Monochromatic K_r-Decompositions of Graphs

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Abstract

Given graphs G and H, and a colouring of the edges of G with k colours, a *monochromatic H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a monochromatic graph isomorphic to H. Let $\phi_k(n, H)$ be the smallest number ϕ such that any k-edge-coloured graph G of order n, admits a monochromatic H-decomposition with at most ϕ parts. Here we study the function $\phi_k(n, K_r)$ for $k \geq 2$ and $r \geq 3$.

1 Introduction

Given two graphs G and H, an *H-decomposition* of G is a partition of its edge set, such that, each part is either a single edge or forms an H-subgraph, i.e., a graph isomorphic to H. Let $\phi(G, H)$ be the smallest possible number of parts in an H-decomposition of G. It is easy to see that, for non-empty H, $\phi(G, H) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint copies of H in G. Consider the function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\}.$$

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which is the smallest number such that any graph G of order n admits an H-decompositions with at most $\phi(n, H)$ parts. This function was first studied, in 1966, by Erdős, Goodman and Pósa [2], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n, K_3) = t_2(n)$, where $t_{r-1}(n)$ denotes the number of edges in the Turán graph of order n, $T_{r-1}(n)$, which is the unique complete $(r-1)$-partite graph on n vertices that has the maximum number of edges and contains no complete subgraph of order r. Later, Bollobás [1], proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

General graphs H were only considered recently by Pikhurko and Sousa [8] who proved the following result.

Theorem 1.1. *[8] Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

However, the exact value of the function $\phi(n, H)$ is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order $r \geq 4$ $(n \geq r)$ [10] and the cycles of length 5 $(n \geq 6)$ and 7 $(n \geq 10)$ [9, **?**]. Later, Özkahya and Person [7] determined it for all edge-critical graphs with chromatic number $r \geq 3$ and n sufficiently large. Let $\mathrm{ex}(n, H)$ denote the maximum number of edges in a graph of order n, that does not contain H as a subgraph. They proved the following result.

Theorem 1.2. *[7] Let H be any edge-critical graph with chromatic number $r \geq 3$. Then, there exists n_0 such that $\phi(n, H) = \mathrm{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\phi(n, H)$ is the Turán graph $T_{r-1}(n)$.*

We consider a coloured version of the H-decomposition problem. We define the problem more precisely.

A *k-edge-colouring* of a graph G is a function $c : E(G) \to \{1, \dots, k\}$. Given a fixed graph H, a graph G of order n and a k-edge-colouring of the edges of G, a *monochromatic H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or a monochromatic copy of H. Let $\phi_k(G, H)$ be the smallest number such that, for any k-edge-colouring of G, there exists a monochromatic H-decomposition of G with at most $\phi_k(G, H)$ elements. The goal is to study the function

$$\phi_k(n, H) = \max\{\phi_k(G, H) \mid v(G) = n\},$$

which is the smallest number such that, any k-edge-coloured graph of order n admits a monochromatic H-decomposition with at most $\phi_k(n, H)$ elements.

In this note we study the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 3$.

2 Monochromatic K_r-decompositions of graphs

In this work we will determine the asymptotic value of the function $\phi_k(n, K_3)$ for all $k \geq 2$ (see Theorem 2.1) and the exact value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$

(see Theorem 2.2). Our results involve the Ramsey numbers and the Turán numbers. Recall that for $r \geq 3$ and $k \geq 2$, the *Ramsey number for* K_r, denoted by $R_k(r)$, is the smallest value of s for which every k-edge-colouring of K_s contains a monochromatic K_r. The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite for all $r \geq 3$ and $k \geq 2$. In fact, for the Ramsey numbers $R_k(r)$, only three of them are currently known. In 1955, Greenwood and Gleason [3] were the first to determine $R_2(3) = 6$, $R_3(3) = 17$ and $R_2(4) = 18$.

Theorem 2.1. *For all $k \geq 2$, we have*

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2).$$

For larger cliques we are able to find the exact value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$.

Theorem 2.2. *Let $r \geq 4$, $k \geq 2$. There is an $n_0 = n_0(r, k)$ such that, for all $n \geq n_0$, we have*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

Moreover, the only graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$.

Before presenting the proofs, we need to introduce the tools and some auxiliary results.

A K_r*-cover* in a graph is a set of edges meeting all K_r's, that is, the removal of a K_r-cover results in a K_r-free graph. A K_r*-packing* in a graph is a set of pairwise edge-disjoint K_r's. The K_r*-covering number* of a graph G, denoted by $\tau_r(G)$, is the minimum size of a K_r-cover of G and the K_r*-packing number* of G, denoted by $\nu_r(G)$, is the maximum size of a K_r-packing of G.

A long-standing conjecture of Tuza, states the following.

Conjecture 2.3. *[11] For every graph G, we have $\tau_3(G) \leq 2\nu_3(G)$.*

Conjecture 2.3 remains open, and many partial results have been proved. By combining results of Krivelevich [6], and Haxell and Rödl [5], Yuster [12] observed that, asymptotically, Tuza's conjecture holds. We have the following result which is crucial to the proofs of Theorems 2.1 and 2.2.

Theorem 2.4. *[12] Let G be a graph on n vertices. Then,*

(i) $\tau_3(G) \leq 2\nu_3(G) + o(n^2)$*;*

(ii) $\tau_r(G) \leq \lfloor \frac{r^2}{4} \rfloor \nu_r(G) + o(n^2)$*, for $r \geq 4$.*

Next, we recall the following result of Győri [4] about the existence of edge-disjoint copies of K_r in graphs on n vertices with more than $t_{r-1}(n)$ edges.

Theorem 2.5. *[4] Let $r \geq 3$, and G be a graph on n vertices with $e(G) = t_{r-1}(n) + m$, where $m = o(n^2)$. Then G contains $m - O(\frac{m^2}{n^2}) = (1 - o(1))m$ edge-disjoint copies of K_r.*

We are now able to prove Theorem 2.1 and Theorem 2.2. We will start by proving the lower bound for both theorems.

Proof of the lower bound in Theorems 2.1 and 2.2: By the definition of $R_k(r)$, there exists a k-edge-colouring f' of the complete graph $K_{R_k(r)-1}$ with no monochromatic K_r. Now, consider the Turán graph $T_{R_k(r)-1}(n)$ with a k-edge-colouring f which is a blow-up of f'. Then the graph $T_{R_k(r)-1}(n)$ with the k-edge-colouring f has no monochromatic K_r and thus $\phi_k(n, K_r) \geq \phi_k(T_{R_k(r)-1}(n), K_r) = t_{R_k(r)-1}(n)$. $\square$

Proof of the upper bound in Theorem 2.1: Let $k \geq 2$ be fixed, let $\varepsilon > 0$ be arbitrary and let n_0 be sufficiently large. Let G be a k-edge-coloured graph on $n \geq n_0$ vertices. For the sake of simplicity, let $R = R_k(3)$. We will show that G admits a monochromatic K_3-decomposition with at most $t_{R-1}(n) + \varepsilon n^2$ parts.

Let $e(G) = t_{R-1}(n) + \varepsilon n^2 + m$, where m is an integer. If $m \leq 0$ then G can be decomposed into single edges and we are done.

Suppose that $m > 0$. Observe that it suffices to show that we can find at least $\frac{m}{2}$ edge-disjoint monochromatic copies of K_3, since then G admits a monochromatic K_3-decomposition with at most $e(G) - 2 \cdot \frac{m}{2} = t_{R-1}(n) + \varepsilon n^2$ parts, as required. Therefore, and in order to get a contradiction, assume that the maximum number of edge-disjoint monochromatic copies of K_3 in our graph G is at most $\frac{m}{2}$. For $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices, containing all the edges in colour i. Our assumption implies that $\sum_{i=1}^{k} \nu_3(G_i) \leq \frac{m}{2}$. By Theorem 2.4, we have $\tau_3(G_i) \leq 2\nu_3(G_i) + \frac{\varepsilon}{2k} n^2$ for every $1 \leq i \leq k$. Therefore, we have

$$\sum_{i=1}^{k} \tau_3(G_i) \leq \sum_{i=1}^{k} \left(2\nu_3(G_i) + \frac{\varepsilon}{2k} n^2\right) \leq m + \frac{\varepsilon}{2} n^2.$$

That is, by deleting at most $m + \frac{\varepsilon}{2} n^2$ edges from G, we obtain a subgraph G' which does not contain a monochromatic copy of K_3. On the other hand, we have $e(G') \geq t_{R-1}(n) + \frac{\varepsilon}{2} n^2 > t_{R-1}(n)$. Turán's Theorem implies that G' must contain K_R as a subgraph and hence G' contains a monochromatic copy of K_3. We have a contradiction, and the upper bound of Theorem 2.1 follows. $\square$

Proof of the upper bound in Theorem 2.2: Let $n_0 = n_0(r, k)$ be sufficiently large, let $n \geq n_0$ and let G be any k-edge-coloured graph on n vertices. For the sake of simplicity, let $R = R_k(r)$. We will show that $\phi_k(G, K_r) \leq t_{R-1}(n)$ with equality if and only if $G = T_{R-1}(n)$.

Let $e(G) = t_{R-1}(n) + m$, where m is an integer. If $m < 0$, we can decompose G into single edges and there is nothing to prove. If $m = 0$ and G contains a monochromatic copy of K_r then G admits a monochromatic K_r-decomposition with at most $t_{R-1}(n) - \binom{r}{2} + 1$ parts and we are done. If G does not contain a monochromatic K_r, then the definition of the Ramsey number implies that G does not contain a copy of K_R. Therefore, $G = T_{R-1}(n)$ by Turán's Theorem. Now, let $m > 0$ and let ℓ be the maximum number of edge-disjoint monochromatic K_r's in G. If $\ell > \frac{m}{\binom{r}{2}-1}$, then

$$\phi_k(G, K_r) \leq \ell + e(G) - \binom{r}{2}\ell < t_{R-1}(n).$$

Therefore, it suffices to show that $\ell > \frac{m}{\binom{r}{2}-1}$.

Consider first the case $m = o(n^2)$. By Theorem 2.5 the graph G contains $(1 - o(1))m$ edge-disjoint copies of K_R. Since each K_R contains a monochromatic copy of K_r, this implies that $\ell > \frac{m}{\binom{r}{2}-1}$ and we are done.

Finally, assume that $m \geq Cn^2$, for some constant $C > 0$. In order to get a contradiction, suppose that $\ell \leq \frac{m}{\binom{r}{2}-1}$. For $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices that contains all edges coloured with colour i. By Theorem 2.4, our assumption implies that

$$\begin{aligned}
\sum_{i=1}^{k} \tau_r(G_i) \leq \sum_{i=1}^{k} \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G_i) + o(n^2) &\leq \left\lfloor \frac{r^2}{4} \right\rfloor \ell + o(n^2) \\
&\leq \left\lfloor \frac{r^2}{4} \right\rfloor \frac{m}{\binom{r}{2} - 1} + o(n^2) \\
&\leq \frac{4}{5} m + o(n^2), \text{ since } r \geq 4.
\end{aligned}$$

That is, by deleting at most $\frac{4}{5}m + o(n^2)$ edges from G, we obtain a subgraph G' that does not contain a monochromatic copy of K_r. But

$$e(G') \geq e(G) - \frac{4}{5}m - o(n^2) \geq t_{R-1}(n) + \frac{1}{5}m - o(n^2) > t_{R-1}(n).$$

Therefore, Turán's Theorem implies that G' must contain a copy of K_R which contains a monochromatic copy of K_r. This is a contradiction and our proof is complete. □

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