

Monochromatic Clique Decompositions of Graphs

Henry Liu

Centro de Matemática e Aplicações

Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa

Campus de Caparica, 2829-516 Caparica, Portugal

`h.liu@fct.unl.pt`

Oleg Pikhurko

Mathematics Institute and DIMAP

University of Warwick

Coventry CV4 7AL, United Kingdom

<http://homepages.warwick.ac.uk/staff/O.Pikhurko>

Teresa Sousa

Departamento de Matemática and Centro de Matemática e Aplicações

Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa

Campus de Caparica, 2829-516 Caparica, Portugal

`tmjs@fct.unl.pt`

10 November, 2014

Abstract

Let G be a graph whose edges are coloured with k colours, and $\mathcal{H} = (H_1, \dots, H_k)$ be a k -tuple of graphs. A *monochromatic \mathcal{H} -decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a monochromatic copy of H_i in colour i , for some $1 \leq i \leq k$. Let $\phi_k(n, \mathcal{H})$ be the smallest number ϕ , such that, for every order- n graph and every k -edge-colouring, there is a monochromatic \mathcal{H} -decomposition with at most ϕ elements. Extending the previous results of Liu and Sousa [“Monochromatic K_r -decompositions of graphs”, *Journal of Graph Theory*, 76:89–100, 2014], we solve this problem when each graph in \mathcal{H} is a clique and $n \geq n_0(\mathcal{H})$ is sufficiently large.

Keywords: Monochromatic graph decomposition; Turán Number; Ramsey Number

1 Introduction

All graphs in this paper are finite, undirected and simple. For standard graph-theoretic terminology the reader is referred to [3].

Given two graphs G and H , an H -decomposition of G is a partition of the edge set of G such that each part is either a single edge or forms a subgraph isomorphic to H . Let $\phi(G, H)$ be the smallest possible number of parts in an H -decomposition of G . It is easy to see that, if H is non-empty, we have $\phi(G, H) = e(G) - \nu_H(G)(e(H) - 1)$, where $\nu_H(G)$ is the maximum number of pairwise edge-disjoint copies of H that can be packed into G . Dor and Tarsi [4] showed that if H has a component with at least 3 edges then it is NP-complete to determine if a graph G admits a partition into copies of H . Thus, it is NP-hard to compute the function $\phi(G, H)$ for such H . Nonetheless, many exact results were proved about the extremal function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number such that any graph G of order n admits an H -decomposition with at most $\phi(n, H)$ elements.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [6], who proved that $\phi(n, K_3) = t_2(n)$, where K_s denotes the complete graph (clique) of order s , and $t_{r-1}(n)$ denotes the number of edges in the *Turán graph* $T_{r-1}(n)$, which is the unique $(r-1)$ -partite graph on n vertices that has the maximum number of edges. A decade later, Bollobás [2] proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

Recently Pikhurko and Sousa [13] studied $\phi(n, H)$ for arbitrary graphs H . Their result is the following.

Theorem 1.1. [13] *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

Let $\text{ex}(n, H)$ denote the maximum number of edges in a graph on n vertices not containing H as a subgraph. The result of Turán [20] states that $T_{r-1}(n)$ is the unique extremal graph for $\text{ex}(n, K_r)$. The function $\text{ex}(n, H)$ is usually called the *Turán function* for H . Pikhurko and Sousa [13] also made the following conjecture.

Conjecture 1.2. [13] *For any graph H of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$ for all $n \geq n_0$.*

A graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the *chromatic number* of H . For $r \geq 4$, a *clique-extension of order r* is a connected graph that consists of a K_{r-1} plus another vertex, say v , adjacent to at most $r - 2$ vertices of K_{r-1} . Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [18] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [17, 19]. Later, Özkahya and Person [12] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

Theorem 1.3. [12] *For any edge-critical graph H with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\text{ex}(n, H)$ is the Turán graph $T_{r-1}(n)$.*

Recently, as an extension of Özkahya and Person's work (and as further evidence supporting Conjecture 1.2), Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$ for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.3 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph H .

Motivated by the recent work about H -decompositions of graphs, a natural problem to consider is the Ramsey (or coloured) version of this problem. More precisely, let G be a graph on n vertices whose edges are coloured with k colours, for some $k \geq 2$ and let $\mathcal{H} = (H_1, \dots, H_k)$ be a k -tuple of fixed graphs, where repetition is allowed. A *monochromatic \mathcal{H} -decomposition* of G is a partition of its edge set such that each part is either a single edge, or forms a monochromatic copy of H_i in colour i , for some $1 \leq i \leq k$. Let $\phi_k(G, \mathcal{H})$ be the smallest number, such that, for any k -edge-colouring of G , there exists a monochromatic \mathcal{H} -decomposition of G with at most $\phi_k(G, \mathcal{H})$ elements. Our goal is to study the function

$$\phi_k(n, \mathcal{H}) = \max\{\phi_k(G, \mathcal{H}) \mid v(G) = n\},$$

which is the smallest number ϕ such that, any k -edge-coloured graph of order n admits a monochromatic \mathcal{H} -decomposition with at most ϕ elements. In the case when $H_i \cong H$ for every $1 \leq i \leq k$, we simply write $\phi_k(G, H) = \phi_k(G, \mathcal{H})$ and $\phi_k(n, H) = \phi_k(n, \mathcal{H})$.

The function $\phi_k(n, K_r)$, for $k \geq 2$ and $r \geq 3$, has been studied by Liu and Sousa [11], who obtained results involving the Ramsey numbers and the Turán numbers. Recall that for $k \geq 2$ and integers $r_1, \dots, r_k \geq 3$, the *Ramsey number for K_{r_1}, \dots, K_{r_k}* , denoted by $R(r_1, \dots, r_k)$, is the smallest value of s , such that, for every k -edge-colouring of K_s , there exists a monochromatic K_{r_i} in colour i , for some $1 \leq i \leq k$. For the case when $r_1 = \dots = r_k = r$, for some $r \geq 3$, we simply write $R_k(r) = R(r_1, \dots, r_k)$. Since $R(r_1, \dots, r_k)$ does not change under any permutation of r_1, \dots, r_k , without loss of generality, we assume throughout that $3 \leq r_1 \leq \dots \leq r_k$.

The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite [15]. To this date, the values of $R(3, r_2)$ have been determined exactly only for $3 \leq r_2 \leq 9$, and these are shown in the following table [14].

r_2	3	4	5	6	7	8	9
$R(3, r_2)$	6	9	14	18	23	28	36

The remaining Ramsey numbers that are known exactly are $R(4, 4) = 18$, $R(4, 5) = 25$, and $R(3, 3, 3) = 17$. The gap between the lower bound and the upper bound for other Ramsey numbers is generally quite large.

For the case $R(3, 3) = 6$, it is easy to see that the only 2-edge-colouring of K_5 not containing a monochromatic K_3 is the one where each colour induces a cycle of length 5. From this 2-edge-colouring, observe that we may take a ‘blow-up’ to obtain a 2-edge-colouring of the Turán graph $T_5(n)$, and easily deduce that $\phi_2(n, K_3) \geq t_5(n)$. See Figure 1.

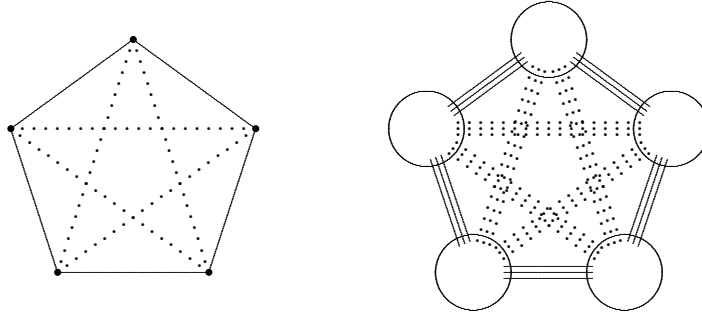


Figure 1. The 2-edge-colouring of K_5 , and its blow-up

This example was the motivation for Liu and Sousa [11] to study K_r -monochromatic decompositions of graphs, for $r \geq 3$ and $k \geq 2$. They have recently proved the following result.

Theorem 1.4. [11]

- (a) $\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2)$;
- (b) $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ for $k = 2, 3$ and n sufficiently large;
- (c) $\phi_k(n, K_r) = t_{R_k(r)-1}(n)$, for $k \geq 2$, $r \geq 4$ and n sufficiently large.

Moreover, the only graph attaining $\phi_k(n, K_r)$ in cases (b) and (c) is the Turán graph $T_{R_k(r)-1}(n)$.

They also made the following conjecture.

Conjecture 1.5. [11] *Let $k \geq 4$. Then $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ for $n \geq R_k(3)$.*

Here, we will study an extension of the monochromatic K_r -decomposition problem when the clique K_r is replaced by a fixed k -tuple of cliques $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$. Our main result, stated in Theorem 1.6, is clearly an extension of Theorem 1.4. Also, it verifies Conjecture 1.5 for sufficiently large n .

Theorem 1.6. *Let $k \geq 2$, $3 \leq r_1 \leq \dots \leq r_k$, and $R = R(r_1, \dots, r_k)$. Let $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$. Then, there is an $n_0 = n_0(r_1, \dots, r_k)$ such that, for all $n \geq n_0$, we have*

$$\phi_k(n, \mathcal{C}) = t_{R-1}(n).$$

Moreover, the only order- n graph attaining $\phi_k(n, \mathcal{C})$ is the Turán graph $T_{R-1}(n)$ (with a k -edge-colouring that does not contain a colour- i copy of K_{r_i} for any $1 \leq i \leq k$).

The upper bound of Theorem 1.6 is proved in Section 2. The lower bound follows easily by the definition of the Ramsey number. Indeed, take a k -edge-colouring f' of the complete graph K_{R-1} without a monochromatic K_{r_i} in colour i , for all $1 \leq i \leq k$. Note that f' exists by definition of the Ramsey number $R = R(r_1, \dots, r_k)$. Let u_1, \dots, u_{R-1} be the vertices of the K_{R-1} . Now, consider the Turán graph $T_{R-1}(n)$ with a k -edge-colouring f which is a ‘blow-up’ of f' . That is, if $T_{R-1}(n)$ has partition classes V_1, \dots, V_{R-1} , then for $v \in V_j$ and $w \in V_\ell$ with $j \neq \ell$, we define $f(vw) = f'(u_j u_\ell)$. Then, $T_{R-1}(n)$ with this k -edge-colouring has no monochromatic K_{r_i} in colour i , for every $1 \leq i \leq k$. Therefore, $\phi_k(n, \mathcal{C}) \geq \phi_k(T_{R-1}(n), \mathcal{C}) = t_{R-1}(n)$ and the lower bound in Theorem 1.6 follows.

In particular, when all the cliques in \mathcal{C} are equal, Theorem 1.6 completes the results obtained previously by Liu and Sousa in Theorem 1.4. In fact, we get the following direct corollary from Theorem 1.6.

Corollary 1.7. *Let $k \geq 2$, $r \geq 3$ and n be sufficiently large. Then,*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

Moreover, the only order- n graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$ (with a k -edge-colouring that does not contain a monochromatic copy of K_r).

2 Proof of Theorem 1.6

In this section we will prove the upper bound in Theorem 1.6. Before presenting the proof we need to introduce the tools. Throughout this section, let $k \geq 2$, $3 \leq$

$r_1 \leq \dots \leq r_k$ be an increasing sequence of integers, $R = R(r_1, \dots, r_k)$ be the Ramsey number for K_{r_1}, \dots, K_{r_k} , and $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$ be a fixed k -tuple of cliques.

We first recall the following stability theorem of Erdős and Simonovits [5, 16].

Theorem 2.1 (Stability Theorem [5, 16]). *Let $r \geq 3$, and G be a graph on n vertices with $e(G) \geq t_{r-1}(n) + o(n^2)$ and not containing K_r as a subgraph. Then, there exists an $(r-1)$ -partite graph G' on n vertices with partition classes V_1, \dots, V_{r-1} , where $|V_i| = \frac{n}{r-1} + o(n)$ for $1 \leq i \leq r-1$, that can be obtained from G by adding and subtracting $o(n^2)$ edges.*

Next, we recall the following result of Győri [7, 8] about the existence of edge-disjoint copies of K_r in graphs on n vertices with more than $t_{r-1}(n)$ edges.

Theorem 2.2. [7, 8] *For every $r \geq 3$ there is C such that every graph G with $n \geq C$ vertices and $e(G) = t_{r-1}(n) + m$ edges, where $m \leq \binom{n}{2}/C$, contains at least $m - Cm^2/n^2$ edge-disjoint copies of K_r .*

Now, we will consider coverings and packings of cliques in graphs. Let $r \geq 3$ and G be a graph. Let \mathcal{K} be the set of all K_r -subgraphs of G . A K_r -cover is a set of edges of G meeting all elements in \mathcal{K} , that is, the removal of a K_r -cover results in a K_r -free graph. A K_r -packing in G is a set of pairwise edge-disjoint copies of K_r . The K_r -covering number of G , denoted by $\tau_r(G)$, is the minimum size of a K_r -cover of G , and the K_r -packing number of G , denoted by $\nu_r(G)$, is the maximum size of a K_r -packing of G . Next, a *fractional K_r -cover* of G is a function $f : E(G) \rightarrow \mathbb{R}_+$, such that $\sum_{e \in E(H)} f(e) \geq 1$ for every $H \in \mathcal{K}$, that is, for every copy of K_r in G the sum of the values of f on its edges is at least 1. A *fractional K_r -packing* of G is a function $p : \mathcal{K} \rightarrow \mathbb{R}_+$ such that $\sum_{H \in \mathcal{K}: e \in E(H)} p(H) \leq 1$ for every $e \in E(G)$, that is, the total weight of K_r 's that cover any edge is at most 1. Here, \mathbb{R}_+ denotes the set of non-negative real numbers. The *fractional K_r -covering number* of G , denoted by $\tau_r^*(G)$, is the minimum of $\sum_{e \in E(G)} f(e)$ over all fractional K_r -covers f , and the *fractional K_r -packing number* of G , denoted by $\nu_r^*(G)$, is the maximum of $\sum_{H \in \mathcal{K}} p(H)$ over all fractional K_r -packings p .

One can easily observe that

$$\nu_r(G) \leq \tau_r(G) \leq \binom{r}{2} \nu_r(G).$$

For $r = 3$, we have $\tau_3(G) \leq 3\nu_3(G)$. A long-standing conjecture of Tuza [21] from 1981 states that this inequality can be improved as follows.

Conjecture 2.3. [21] *For every graph G , we have $\tau_3(G) \leq 2\nu_3(G)$.*

Conjecture 2.3 remains open although many partial results have been proved. By using the earlier results of Krivelevich [10], and Haxell and Rödl [9], Yuster [22] proved the following theorem which will be crucial to the proof of Theorem 1.6. In the case $r = 3$, it is an asymptotic solution of Tuza's conjecture.

Theorem 2.4. [22] *Let $r \geq 3$ and G be a graph on n vertices. Then*

$$\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G) + o(n^2). \quad (2.1)$$

We now prove the following lemma which states that a graph G with n vertices and at least $t_{R-1}(n) + \Omega(n^2)$ edges falls quite short of being optimal.

Lemma 2.5. *For every $k \geq 2$ and $c_0 > 0$ there are $c_1 > 0$ and n_0 such that for every graph G of order $n \geq n_0$ with at least $t_{R-1}(n) + c_0 n^2$ edges, we have $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) - c_1 n^2$.*

Proof. Suppose that the lemma is false, that is, there is $c_0 > 0$ such that for some increasing sequence of n there is a graph G on n vertices with $e(G) \geq t_{R-1}(n) + c_0 n^2$ and $\phi_k(G, \mathcal{C}) \geq t_{R-1}(n) + o(n^2)$. Fix a k -edge-colouring of G and, for $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices that contains all edges with colour i .

Let $m = e(G) - t_{R-1}(n)$, and let $s \in \{0, \dots, k\}$ be the maximum such that

$$r_1 = \dots = r_s = 3.$$

Let us very briefly recall the argument from [11] that shows $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) + o(n^2)$, adopted to our purposes. If we remove a K_{r_i} -cover from G_i for every $1 \leq i \leq k$, then we destroy all copies of K_R in G . By Turán's theorem, at most $t_{R-1}(n)$ edges remain. Thus,

$$\sum_{i=1}^k \tau_{r_i}(G_i) \geq m. \quad (2.2)$$

By Theorem 2.4, if we decompose G into a maximum K_{r_i} -packing in each G_i and the remaining edges, we obtain that

$$\begin{aligned} \phi_k(G, \mathcal{C}) &\leq e(G) - \sum_{i=1}^k \left(\binom{r_i}{2} - 1 \right) \nu_{r_i}(G_i) \\ &\leq t_{R-1}(n) + m - \sum_{i=1}^k \frac{\binom{r_i}{2} - 1}{\lfloor r_i^2/4 \rfloor} \tau_{r_i}(G_i) + o(n^2) \\ &\leq t_{R-1}(n) + m - \sum_{i=1}^k \tau_{r_i}(G_i) - \frac{1}{4} \sum_{i=s+1}^k \tau_{r_i}(G_i) + o(n^2) \leq t_{R-1}(n) + o(n^2). \end{aligned} \quad (2.3)$$

The third inequality holds since $((\binom{r}{2} - 1)/\lfloor r^2/4 \rfloor) \geq 5/4$ for $r \geq 4$ and is equal to 1 for $r = 3$.

Let us derive a contradiction from this by looking at the properties of our hypothetical counterexample G . First, all inequalities that we saw have to be equalities within an additive term $o(n^2)$. In particular, the slack in (2.2) is $o(n^2)$, that is,

$$\sum_{i=1}^k \tau_{r_i}(G_i) = m + o(n^2). \quad (2.4)$$

Also, $\sum_{i=s+1}^k \tau_{r_i}(G_i) = o(n^2)$. In particular, we have that $s \geq 1$. To simplify the later calculations, let us re-define G by removing a maximum K_{r_i} -packing from G_i for each $i \geq s+1$. The new graph is still a counterexample to the lemma if we decrease c_0 slightly, since the number of edges removed is at most $\sum_{i=s+1}^k \binom{r_i}{2} \tau_{r_i}(G_i) = o(n^2)$.

Suppose that we remove, for each $i \leq s$, an arbitrary (not necessarily minimum) K_3 -cover F_i from G_i such that

$$\sum_{i=1}^s |F_i| \leq m + o(n^2). \quad (2.5)$$

Let $G' \subseteq G$ be the obtained K_R -free graph. (Recall that we assumed that G_i is K_{r_i} -free for all $i \geq s+1$.) Let $G'_i \subseteq G_i$ be the colour classes of G' . We know by (2.5) that $e(G') \geq t_{R-1}(n) + o(n^2)$. Since G' is K_R -free, we conclude by the Stability Theorem (Theorem 2.1) that there is a partition $V(G) = V(G') = V_1 \dot{\cup} \dots \dot{\cup} V_{R-1}$ such that

$$\forall i \in \{1, \dots, R-1\}, \quad |V_i| = \frac{n}{R-1} + o(n) \quad \text{and} \quad |E(T) \setminus E(G')| = o(n^2), \quad (2.6)$$

where T is the complete $(R-1)$ -partite graph with parts V_1, \dots, V_{R-1} .

Next, we essentially expand the proof of (2.1) for $r = 3$ and transform it into an algorithm that produces K_3 -coverings F_i of G_i , with $1 \leq i \leq s$, in such a way that (2.5) holds but (2.6) is impossible whatever V_1, \dots, V_{R-1} we take, giving the desired contradiction.

Let H be an arbitrary graph of order n . By the LP duality, we have that

$$\tau_r^*(H) = \nu_r^*(H). \quad (2.7)$$

By the result of Haxell and Rödl [9] we have that

$$\nu_r^*(H) = \nu_r(H) + o(n^2). \quad (2.8)$$

Krivelevich [10] showed that

$$\tau_3(H) \leq 2\tau_3^*(H). \quad (2.9)$$

Thus, $\tau_3(H) \leq 2\nu_3(H) + o(n^2)$ giving (2.1) for $r = 3$.

The proof of Krivelevich [10] of (2.9) is based on the following result.

Lemma 2.6. *Let H be an arbitrary graph and $f : E(H) \rightarrow \mathbb{R}_+$ be a minimum fractional K_3 -cover. Then $\tau_3(H) \leq \frac{3}{2}\tau_3^*(H)$ or there is $xy \in E(H)$ with $f(xy) = 0$ that belongs to at least one triangle of H .*

Proof. If there is an edge $xy \in E(H)$ that does not belong to a triangle, then necessarily $f(xy) = 0$ and xy does not belong to any optimal fractional or integer K_3 -cover. We can remove xy from $E(H)$ without changing the validity of the lemma. Thus, we can assume that every edge of H belongs to a triangle.

Suppose that $f(xy) > 0$ for every edge xy of H , for otherwise we are done. Take a maximum fractional K_3 -packing p . Recall that it is a function that assigns a weight $p(xyz) \in \mathbb{R}_+$ to each triangle xyz of H such that for every edge xy the sum of weights over all K_3 's of H containing xy is at most 1, that is,

$$\sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) \leq 1, \quad (2.10)$$

where $\Gamma(v)$ denotes the set of neighbours of the vertex v in H .

This is the dual LP to the minimum fractional K_3 -cover problem. By the complementary slackness condition (since f and p are optimal solutions), we have equality in (2.10) for every $xy \in E(H)$. This and the LP duality imply that

$$\tau_3^*(H) = \nu_3^*(H) = \sum_{\text{triangle } xyz} p(xyz) = \frac{1}{3} \sum_{xy \in E(H)} \sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) = \frac{1}{3} e(H).$$

On the other hand $\tau_3(H) \leq \frac{1}{2} e(H)$: take a bipartite subgraph of H with at least half of the edges; then the remaining edges form a K_3 -cover. Putting the last two inequalities together, we obtain the required result. \square

Let $1 \leq i \leq s$. We now describe an algorithm for finding a K_3 -cover F_i in G_i . Initially, let $H = G_i$ and $F_i = \emptyset$. Repeat the following.

Take a minimum fractional K_3 -cover f of H . If the first alternative of Lemma 2.6 is true, pick a K_3 -cover of H of size at most $\frac{3}{2}\tau_3^*(H)$, add it to F_i and stop. Otherwise, fix some edge $xy \in E(H)$ returned by Lemma 2.6. Let F' consist of all pairs xz and

yz over $z \in \Gamma(x) \cap \Gamma(y)$. Add F' to F_i and remove F' from $E(H)$. Repeat the whole step (with the new H and f).

Consider any moment during this algorithm, when we had $f(xy) = 0$ for some edge xy of H . Since f is a fractional K_3 -cover, we have that $f(xz) + f(yz) \geq 1$ for every $z \in \Gamma(x) \cap \Gamma(y)$. Thus, if H' is obtained from H by removing 2ℓ such pairs, where $\ell = |\Gamma(x) \cap \Gamma(y)|$, then $\tau_3^*(H') \leq \tau_3^*(H) - \ell$ because f when restricted to $E(H')$ is still a fractional cover (although not necessarily an optimal one). Clearly, $|F_i|$ increases by 2ℓ during this operation. Thus, indeed we obtain, at the end, a K_3 -cover F_i of G_i of size at most $2\tau_3^*(G_i)$.

Also, by (2.7) and (2.8) we have that

$$\sum_{i=1}^s |F_i| \leq 2 \sum_{i=1}^s \nu_3(G_i) + o(n^2).$$

Now, since all slacks in (2.3) are $o(n^2)$, we conclude that

$$\sum_{i=1}^s \nu_3(G_i) \leq \frac{m}{2} + o(n^2)$$

and (2.5) holds. In fact, (2.5) is equality by (2.4).

Recall that G'_i is obtained from G_i by removing all edges of F_i and G' is the edge-disjoint union of the graphs G'_i . Suppose that there exist V_1, \dots, V_{R-1} satisfying (2.6). Let $M = E(T) \setminus E(G')$ consist of *missing* edges. Thus, $|M| = o(n^2)$.

Let

$$X = \{x \in V(T) \mid \deg_M(x) \geq c_2 n\},$$

where we define $c_2 = (4(R-1))^{-1}$. Clearly,

$$|X| \leq 2|M|/c_2 n = o(n).$$

Observe that, for every $1 \leq i \leq s$, if the first alternative of Lemma 2.6 holds at some point, then the remaining graph H satisfies $\tau_3^*(H) = o(n^2)$. Indeed, otherwise by $\tau_3(G_i) \leq 2\tau_3^*(G_i) - \tau_3^*(H)/2 + o(n^2)$ we get a strictly smaller constant than 2 in (2.9) and thus a gap of $\Omega(n^2)$ in (2.3), a contradiction. Therefore, all but $o(n^2)$ edges in F_i come from some *parent edge* xy that had f -weight 0 at some point.

When our algorithm adds pairs xz and yz to F_i with the same parent xy , then it adds the same number of pairs incident to x as those incident to y . Let \mathcal{P} consist of pairs xy that are disjoint from X and were a parent edge during the run of the algorithm. Since the total number of pairs in F_i incident to X is at most $n|X| = o(n^2)$, there are $|F_i| - o(n^2)$ pairs in F_i such that their parent is in \mathcal{P} .

Let us show that y_0 and y_1 belong to different parts V_j for every pair $y_0y_1 \in \mathcal{P}$. Suppose on the contrary that, say, $y_0, y_1 \in V_1$. For each $2 \leq j \leq R-1$ pick an arbitrary $y_j \in V_j \setminus (\Gamma_M(y_0) \cup \Gamma_M(y_1))$. Since $y_0, y_1 \notin X$, the possible number of choices for y_j is at least

$$\frac{n}{R-1} - 2c_2n + o(n) \geq \frac{n}{R-1} - 3c_2n.$$

Let

$$Y = \{y_0, \dots, y_{R-1}\}.$$

By the above, we have at least $(\frac{n}{R-1} - 3c_2n)^{R-2} = \Omega(n^{R-2})$ choices of Y . Note that by the definition, all edges between $\{y_0, y_1\}$ and the rest of Y are present in $E(G')$. Thus, the number of sets Y containing at least one edge of M different from y_0y_1 is at most

$$|M| \times n^{R-4} = o(n^{R-2}).$$

This is $o(1)$ times the number of choices of Y . Thus, for almost every Y , $H = G'[Y]$ is a clique (except perhaps the pair y_0y_1). In particular, there is at least one such choice of Y ; fix it. Let $i \in \{1, \dots, k\}$ be arbitrary. Adding back the pair y_0y_1 coloured i to H (if it is not there already), we obtain a k -edge-colouring of the complete graph H of order R . By the definition of $R = R(r_1, \dots, r_k)$, there must be a monochromatic triangle on abc of colour $h \leq s$. (Recall that we assumed at the beginning that G_j is K_{r_j} -free for each $j > s$.) But abc has to contain an edge from the K_3 -cover F_h , say ab . This edge ab is not in G' (it was removed from G). If a, b lie in different parts V_j , then $ab \in M$, a contradiction to the choice of Y . The only possibility is that $ab = y_0y_1$. Then $h = i$. Since both y_0c and y_1c are in G'_i , they were never added to the K_3 -cover F_i by our algorithm. Therefore, y_0y_1 was never a parent, which is the desired contradiction.

Thus, every $xy \in \mathcal{P}$ connects two different parts V_j . For every such parent xy , the number of its children in M is at least half of all its children. Indeed, for every pair of children xz and yz , at least one connects two different parts; this child necessarily belongs to M . Thus,

$$|F_i \cap M| \geq \frac{1}{2} |F_i| + o(n^2).$$

(Recall that parent edges that intersect X produce at most $2n|X| = o(n^2)$ children.) Therefore,

$$|M| \geq \frac{1}{2} \sum_{i=1}^s |F_i| + o(n^2) \geq \frac{m}{2} + o(n^2) = \Omega(n^2),$$

contradicting (2.6). This contradiction proves Lemma 2.5. \square

We are now able to prove Theorem 1.6.

Proof of the upper bound in Theorem 1.6. Let C be the constant returned by Theorem 2.2 for $r = R$. Let $n_0 = n_0(r_1, \dots, r_k)$ be sufficiently large to satisfy all the inequalities we will encounter. Let G be a k -edge-coloured graph on $n \geq n_0$ vertices. We will show that $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n)$ with equality if and only if $G = T_{R-1}(n)$, and G does not contain a monochromatic copy of K_{r_i} in colour i for every $1 \leq i \leq k$.

Let $e(G) = t_{R-1}(n) + m$, where m is an integer. If $m < 0$, we can decompose G into single edges and there is nothing to prove.

Suppose $m = 0$. If G contains a monochromatic copy of K_{r_i} in colour i for some $1 \leq i \leq k$, then G admits a monochromatic \mathcal{C} -decomposition with at most $t_{R-1}(n) - \binom{r_i}{2} + 1 < t_{R-1}(n)$ parts and we are done. Otherwise, the definition of R implies that G does not contain a copy of K_R . Therefore, $G = T_{R-1}(n)$ by Turán's theorem and $\phi_k(G, \mathcal{C}) = t_{R-1}(n)$ as required.

Now suppose $m > 0$. We can also assume that $m < \binom{n}{2}/C$ for otherwise we are done: $\phi_k(G, \mathcal{C}) < t_{R-1}(n)$ by Lemma 2.5. Thus, by Theorem 2.2, the graph G contains at least $m - Cm^2/n^2 > \frac{m}{2}$ edge-disjoint copies of K_R . Since each K_R contains a monochromatic copy of K_{r_i} in the colour- i graph G_i , for some $1 \leq i \leq k$, we conclude that $\sum_{i=1}^k \nu_{r_i}(G_i) > \frac{m}{2}$, so that $\sum_{i=1}^k (\binom{r_i}{2} - 1)\nu_{r_i}(G_i) \geq \sum_{i=1}^k 2\nu_{r_i}(G_i) > m$. We have

$$\phi_k(G, \mathcal{C}) = e(G) - \sum_{i=1}^k \binom{r_i}{2} \nu_{r_i}(G_i) + \sum_{i=1}^k \nu_{r_i}(G_i) < t_{R-1}(n),$$

giving the required. \square

Remark. By analysing the above argument, one can also derive the following stability property for every fixed family \mathcal{C} of cliques as $n \rightarrow \infty$: every graph G on n vertices with $\phi_k(G, \mathcal{C}) = t_{R-1}(n) + o(n^2)$ is $o(n^2)$ -close to the Turán graph $T_{R-1}(n)$ in the edit distance.

Acknowledgements

Henry Liu and Teresa Sousa acknowledge the support from FCT - Fundação para a Ciência e a Tecnologia (Portugal), through the projects PTDC/MAT/113207/2009 and PEst-OE/MAT/UI0297/2011 (CMA). Oleg Pikhurko was supported by ERC grant 306493 and EPSRC grant EP/K012045/1.

The authors thank the anonymous referees for the careful reading of the manuscript.

References

- [1] P. Allen, J. Böttcher, and Y. Person. An improved error term for minimum H -decompositions of graphs. *J. Combin. Theory Ser. B*, 108:92–101, 2014.
- [2] B. Bollobás. On complete subgraphs of different orders. *Math. Proc. Cambridge Philos. Soc.*, 79:19–24, 1976.
- [3] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- [4] D. Dor and M. Tarsi. Graph decomposition is NP-complete: a complete proof of Holyer’s conjecture. *SIAM J. Comput.*, 26:1166–1187, 1997.
- [5] P. Erdős. Some recent results on extremal problems in graph theory. (Results). Theory Graphs, Int. Symp. Rome 1966, 117-123 (English), 124-130 (French), 1967.
- [6] P. Erdős, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canad. J. Math.*, 18:106–112, 1966.
- [7] E. Győri. On the number of edge-disjoint triangles in graphs of given size. In *Combinatorics (Eger, 1987)*, Colloq. Math. Soc. János Bolyai, pages 267–276. North-Holland, Amsterdam, 1988.
- [8] E. Győri. On the number of edge disjoint cliques in graphs of given size. *Combinatorica*, 11:231–243, 1991.
- [9] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21:13–38, 2001.
- [10] M. Krivelevich. On a conjecture of Tuza about packing and covering of triangles. *Discrete Math.*, 142:281–286, 1995.
- [11] H. Liu and T. Sousa. Monochromatic K_r -decompositions of graphs. *J. Graph Theory*, 76:89–100, 2014.
- [12] L. Özkahya and Y. Person. Minimum H -decompositions of graphs: edge-critical case. *J. Combin. Theory Ser. B*, 102:715–725, 2012.
- [13] O. Pikhurko and T. Sousa. Minimum H -decompositions of graphs. *J. Combin. Theory Ser. B*, 97:1041–1055, 2007.
- [14] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, DS01:Dynamic Survey, Version of 12 January, 2014.
- [15] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.

- [16] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.
- [17] T. Sousa. Decompositions of graphs into 5-cycles and other small graphs. *Electron. J. Combin.*, 12:Research Paper 49, 7 pp. (electronic), 2005.
- [18] T. Sousa. Decompositions of graphs into a given clique-extension. *Ars Combin.*, 100:465–472, 2011.
- [19] T. Sousa. Decompositions of graphs into cycles of length seven and single edges. *Ars Combin.*, to appear.
- [20] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [21] Zs. Tuza. In *Finite and Infinite Sets*, volume 37 of *Colloquia Mathematica Societatis János Bolyai*, page 888. North-Holland Publishing Co., Amsterdam, 1984.
- [22] R. Yuster. Dense graphs with a large triangle cover have a large triangle packing. *Combin. Probab. Comput.*, 21:952–962, 2012.