal{H})$ and $E(\mathcal{H}') = \{e \in E(\mathcal{H}) : e$ does not contain $V_{t-2} \cup V_{t-1}\}$. Then $\mathcal{H}' \cong \mathcal{K}^r_{n'_1,\dots,n'_{t-1}}$, with $n'_1 = \cdots = n'_{t-3} = 1$, $n'_{t-2} = 2$ and $n'_{t-1} = n_t \geq 2$. Hence, $rc(\mathcal{H}) \leq rc(\mathcal{H}') = 2$.

Finally, let $t = r$ and $n_{t-1} = 1$. Then $\mathcal{H}$ is minimally connected, and by Theorem 1 we have $rc(\mathcal{H}) = e(\mathcal{H}) = n_t$. $\square$

We now consider $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s)$, for $t \geq r \geq 3$ and $1 \leq s < r$, which will be more complicated. Firstly, $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s)$ may not always exist. The next lemma characterises precisely when $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s)$ exists.

Lemma 6. *Let $t \geq r \geq 3$, $1 \leq s < r$ and $1 \leq n_1 \leq \cdots \leq n_t$. Then, $\mathcal{K}^r_{n_1,\dots,n_t} \in \mathcal{F}_{r,s}$ if and only if $n_t = 1$, or $n_{2(t-r)+s+1} \geq 2$ (and $2(t-r)+s+1 \leq t$), or $2(t-r)+s+1 \geq t$.*

Proof. Clearly $\mathcal{K}^r_{n_1,\dots,n_t} \in \mathcal{F}_{r,s}$ if $n_t = 1$. Now, let $n_t \geq 2$ and $p = 2(t-r)+s+1$. If $p < t$ and $n_p = 1$, then for $x, y \in V_t$ and edges e, f with $x \in e$ and $y \in f$, we have $|e \cap f| \geq 2(r-(t-p)) - p > s$. Hence, there is no $x - y$ (r,s)-path, and $\mathcal{K}^r_{n_1,\dots,n_t} \notin \mathcal{F}_{r,s}$. On the other hand, suppose that $n_p \geq 2$ (and $p \leq t$), or $p \geq t$. Any two vertices in different classes of $\mathcal{K}^r_{n_1,\dots,n_t}$ are connected by an edge. Now, for any $x, y \in V_i$ for some class V_i, there are at least $m = \max(t-p, 0)$ classes V_j, $j \neq i$, with $n_j \geq 2$. Let $u_1, v_1, \dots, u_m, v_m$ be vertices from these classes, with each pair u_k, v_k from the same class. There are $t-m-1$ remaining classes (excluding V_i), and it is not difficult to check that $t-m-1 \geq 2(r-m-1)-s \geq r-m-1 \geq s$. Let $w_1, \dots, w_{2(r-m-1)-s}$ be vertices from these $t-m-1$ remaining classes, with one vertex from each class. Consider the edges

$$g = xu_1 \cdots u_m w_1 \cdots w_{r-m-1} \quad \text{and} \quad h = yv_1 \cdots v_m w_1 \cdots w_s w_{r-m} \cdots w_{2(r-m-1)-s}.$$

Then $|g \cap h| = s$, and $\{g, h\}$ is an $x - y$ (r,s)-path. Hence, $\mathcal{K}^r_{n_1,\dots,n_t} \in \mathcal{F}_{r,s}$. $\square$

We remark that Lemma 6 implies that $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s)$ exists if $t \geq 2r - s - 1$, and in particular, $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, r-1)$ always exists. We now determine $rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s)$ exactly, whenever we have existence. We first consider the case when $1 \leq s \leq r-2$.

Theorem 7. *Let $t \geq r \geq 3$, $1 \leq s \leq r-2$ and $1 \leq n_1 \leq \cdots \leq n_t$. Suppose that one of the following holds.*

(i) $n_t = 1$.

(ii) $n_{2(t-r)+s+1} \geq 2$ *(and $2(t-r)+s+1 \leq t$).*

(iii) $2(t-r)+s+1 \geq t$.

Then,

$$rc(\mathcal{K}^r_{n_1,\dots,n_t}, r, s) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_t \geq 2. \end{cases}$$

Proof. Write $\mathcal{H}$ for $\mathcal{K}^r_{n_1,\dots,n_t}$. If (i) holds, then $\mathcal{H} \cong \mathcal{K}^r_t$, and thus $rc(\mathcal{H}, r, s) = 1$.

Now, suppose that (i) does not hold, so that $n_t \geq 2$, and $rc(\mathcal{H}, r, s) \geq 2$. Suppose firstly that $t = r$, which means that (ii) holds. We have $n_{s+1} \geq 2$, so that $n_{s+1}, \dots, n_t = n_r \geq 2$. For every $1 \leq i \leq t$, let $V_i = \{v^i_1, v^i_2, \dots, v^i_{n_i}\}$. We colour an edge $v^1_{k_1} v^2_{k_2} \cdots v^r_{k_r}$ with colour 1 if $k_j = 1$ for some $j \geq s+1$, and with colour 2 otherwise. Clearly, two vertices in two different classes of $\mathcal{H}$ are connected by an edge. Now, for a class V_i, let $v^i_p, v^i_q \in V_i$ with $1 \leq p < q \leq n_i$. If $i \geq s+1$, we take the (r,s)-path $\{e, f\}$ where

$$e = v^1_1 v^2_1 \cdots v^{i-1}_1 v^i_p v^{i+1}_1 \cdots v^r_1 \quad \text{and} \quad f = v^1_1 \cdots v^s_1 v^{s+1}_2 \cdots v^{i-1}_2 v^i_q v^{i+1}_2 \cdots v^r_2.$$

Note that since $s \leq r-2$, we have $i \neq s+1$ or $i \neq r$, thus $v^{s+1}_1 \in e$ or $v^r_1 \in e$, and e has colour 1. Since $q \geq 2$, it is clear that f has colour 2.

If $i \le s$, we take the (r,s)-path $\{g,h\}$ where

$$\begin{aligned} g &= v_1^1 \cdots v_1^{i-1} v_p^i v_1^{i+1} \cdots v_1^s v_2^{s+1} v_1^{s+2} \cdots v_1^r \quad \text{and} \\ h &= v_1^1 \cdots v_1^{i-1} v_q^i v_1^{i+1} \cdots v_1^s v_2^{s+1} v_2^{s+2} \cdots v_2^r. \end{aligned}$$

Again, since $s \le r-2$, we have $v_1^r \in g$, and hence g has colour 1. Clearly, h has colour 2. Hence in both cases, we have a rainbow $v_p^i - v_q^i$ (r,s)-path of length 2, and $rc(\mathcal{H}, r, s) \le 2$.

Now let $t > r$. We obtain the subhypergraph $\mathcal{H}' \subset \mathcal{H}$ with r classes and $V(\mathcal{H}') = V(\mathcal{H})$, as follows. If (ii) holds (which implies that $t < 2r$), or (iii) holds with $t < 2r$, then let the classes of $\mathcal{H}'$ be

$$V_1 \cup V_2, V_3 \cup V_4, \ldots, V_{2(t-r)-1} \cup V_{2(t-r)}, V_{2(t-r)+1}, \ldots, V_t.$$

If (iii) holds with $t \ge 2r$, then let the classes of $\mathcal{H}'$ be

$$V_1 \cup V_2, V_3 \cup V_4, \ldots, V_{2(r-1)-1} \cup V_{2(r-1)}, V_{2r-1} \cup \cdots \cup V_t.$$

In each case, let the edge set $E(\mathcal{H}')$ consist of those edges of $\mathcal{H}$ that meet each class of $\mathcal{H}'$ in exactly one vertex. This means that $\mathcal{H}'$ is a complete multipartite hypergraph with r classes, say $\mathcal{H}' \cong \mathcal{K}^r_{n'_1,\ldots,n'_r}$ for some $1 \le n'_1 \le \cdots \le n'_r$, with $n'_1 + \cdots + n'_r = n_1 + \cdots + n_t$. If (ii) holds, then the condition $n_{2(t-r)+s+1} \ge 2$ implies that, the number of classes of $\mathcal{H}'$ with at least two vertices is at least $t - (2(t-r)+s) + (t-r) = r - s$, and thus $n'_{s+1} \ge 2$. If (iii) holds and $t < 2r$, then the number of such classes of $\mathcal{H}'$ is at least $t - r + 1 \ge r - s$, which again implies $n'_{s+1} \ge 2$. Clearly, if (iii) holds and $t \ge 2r$, then $n'_{s+1} \ge 2$. Hence in every case, we have $rc(\mathcal{H}, r, s) \le rc(\mathcal{H}', r, s) = 2$. $\square$

Now, we consider the case when $s = r-1$. Recall that by Lemma 6, $rc(\mathcal{K}^r_{n_1,\ldots,n_t}, r, r-1)$ always exists (this is also easy to see directly). It is a little surprising that the case when $s = r-1$ alone is much more difficult than every other case, when $1 \le s \le r-2$.

Theorem 8. *Let $t \ge r \ge 3$, $1 \le n_1 \le \cdots \le n_t$, $n = n_t$ and $b = \sum_{S \in [t-1]^{(r-1)}} \prod_{i \in S} n_i$, where $[t-1]^{(r-1)}$ denotes the family of subsets of $\{1,\ldots,t-1\}$ with size $r-1$. Then,*

$$rc(\mathcal{K}^r_{n_1,\ldots,n_t}, r, r-1) = \begin{cases} \lceil \sqrt[b]{n}\, \rceil & \text{if } t = r \text{ and } n_1 = 1, \\ \min(\lceil \sqrt[b]{n}\, \rceil, r+2) & \text{if } t = r \text{ and } n_1 \ge 2, \\ \min(\lceil \sqrt[b]{n}\, \rceil, 3) & \text{if } t > r. \end{cases} \tag{4}$$

Proof. Throughout this proof, we write $\mathcal{H}$ for $\mathcal{K}^r_{n_1,\ldots,n_t}$, and $[N] = \{1,\ldots,N\}$ for a positive integer N. As before, whenever we have constructed a colouring for $\mathcal{H}$ and want to prove that it is $(r, r-1)$-rainbow connected, we only have to show that all pairs of vertices in the same class of $\mathcal{H}$ are connected by a rainbow $(r, r-1)$-path, since any pair of vertices in different classes are connected by an edge of $\mathcal{H}$.

If $n_t = 1$, then $\mathcal{H} \cong \mathcal{K}^r_t$, and hence $rc(\mathcal{H}, r, r-1) = 1$, which agrees with the theorem. From now on, we assume that $n_t \ge 2$, which implies that $\lceil \sqrt[b]{n}\, \rceil \ge 2$. We proceed by proving several claims. In Claims 9 to 11 below, we prove some upper bounds for $rc(\mathcal{H}, r, r-1)$.

Claim 9. $rc(\mathcal{H}, r, r-1) \le \lceil \sqrt[b]{n}\, \rceil$.

Proof. Let $k = \lceil \sqrt[b]{n} \rceil \geq 2$. We have $(k-1)^b < n_t \leq k^b$. For each $w \in V_t$, we assign a $\binom{t-1}{r-1}$-tuple of functions $\{W^S : S \in [t-1]^{(r-1)}\}$ to w such that, if $S \in [t-1]^{(r-1)}$ with $S = \{p_1, \ldots, p_{r-1}\}$ and $1 \leq p_1 < \cdots < p_{r-1} \leq t-1$, we have W^S is a function from $[n_{p_1}] \times \cdots \times [n_{p_{r-1}}]$ to $[k]$. We assign such $\binom{t-1}{r-1}$-tuples of functions to all vertices of V_t in the following way.

- First, choose n_{t-1} vertices from V_t, say $w_1, \ldots, w_{n_{t-1}} \in V_t$. For $1 \leq q \leq n_{t-1}$, the vertex w_q is assigned the $\binom{t-1}{r-1}$-tuple $\{W_q^S : S \in [t-1]^{(r-1)}\}$, where for $S \in [t-1]^{(r-1)}$, we have $W_q^S(i_1, \ldots, i_{r-1}) = 2$ if $i_j = q$ for some $1 \leq j \leq r-1$, and $i_{j'} = 1$ for all $1 \leq j' \leq r-1$, $j' \neq j$; and W_q^S takes the value 1 elsewhere. This can be done since $n_t \geq n_{t-1}$.
- Then, we assign $\binom{t-1}{r-1}$-tuples of functions to the remaining vertices of V_t in such a way that all vertices of V_t will be assigned with distinct sets of functions. That is, for all $w, x \in V_t$, there exists $S \in [t-1]^{(r-1)}$ such that $W^S \neq X^S$, where W^S and X^S are the functions corresponding to S in the $\binom{t-1}{r-1}$-tuples of w and x respectively. This can be done since $n_t \leq k^b$.

For every $1 \leq p \leq t-1$, let $V_p = \{v_1^p, \ldots, v_{n_p}^p\}$. We define a colouring c_1 on $\mathcal{H}$ with k colours as follows. Let $w \in V_t$ and $S \in [t-1]^{(r-1)}$, with $S = \{p_1, \ldots, p_{r-1}\}$ and $1 \leq p_1 < \cdots < p_{r-1} \leq t-1$. For $i_1 \in [n_{p_1}], \ldots, i_{r-1} \in [n_{p_{r-1}}]$, let

$$c_1(wv_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}}) = W^S(i_1, \ldots, i_{r-1}),$$

where W^S is the function corresponding to S assigned to w. We also colour the remaining edges arbitrarily (using the k available colours). We claim that the colouring c_1 is $(r, r-1)$-rainbow connected.

- If $x, y \in V_p$ for some $1 \leq p \leq t-1$, then let $x = v_i^p$ and $y = v_{i'}^p$ for some $1 \leq i < i' \leq n_p$. Consider the vertex $w_i \in V_t$, and choose $S \in [t-1]^{(r-1)}$ such that $p \in S$. Let $p_1, \ldots, p_{r-2}$ be the other elements of S. Then, we may take the $(r, r-1)$-path $\{v_i^p v_1^{p_1} \cdots v_1^{p_{r-2}} w_i, v_1^{p_1} \cdots v_1^{p_{r-2}} w_i v_{i'}^p\}$, which has colours 2 and 1.
- If $x, y \in V_t$, then choose $S \in [t-1]^{(r-1)}$ such that $X^S \neq Y^S$, with $X^S(i_1, \ldots, i_{r-1}) \neq Y^S(i_1, \ldots, i_{r-1})$. Take $\{xv_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}}, v_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}} y\}$, where $S = \{p_1, \ldots, p_{r-1}\}$. This $(r, r-1)$-path has colours $X^S(i_1, \ldots, i_{r-1})$ and $Y^S(i_1, \ldots, i_{r-1})$.

In each case, we have a rainbow $x - y$ $(r, r-1)$-path. Hence, c_1 is $(r, r-1)$-rainbow connected, and we have $rc(\mathcal{H}, r, r-1) \leq k = \lceil \sqrt[b]{n} \rceil$. □

Claim 10. *For $t = r$ and $n_1 \geq 2$, we have $rc(\mathcal{H}, r, r-1) \leq r+2$.*

Proof. If $\lceil \sqrt[b]{n} \rceil \leq r+2$, then the claim follows by using the colouring c_1 in Claim 9. Now, let $\lceil \sqrt[b]{n} \rceil \geq r+3$, so that $n_t > (r+2)^b$. Partition $V_t = U \dot\cup U'$ such that $|U| = (r+2)^b$, so that $U' \neq \emptyset$. Assign to the vertices of U the $(r+2)^b$ distinct functions from $[n_1] \times \cdots \times [n_{t-1}]$ to $[r+2]$, noting that $b = n_1 \cdots n_{t-1}$. For $1 \leq p \leq t-1$, let $V_p = \{v_1^p, \ldots, v_{n_p}^p\}$. We define the colouring c_2 on $\mathcal{H}$ with $r+2$ colours as follows. For $i_1 \in [n_1], \ldots, i_{t-1} \in [n_{t-1}]$ and $w \in U$, let

$$c_2(wv_{i_1}^1 \cdots v_{i_{t-1}}^{t-1}) = W(i_1, \ldots, i_{t-1}),$$

where W is the function assigned to w. For $i_1 \in [n_1], \ldots, i_{t-1} \in [n_{t-1}]$ and $w' \in U'$, let

$$c_2(w'v^1_{i_1} \cdots v^{t-1}_{i_{t-1}}) = \begin{cases} 1 & \text{if } (i_1, \ldots, i_{t-1}) = (1, \ldots, 1), \\ 2 & \text{otherwise.} \end{cases}$$

We claim that the colouring c_2 is $(r, r-1)$-rainbow connected. Note that in the subhypergraph on $V(\mathcal{H}) \setminus U'$ (with all the edges lying inside $V(\mathcal{H}) \setminus U'$), c_2 is $(r, r-1)$-rainbow connected, since c_2 becomes the same type of colouring as c_1 in Claim 9. Hence, it suffices to check that every $x \in U'$ and $y \in V_t$ are connected by a rainbow $(r, r-1)$-path. Let $z \in U$ be the vertex where the function Z has $Z(1, \ldots, 1) = 1$, and Z takes the value 2 elsewhere.

- If $y \in U \setminus \{z\}$, then either $Y(1, \ldots, 1) \neq 1$, or there exist $i_1 \in [n_1], \ldots, i_{t-1} \in [n_{t-1}]$ with $(i_1, \ldots, i_{t-1}) \neq (1, \ldots, 1)$ and $Y(i_1, \ldots, i_{t-1}) \neq 2$. If the former, then we take the $(r, r-1)$-path $\{xv^1_1 \cdots v^{t-1}_1, v^1_1 \cdots v^{t-1}_1 y\}$, which has colours 1 and $Y(1, \ldots, 1)$. If the latter, then we take the $(r, r-1)$-path $\{xv^1_{i_1} \cdots v^{t-1}_{i_{t-1}}, v^1_{i_1} \cdots v^{t-1}_{i_{t-1}} y\}$, which has colours 2 and $Y(i_1, \ldots, i_{t-1})$.
- If $y \in \{z\} \cup U'$, then since $n_1 \geq 2$, we can choose a vertex $w \in U \setminus \{z\}$ such that, the function W of w satisfies

  $$\begin{aligned} W(1, \ldots, 1) &= 3, \\ W(2, 1, \ldots, 1) &= 4, \\ W(2, 2, 1, \ldots, 1) &= 5, \\ &\vdots \\ W(2, \ldots, 2, 1) &= r+1, \\ W(2, \ldots, 2) &= r+2. \end{aligned}$$

  We take the $(r, r-1)$-path

  $$\{xv^1_1 \cdots v^{t-1}_1, v^1_1 \cdots v^{t-1}_1 w, v^2_1 \cdots v^{t-1}_1 w v^1_2, \ldots, v^{t-1}_1 w v^1_2 \cdots v^{t-2}_2, \\ wv^1_2 \cdots v^{t-1}_2, v^1_2 \cdots v^{t-1}_2 y\},$$

  which has colours $1, 3, 4, \ldots, r+1, r+2$ and 2.

In each case, we have a rainbow $x-y$ $(r, r-1)$-path. Hence, c_2 is $(r, r-1)$-rainbow connected, and we have $rc(\mathcal{H}, r, r-1) \leq r+2$. □

Claim 11. *For $t > r$, we have $rc(\mathcal{H}, r, r-1) \leq 3$.*

Proof. If $\lceil \sqrt[b]{n} \rceil \leq 3$, then the claim follows by using the colouring c_1 in Claim 9. Now, let $\lceil \sqrt[b]{n} \rceil \geq 4$, so that $n_t > 3^b$. Partition $V_t = U \dot\cup U'$ such that $|U| = 3^b$, so that $U' \neq \emptyset$. For $w \in U$, we assign a $\binom{t-1}{r-1}$-tuple of functions $\{W^S : S \in [t-1]^{(r-1)}\}$ to w such that, if $S \in [t-1]^{(r-1)}$ with $S = \{p_1, \ldots, p_{r-1}\}$ and $1 \leq p_1 < \cdots < p_{r-1} \leq t-1$, we have W^S is a function from $[n_{p_1}] \times \cdots \times [n_{p_{r-1}}]$ to $\{1, 2, 3\}$. We assign all 3^b such $\binom{t-1}{r-1}$-tuples of functions to the vertices of U.

For $1 \leq p \leq t-1$, let $V_p = \{v^p_1, \ldots, v^p_{n_p}\}$. We define the colouring c_3 on $\mathcal{H}$, using three colours, as follows. Let $S \in [t-1]^{(r-1)}$, with $S = \{p_1, \ldots, p_{r-1}\}$ and $1 \leq p_1 < \cdots < p_{r-1} \leq t-1$. For $i_1 \in [n_{p_1}], \ldots, i_{r-1} \in [n_{p_{r-1}}]$ and $w \in U$, let

$$c_3(wv^{p_1}_{i_1} \cdots v^{p_{r-1}}_{i_{r-1}}) = W^S(i_1, \ldots, i_{r-1}).$$

For $i_1 \in [n_{p_1}], \ldots, i_{t-1} \in [n_{p_{r-1}}]$ and $w' \in U'$, let

$$c_3(w' v_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}}) = \begin{cases} 1 & \text{if } (p_1, \ldots, p_{r-1}) = (1, \ldots, r-1) \\ & \text{and } (i_1, \ldots, i_{r-1}) = (1, \ldots, 1), \\ 2 & \text{otherwise.} \end{cases}$$

Finally, let $c_3(e) = 3$ for every edge e with vertices in r of $V_1, \ldots, V_{t-1}$, noting that such edges exist since $r \leq t-1$.

We claim that the colouring c_3 is $(r, r-1)$-rainbow connected. As in Claim 10, c_3 is $(r, r-1)$-rainbow connected for the subhypergraph on $V(\mathcal{H}) \setminus U'$, and it suffices to check that every $x \in U'$ and $y \in V_t$ are connected by a rainbow $(r, r-1)$-path. Let $z \in U$ be the vertex such that, the function $Z^{[r-1]}$ has $Z^{[r-1]}(1, \ldots, 1) = 1$, and $Z^{[r-1]}$ takes the value 2 elsewhere; and every other function in the $\binom{t-1}{r-1}$-tuple of z is identically equal to 2.

- If $y \in U \setminus \{z\}$, then either $Y^{[r-1]}(1, \ldots, 1) \neq 1$, or there exist $S \in [t-1]^{(r-1)}$, where $S = \{p_1, \ldots, p_{r-1}\}$, $1 \leq p_1 < \cdots < p_{r-1} \leq t-1$, and $i_1 \in [n_{p_1}], \ldots, i_{r-1} \in [n_{p_{r-1}}]$, with $(S, i_1, \ldots, i_{r-1}) \neq ([r-1], 1, \ldots, 1)$ and $Y^S(i_1, \ldots, i_{r-1}) \neq 2$. If the former, then we take the $(r, r-1)$-path $\{x v_1^1 \cdots v_1^{r-1}, v_1^1 \cdots v_1^{r-1} y\}$, which has colours 1 and $Y^{[r-1]}(1, \ldots, 1)$. If the latter, then we take the $(r, r-1)$-path $\{x v_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}}, v_{i_1}^{p_1} \cdots v_{i_{r-1}}^{p_{r-1}} y\}$, which has colours 2 and $Y^S(i_1, \ldots, i_{r-1})$.
- If $y \in \{z\} \cup U'$, then choose a vertex $v \in V_r$ (note that $V_r \neq V_t$ by hypothesis). We take the $(r, r-1)$-path

$$\{x v_1^1 \cdots v_1^{r-1}, v_1^1 \cdots v_1^{r-1} v, v_1^2 \cdots v_1^{r-1} v y\},$$

  which has colours $1, 3$ and 2.

In each case, we have a rainbow $x - y$ $(r, r-1)$-path. Hence, c_3 is $(r, r-1)$-rainbow connected, and we have $rc(\mathcal{H}, r, r-1) \leq 3$. □

Now, in Claim 12 below, we prove some lower bounds for $rc(\mathcal{H}, r, r-1)$.

Claim 12.

(a) If $t = r$ and $n_1 = 1$, then $rc(\mathcal{H}, r, r-1) \geq \lceil \sqrt[b]{n} \rceil$.

(b) If $t = r$ and $n_1 \geq 2$, then $rc(\mathcal{H}, r, r-1) \geq \min(\lceil \sqrt[b]{n} \rceil, r+2)$.

(c) If $t > r$, then $rc(\mathcal{H}, r, r-1) \geq \min(\lceil \sqrt[b]{n} \rceil, 3)$.

Proof. Let $k = \lceil \sqrt[b]{n} \rceil \geq 2$. For $1 \leq p \leq t-1$, let $V_p = \{v_1^p, \ldots, v_{n_p}^p\}$.

(a) and (b) Suppose that we have a colouring c_4 on $\mathcal{H}$, using $k_1 < k$ colours for (a), and $k_1 < \min(k, r+2)$ colours for (b). For every $w \in V_t$, we associate w with the function $W : [n_1] \times \cdots \times [n_{t-1}] \to [k_1]$, where

$$W(i_1, \ldots, i_{t-1}) = c_4(w v_{i_1}^1 \cdots v_{i_{t-1}}^{t-1})$$

for $i_1 \in [n_1], \ldots, i_{t-1} \in [n_{t-1}]$ (note that $i_1 = 1$ for (a)). Then, since $k_1^b \leq (k-1)^b < n = n_t$, this means that there exist $x, y \in V_t$ such that, the functions X and Y are identical, so that every $x - y$ $(r, r-1)$-path of length 2 is monochromatic. Now, observe that in the hypergraph

$\mathcal{K}^r_{n'_1,\dots,n'_r}$, where $1 \le n'_1 \le \dots \le n'_r$, any $(r, r-1)$-path connecting two vertices in the same class with length greater than 2 has length at least $r+2$, and such an $(r, r-1)$-path can only exist if $n'_1 \ge 2$. It follows that there is no rainbow $x-y$ $(r, r-1)$-path in $\mathcal{H}$. Therefore, $rc(\mathcal{H}, r, r-1) \ge k = \lceil \sqrt[b]{n}\, \rceil$ for (a), and $rc(\mathcal{H}, r, r-1) \ge \min(k, r+2) = \min(\lceil \sqrt[b]{n}\, \rceil, r+2)$ for (b).

(c) Suppose that we have a colouring c_5 on $\mathcal{H}$, using fewer than $\min(k, 3)$ colours. Then c_5 uses two colours, and $2^b < n = n_t$. For every $w \in V_t$, we associate w with the $\binom{t-1}{r-1}$-tuple of functions $\{W^S : S \in [t-1]^{(r-1)}\}$, where for $S \in [t-1]^{(r-1)}$ with $S = \{p_1, \dots, p_{r-1}\}$ and $1 \le p_1 < \dots < p_{r-1} \le t-1$, we have W^S is a function from $[n_{p_1}] \times \dots \times [n_{p_{r-1}}]$ to $\{1, 2\}$, given by

$$W^S(i_1, \dots, i_{r-1}) = c_5(wv^{p_1}_{i_1} \cdots v^{p_{r-1}}_{i_{r-1}}),$$

for $i_1 \in [n_{p_1}], \dots, i_{r-1} \in [n_{p_{r-1}}]$. Then, since $n_t > 2^b$, there exist $x, y \in V_t$ such that the $\binom{t-1}{r-1}$-tuples of functions of x and y are the same. This means that every $x-y$ $(r, r-1)$-path of length 2 in $\mathcal{H}$ is monochromatic, and hence there does not exist a rainbow $x-y$ $(r, r-1)$-path. Therefore, $rc(\mathcal{H}, r, r-1) \ge \min(\lceil \sqrt[b]{n}\, \rceil, 3)$. □

We can now easily complete the proof of Theorem 8. For each case of (4), the upper bound follows from some combination of Claims 9 to 11, and the lower bound follows from Claim 12. □

3 Separation of Rainbow Connection Numbers

In this section, we prove that the functions $rc(\mathcal{H})$, $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ are separated from one another, as stated in the introduction.

Theorem 13. *Let $a > 0$, $r \ge 3$ and $1 \le s \ne s' < r$.*

(a) There exists an r-uniform hypergraph $\mathcal{H} \in \mathcal{F}_{r,s}$ such that $rc(\mathcal{H}, r, s) \ge a$ and $rc(\mathcal{H}) = 2$.

(b) There exists an r-uniform hypergraph $\mathcal{H} \in \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$ such that $rc(\mathcal{H}, r, s) \ge a$ and $rc(\mathcal{H}, r, s') = 2$.

Proof. We first prove (b), and then we deduce (a). To prove (b), we consider two cases.

Case 1. $r - s \nmid s - s'$.

Note that in particular, this case holds if $s < s'$. We construct an r-uniform hypergraph $\mathcal{H}$ as follows. Take an (r, s)-path $\mathcal{P}$ of length $\ell \ge 2$. Let $V(\mathcal{P}) = \{v_1, \dots, v_n\}$ and $E(\mathcal{P}) = \{e_1, \dots, e_\ell\}$, where $n = (\ell-1)(r-s) + r$ and $e_i = v_{(i-1)(r-s)+1} \cdots v_{(i-1)(r-s)+r}$ for $1 \le i \le \ell$. For $1 \le i \le \ell - 1$, let $e'_i = v_{(i-1)(r-s)+r-s'+1} \cdots v_{(i-1)(r-s)+r}$, "the last s' vertices" of e_i, and $e'_\ell = v_{n-r+s'+2} \cdots v_n$, "the last $r - s' - 1$ vertices" of e_ℓ. Note that $e'_\ell = \emptyset$ if and only if $s' = r - 1$. Now, let ℓ be sufficiently large such that $r < n - r + s'$. We add to $\mathcal{P}$ the edges

$$\{f_{i,j} = e'_i \cup \{v_j\} \cup e'_\ell : (i-1)(r-s) + r < n - r + s' \text{ and } (i-1)(r-s) + r < j < n - r + s' + 2\}.$$

Note that whenever we have

$$(i-1)(r-s) + r < n - r + s', \tag{5}$$

then there are at least two vertices of $\mathcal{P}$ between $v_{(i-1)(r-s)+r}$ (the "last vertex" of e'_i) and $v_{n-r+s'+2}$ (the "first vertex" of e'_ℓ), and an edge $f_{i,j}$ is obtained by adding a vertex v_j between $v_{(i-1)(r-s)+r}$ and $v_{n-r+s'+2}$ to $e'_i \cup e'_\ell$. The condition on ℓ implies that there exists at least one edge $f_{1,j}$. In particular, if $s < s'$, then $\ell \geq 2$, and for every $1 \leq i \leq \ell - 1$, we have $(i-1)(r-s)+r < n-r+s'$, and there exists at least one edge $f_{i,j}$.

Now, we show that $rc(\mathcal{H}, r, s') = 2$ and $rc(\mathcal{H}, r, s) = \ell$ (which means that $\mathcal{H} \in \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$). By colouring the edges e_k ($1 \leq k \leq \ell$) with colour 1, and the edges $f_{i,j}$ with colour 2, we claim that this colouring is (r, s')-rainbow connected. Consider vertices v_p and v_q ($1 \leq p < q \leq n$) not in the same edge of $\mathcal{P}$. Let $v_p \in e_i$ and $v_q \in e_{i'}$ with i, i' minimum, so that $1 \leq i < i' \leq \ell$. If $s < s'$, then edges $f_{i,j}$ exist, and v_p and v_q are contained in the rainbow (r, s')-path $\{e_i, f_{i,q}\}$ (if $v_q \notin e'_\ell$), or $\{e_i, f_{i,j}\}$ (if $v_q \in e'_\ell$, for some j). Now, let $s > s'$. We divide into the following cases. In each case, we find a rainbow (r, s')-path of length at most 2 containing v_p and v_q.

- If $v_q \in e'_\ell$, we take $f_{1,p}$ (if $i \geq 2$), or $\{e_1, f_{1,j}\}$ (if $i = 1$, for some j).
- Let $v_q \notin e'_\ell$ and $2s \geq r + 1$. We have $p \leq q - s \leq n - r + s' + 1 - s$. Also, if $i \geq 2$, then the definition of i gives $p > (i-2)(r-s) + r$. Thus $(i-1)(r-s)+r < n-r+s'+1-s+(r-s) \leq n-r+s'$ if $2s \geq r+1$. Inequality (5) holds, so that edges $f_{i,j}$ exist (recall that they also exist for $i = 1$), and we can take $\{e_i, f_{i,q}\}$.
- Let $v_q \notin e'_\ell$ and $2s \leq r$. Since $|e_{\ell-1} \cap e_\ell| + |e'_\ell| = s + (r - s' - 1) \geq r = |e_\ell|$, we have $i' \leq \ell - 1$ and $i \leq \ell - 2$. Thus $(i-1)(r-s)+r \leq (\ell-3)(r-s)+r < n-r+s'$ if $2s \leq r$. Again (5) holds, so that edges $f_{i,j}$ exist, and we can take $\{e_i, f_{i,q}\}$.

Therefore, we have $rc(\mathcal{H}, r, s') \leq 2$. Hence $rc(\mathcal{H}, r, s') = 2$, since no edge of $\mathcal{H}$ contains v_1 and v_n.

On the other hand, we clearly have $rc(\mathcal{H}, r, s) \leq \ell$, since any colouring of $\mathcal{H}$ with ℓ colours where the edges of $\mathcal{P}$ have ℓ distinct colours is (r, s)-rainbow connected. We show that $rc(\mathcal{H}, r, s) \geq \ell$, by showing that $\mathcal{P}$ is the unique $v_1 - v_n$ (r, s)-path. Assume that there exists an alternative $v_1 - v_n$ (r, s)-path $\mathcal{P}' = \{g_1, \ldots, g_{\ell'}\}$, for some $\ell' \geq 2$, with the edges $g_1, \ldots, g_{\ell'}$ in this order. Since e_1 is the unique edge containing v_1, this implies that there exists $k \geq 2$ with $g_1 = e_1, \ldots, g_{k-1} = e_{k-1}$ and $g_k = f_{i,j}$ for some $f_{i,j}$. Now if $i \geq k$ (which implies that $(k-1)(r-s)+r < n-r+s'$, by the existence of $f_{i,j}$), then $|g_k \cap g_{k-1}| = |f_{i,j} \cap e_{k-1}| < s$, a contradiction. If $i < k$, then $|g_k \cap g_i| = |f_{i,j} \cap e_i| = s'$. But for $\mathcal{P}'$ to be an (r, s)-path, we have either $|g_k \cap g_i| = 0$ or $|g_k \cap g_i| = s - m(r-s) > 0$ for some $m \geq 0$. Hence $|g_k \cap g_i| = s'$ is not possible in view of $r - s \nmid s - s'$, and we have another contradiction.

Case 2. $r - s \mid s - s'$.

Note that we necessarily have $s > s'$. Let $s - s' = m(r-s)$ for some $m \geq 1$. We construct a similar r-uniform hypergraph $\mathcal{H}$ as follows. Take an (r, s)-path $\mathcal{P}$ of length ℓ on $n = (\ell-1)(r-s)+r$ vertices, and let the v_i, e_i and e'_i be as in Case 1. Observe that $|e_i \cap e_{i'}| = s'$ if and only if $|i - i'| = m + 1$. Let $\ell \geq 2m + 3$. We add to $\mathcal{P}$ the edges

$$\{f_{i,j} = e'_i \cup \{v_j\} \cup e'_\ell : 1 \leq i \leq \ell - 2m - 2 \text{ and}$$
$$(i+m)(r-s)+r < j < n-r+s'+2\}.$$

Again, if $1 \leq i \leq \ell - 2m - 2$, then there exists at least one edge $f_{i,j}$ (since $(i+m)(r-s)+r \leq n-r+s'$), and in particular, there exists at least one edge $f_{1,j}$. We show that $rc(\mathcal{H}, r, s') = 2$ and $rc(\mathcal{H}, r, s) = \ell$. For an edge e_k where $h(m+1) < k \leq (h+1)(m+1)$ for some $h \geq 0$, we colour e_k with colour 1 if h is even, and colour 2 if h is odd. For an edge $f_{i,j}$, we colour it with colour 1 (respectively, colour 2) if e_i has colour 2 (respectively, colour 1). We claim that this colouring is (r, s')-rainbow connected. Let v_p and v_q ($1 \le p < q \le n$) be two vertices not in the same edge of $\mathcal{P}$. If $v_p \in e_{\ell-m-1} \cup \cdots \cup e_\ell$ then, since $|e_{\ell-m-1} \cap e_\ell| = s'$, the path $\{e_{\ell-m-1}, e_\ell\}$ is a rainbow (r, s')-path containing v_p and v_q. Otherwise, if $v_p \notin e_{\ell-m-1} \cup \cdots \cup e_\ell$ then, since $|e_{\ell-2m-2} \cap e_{\ell-m-1}| = s'$, we have $v_p \in e_i$ for some $1 \le i \le \ell - 2m - 2$, and edges $f_{i,j}$ exist. Then, v_p and v_q are contained in the rainbow (r, s')-path $\{e_i, e_{i+m+1}\}$ (if $v_q \in e_i \cup \cdots \cup e_{i+m+1}$), or $\{e_i, f_{i,q}\}$ (if $v_q \notin e_i \cup \cdots \cup e_{i+m+1} \cup e'_\ell$), or $\{e_i, f_{i,j}\}$ (if $v_q \in e'_\ell$, for some j). Therefore, we have $rc(\mathcal{H}, r, s') \le 2$, and hence $rc(\mathcal{H}, r, s') = 2$.

Finally, as in Case 1, we have $rc(\mathcal{H}, r, s) \le \ell$. We show that $rc(\mathcal{H}, r, s) \ge \ell$, by showing that $\mathcal{P}$ is the unique $v_1 - v_n$ (r, s)-path. Again, assume that there exists an alternative $v_1 - v_n$ (r, s)-path $\mathcal{P}' = \{g_1, \dots, g_{\ell'}\}$, for some $\ell' \ge 2$, with the edges $g_1, \dots, g_{\ell'}$ in this order. There exists $k \ge 2$ with $g_1 = e_1, \dots, g_{k-1} = e_{k-1}$ and $g_k = f_{i,j}$ for some $f_{i,j} = e'_i \cup \{v_j\} \cup e'_\ell$. Since $\mathcal{P}'$ is an (r, s)-path, we have $|f_{i,j} \cap e_{k-1}| = s$ and $|f_{i,j} \setminus (e_1 \cup \cdots \cup e_{k-1})| = r - s$. If $v_j \notin e_1 \cup \cdots \cup e_{k-1}$, then $|f_{i,j} \setminus (e_1 \cup \cdots \cup e_{k-1})| \ge |\{v_j\} \cup e'_\ell| = r - s' > r - s$, a contradiction. It follows that $f_{i,j} \setminus (e_1 \cup \cdots \cup e_{k-1})$ consists of exactly $r - s$ of the vertices of e'_ℓ, with the remaining $|e'_\ell| - (r - s) = s - s' - 1 \ge 0$ vertices of e'_ℓ, along with the vertices of $e'_i \cup \{v_j\}$ (giving s vertices in total), lying in $e_1 \cup \cdots \cup e_{k-1}$. In fact, these latter s vertices must lie in e_{k-1}, in view of $|f_{i,j} \cap e_{k-1}| = s$. In particular, both $v_{(i-1)(r-s)+r-s'+1}$ (the "first vertex" of e'_i) and v_j lie in e_{k-1}. But this is not possible, since $j - ((i-1)(r-s) + r - s' + 1) \ge ((i+m)(r-s) + r + 1) - ((i-1)(r-s) + r - s' + 1) = r$, and the difference of the indices of two vertices of e_{k-1} is at most $r - 1$. We have a final contradiction.

In both cases, (b) follows by taking $\ell \ge a$ to be sufficiently large.

We can now deduce (a). Given $a > 0$, $r \ge 3$ and $1 \le s < r$, take s' with $1 \le s \ne s' < r$ and $\mathcal{H} \in \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$ as described in the constructions above, such that $rc(\mathcal{H}, r, s) \ge a$ and $rc(\mathcal{H}, r, s') = 2$. Then, we have $rc(\mathcal{H}) = rc(\mathcal{H}, r, s') = 2$. $\square$

4 Concluding Remark

We have now obtained some introductory results and remarks in the rainbow connection subject for hypergraphs. It would be interesting to extend the study of rainbow connection for further and larger families of hypergraphs; in particular, for those families of hypergraphs which satisfy a certain condition.

Acknowledgements

Henry Liu, Manuel Silva and Teresa Sousa acknowledge the support from FCT - Fundação para a Ciência e a Tecnologia (Portugal), through Project PEst-OE/MAT/UI0297/2011 (CMA). Henry Liu and Teresa Sousa also acknowledge the support from FCT - Fundação para a Ciência e a Tecnologia (Portugal), through Project PTDC/MAT/113207/2009. The authors would like to thank the anonymous referees for reading the paper carefully and for their insightful comments.

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