Friendship Decompositions of Graphs

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Abstract

We investigate the maximum number of elements in an optimal t-friendship decomposition of graphs of order n. Asymptotic results will be obtained for all fixed $t \geq 4$ and for $t = 2, 3$ exact results will be derived.

1 Introduction and Terminology

For notation and terminology not discussed here the reader is referred to [**?**]. All graphs considered here are finite and simple, i.e., they have no loops or multiple edges. Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set of neighbors of v is denoted by $N_G(v)$ or briefly by $N(v)$ if it is clear which graph is being considered. We always have $v \notin N(v)$. Let $\overline{N}_G(v) = V - (N_G(v) \cup \{v\})$. A *clique* is a complete graph and the complete bipartite graph with parts of size m and n will be denoted by $K_{m,n}$. A graph of the form $K_{1,m}$ with possible some isolated vertices will be called a *star*. The *center of a star* is its vertex of maximum degree.

Let $t \geq 1$ be fixed. A graph that consists of t edge disjoint cliques sharing a vertex v is said to be a *t-friendship graph* (t-fs graph) *with center v*. A *t-friendship decomposition* of a graph G is a set, $\mathcal{F}$, of edge disjoint t-friendship subgraphs of G, such that any edge of G is an edge of exactly one element of $\mathcal{F}$. For $t \geq 1$ let $\phi_t(G)$ denote the minimum number of t-friendship graphs in a t-friendship decomposition of the graph G and $\phi_t(n) = \max \phi_t(G)$ where G runs over all graphs of order n. In [2] Erdös, Goodman and Pósa proved that $\phi_1(n) = \lfloor n^2/4 \rfloor$. In our paper we will study the function $\phi_t(n)$ for every fixed $t \geq 2$. For $t \geq 4$ asymptotic results will be obtained and for $t = 2, 3$ the exact value will be derived.

2 t-Friendship Decompositions

We start by studying the decomposition problem of any graph of order n into t-friendship graphs, for all fixed $t \geq 2$. This problem will be asymptotically solved. However, for $t = 2, 3$ we will be able to improve our proof in order to obtain exact results.

Theorem 2.1. *Any graph of order n admits a t-friendship decomposition with at most $\frac{n^2}{4t} + \frac{n}{4t} + n$ elements, for all fixed $t \geq 2$.*

To prove the theorem we will need the following trivial lemma:

Lemma 2.2. *[3] Let G be an arbitrary graph of order n such that $\deg v \geq d$ for every vertex v of G. Then there exists a partition of $V(G)$ such that the number of the classes of the partition is $n - d$ and such that every class of the partition spans a clique.* $\square$

Proof of Theorem 2.1. By induction on the number of vertices in a graph. The theorem is obviously true for $n = 2$. Let v be a vertex of minimum degree and let $\delta := \delta(G)$.

If $\delta \leq n/2$ we consider the edges incident with v as $\left\lceil \frac{\deg v}{t} \right\rceil$ elements of a t-fs decomposition and consider an optimal decomposition of $G - v$. Then

$$\phi_t(G) \leq \phi_t(G - v) + \left\lceil \frac{\delta}{t} \right\rceil. \tag{2.1}$$

If $\delta > n/2$ then $\deg_{G[N(v)]}(u) \geq \delta - 1 - (n - \delta - 1) = 2\delta - n > 0$, for all $u \in N(v)$. Lemma 2.2 implies that there exists a partition $\{R_1, \ldots, R_{n-\delta}\}$ of $N(v)$ such that $G[R_i]$

is a clique, for $i = 1, \ldots, n-\delta$. Then $G[R_i \cup \{v\}]$ is also a clique for $i = 1, \ldots, n-\delta$. Therefore by grouping all these cliques into t-tuples we obtain $\lceil \frac{n-\delta}{t} \rceil$ t-fs graphs and combining them with an optimal t-fs decomposition of the remaining graph H, where $V(H) = V - \{v\}$ and $E(H) = E(G) - \cup_{i=1}^{n-\delta} E(G[R_i \cup \{v\}])$, we obtain

$$\phi_t(G) \leq \phi_t(H) + \left\lceil \frac{n-\delta}{t} \right\rceil. \tag{2.2}$$

Let $n = 2tk + r$, with $0 \leq r < 2t$. If $\delta \leq n/2$ then from (2.1) we obtain

$$\phi_t(G) \leq \phi_t(G - v) + k + 1.$$

If $\delta > n/2$ then (2.2) implies that

$$\phi_t(G) \leq \phi_t(H) + k + 1.$$

Therefore,

$$\begin{aligned}
\phi_t(G) &\leq \frac{(n-1)^2}{4t} + \frac{n-1}{4t} + n - 1 + k + 1 \\
&\leq \frac{(n-1)^2}{4t} + \frac{n-1}{4t} + n - 1 + \frac{n-r}{2t} + 1 \\
&\leq \frac{n^2}{4t} + \frac{n}{4t} + n.
\end{aligned}$$

□

By considering the graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ we trivially have that $\phi_t(n) \geq \phi_t(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \geq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil / t \geq \frac{n^2-1}{4t}$. Therefore, for fixed $t \geq 2$ the decomposition problem into t-friendship graphs is asymptotically solved and we have the following theorem.

Theorem 2.3.

$$\phi_t(n) = \left(\frac{1}{4t} + o(1) \right) n^2, \ \textit{for all fixed } t \geq 2.$$

3 2-Friendship Decompositions

For the special case of $t = 2$ the proof of Theorem 2.1 can be improved to give us the exact value of $\phi_2(n)$, for all $n \geq 1$.

Theorem 3.1. *Any graph of order n can be decomposed into at most $\lceil n^2/8 \rceil$ 2-friendship graphs if n is even and at most $(n^2-1)/8$ 2-friendship graphs if n is odd. Moreover, this bound is sharp for the bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Proof. By induction on the number of vertices in a graph. Clearly the result holds for all graphs with at most three vertices. Let G be a graph having n vertices where $n \geq 4$. Let v be a vertex of minimum degree and let $\delta := \delta(G)$. Let $n = 4k + r$, with $0 \leq r \leq 3$ and $k \geq 1$. The theorem follows directly from (2.1) or (2.2) and the induction hypothesis for all values of n except for $n = 4k+3$ and $\delta \in \{2k+1, 2k+2\}$.

Let $n = 4k+3$ and $\delta = 2k+1$. If $G[N(v)]$ has an edge, say e, then the edge e and the edges incident with v form at most k 2-fs graphs. Hence

$$\phi_2(G) \leq \phi_2(G - v - e) + k \leq \left\lceil \frac{(n-1)^2}{8} \right\rceil + k = \frac{n^2-1}{8}.$$

Assume that $G[N(v)]$ has no edges and let $u \in N(v)$. Then $\deg\ u \leq 2k+2$ and we have

$$\phi_2(G) \leq \phi_2(G - \{v, u\}) + \left\lceil \frac{\deg\ v - 1}{2} \right\rceil + \left\lceil \frac{\deg\ u}{2} \right\rceil = \frac{n^2-1}{8}.$$

Finally, let $n = 4k+3$ and $\delta = 2k+2$. In this case $\deg_{G[N(v)]}(u) \geq 1$, for all $u \in N(v)$. If $\deg_{G[N(v)]}(u) \geq 2$, for all $u \in N(v)$, then by Lemma 2.2 the edges incident with v can be decomposed into at most k 2-fs graphs and the result follows by the induction hypothesis. Suppose that $\deg_{G[N(v)]}(u) = 1$ for some $u \in N(v)$. In this case observe that $\deg\ u = 2k+2$ and let $w \in N(v)$ be adjacent to u. Consider the 2-fs graph, F, with edges vu, vw, uw and uw' for some $w' \in N(u) - \{v, w\}$. Observe that such w' exists since we assumed that $k \geq 1$. Let G_1 be the graph obtained after deleting the edges of F. Then $\deg_{G_1}(v) = 2k$ and $\deg_{G_1}(u) = 2k-1$. Therefore,

$$\begin{aligned}\phi_2(G) &\leq \phi_2(G_1 - \{v, u\}) + \left\lceil \frac{\deg_{G_1}(v)}{2} \right\rceil + \left\lceil \frac{\deg_{G_1}(u)}{2} \right\rceil + 1 \\ &\leq \frac{(n-2)^2 - 1}{8} + 2k + 1 = \frac{n^2-1}{8}.\end{aligned}$$

To complete the proof it remains to see that the bound is sharp for $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. We trivially have $\phi_2(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \geq \lceil |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|/2 \rceil$, therefore $\phi_2(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \geq \lceil n^2/8 \rceil$ if n is even and $\phi_2(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \geq (n^2-1)/8$ if n is odd, hence the equality. □

4 3-Friendship Decompositions

As we did for 2-friendship decompositions of graphs we can also obtain exact results for 3-friendship decompositions of graphs. However, in this case, the calculations will not be so straightforward as they were for the case $t = 2$. Next theorem is our main result of this section.

Theorem 4.1. *Any graph of order n, except the 5-cycle, can be decomposed into at most $\lceil n^2/12 \rceil$ 3-friendship graphs if n is even and at most $\lceil (n^2-1)/12 \rceil$ 3-friendship graphs if n is odd. Moreover, this bound is sharp for the bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Before proving the general case, and for the sake of simplicity, we will see that Theorem 4.1 holds for some special types of graphs needed later.

Lemma 4.2. *Theorem 4.1 holds for bipartite graphs.*

Proof. Let G be a bipartite graph of order n with parts A and B of size s and t respectively. We will see that $\phi_3(G) \leq \lceil st/3 \rceil$.
Let $s = 3k_1 + r_1$ and $t = 3k_2 + r_2$ with $0 \leq r_1 \leq 2$ and $0 \leq r_2 \leq 2$. Assume $r_1 \leq r_2$. Let $A = A_1 \cup A_2$ where $|A_1| = 3k_1$ and $|A_2| = r_1$ and $B = B_1 \cup B_2$ where $|B_1| = 3k_2$ and $|B_2| = r_2$. Consider the subgraph induced by $A \cup B_1$. Clearly its edges can be decomposed into at most sk_2 3-fs graphs. Now consider the subgraph induced by $A_1 \cup B_2$ and remove all 3-fs graphs with centers in B_2, at most $r_2 k_1$ of them. Finally, observe that the edges left have endpoints in $A_2 \cup B_2$, hence it suffices to remove all 3-fs graphs with centers in A_2, at most r_1 of them. Therefore, $\phi_3(G) \leq sk_2 + r_2 k_1 + r_1 \leq \left\lceil \frac{st}{3} \right\rceil$, where the last inequality follows from the fact that $0 \leq r_1 \leq r_2 \leq 2$. Moreover, $\lceil st/3 \rceil$ is at most $\lceil (s+t)^2/12 \rceil$ if $s+t$ is even and at most $\left\lceil \frac{(s+t)^2-1}{12} \right\rceil$ if $s+t$ is odd.

To complete the proof it remains to observe that for $s = \lfloor n/2 \rfloor$ and $t = \lceil n/2 \rceil$ $\lceil st/3 \rceil$equals $\lceil n^2/12 \rceil$ if n is even and $\lceil (n^2-1)/12 \rceil$ if n is odd. □

Given a graph G and two disjoint sets $A, B \subseteq V(G)$ we denote by $G[A, B]$ the bipartite graph with all the edges of G with one endpoint in A and the other in B. Recall that a star with center y is a graph of the form $K_{1,m}$ with possible some isolated vertices where y is its vertex of maximum degree.

Lemma 4.3. *Let G be a graph with $n = 6k + 5$ vertices and minimum degree $3k + 2$. Let v be a vertex of minimum degree and assume that $G[N(v)]$ has no edges, $G[\overline{N}(v)]$ is a star or an empty graph and $G[N(v), \overline{N}(v)]$ is a complete bipartite graph. Then G admits a 3-friendship decomposition with at most $\lceil (n^2-1)/12 \rceil = 3k^2+5k+2$ elements.*

Proof. Let $N(v) = \{u_1, \ldots, u_{3k+1}, u\}$ and $\overline{N}(v) = \{y_1, \ldots, y_{3k+1}, y\}$ and let y be the center of the star (in case $G[\overline{N}(v)]$ is an empty graph, y is any vertex in $\overline{N}(v)$). For $i = 1 \ldots 3k+1$ remove the 3-fs graph induced by the vertices u_i, y, y_i, v. In this step we removed $3k+1$ 3-fs graphs. Now $G[\overline{N}(v)]$ has no edges and we are left with a bipartite graph. Hence it suffices to remove all 3-fs graphs with centers in $N(v)$. Observe that in the remaining graph $\deg u_i = 3k$ for $i = 1, \ldots 3k+1$ and $\deg u = 3k+3$ so all these edges can be decomposed into at most $k(3k+1)+k+1$ 3-fs graphs. Therefore,

$$\phi_3(G) \leq 3k + 1 + k(3k + 1) + k + 1 = 3k^2 + 5k + 2. \qquad \square$$

Lemma 4.4. *Let G be a graph with $n = 6k+5$ vertices and minimum degree $3k+3$. Let v be a vertex of minimum degree and assume that $G[N(v)] = K_{1,3k+2}$, $G[\overline{N}(v)]$ has no edges, and $G[N(v), \overline{N}(v)]$ is a complete bipartite graph. Then G admits a 3-friendship decomposition with at most $3k^2 + 4k + 2 < \lceil (n^2 - 1)/12 \rceil$ elements.*

Proof. Let $N(v) = \{u_1, \ldots, u_{3k+2}, u\}$, let u be adjacent to all vertices of $N(v)$ and let $\overline{N}(v) = \{y_1, \ldots, y_{3k+1}\}$. The edges incident with u and the edges vu_{3k+2}, u_1y_1, u_2y_2, $\cdots$, $u_{3k+1}y_{3k+1}$ can be decomposed using $k + 1$ 3-fs graphs as described by Figure 1. Remove all edges incident with y_i except the edge y_iu_{3k+2} for $i = 1, \ldots, 3k$ and all edges incident with y_{3k+1} except the edge $y_{3k+1}u_1$. In total we removed $k(3k + 1)$ 3-fs graphs. The edges left can be decomposed using $2k + 1$ 3-fs graphs. Therefore,

$$\phi_3(G) \leq k + 1 + k(3k + 1) + 2k + 1 = 3k^2 + 4k + 2. \qquad \square$$

Lemma 4.5. *Let G be a graph with $n = 6k+5$ vertices and minimum degree $3k + 3$. Let v be a vertex of minimum degree. Assume that $G[\overline{N}(v)]$ has no edges, $G[N(v), \overline{N}(v)]$ is a complete bipartite graph and $G[N(v)]$ looks like the graph described in Figure 2 with the edge $\{w_1, w_2\}$ being present or not. Then G admits a 3-friendship decomposition with at most $\lceil (n^2 - 1)/12 \rceil = 3k^2 + 5k + 2$ elements.*

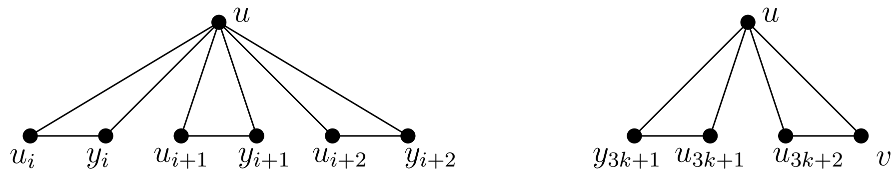


Figure 1: $i \equiv 1 \pmod 3$ and $i \leq 3k-2$

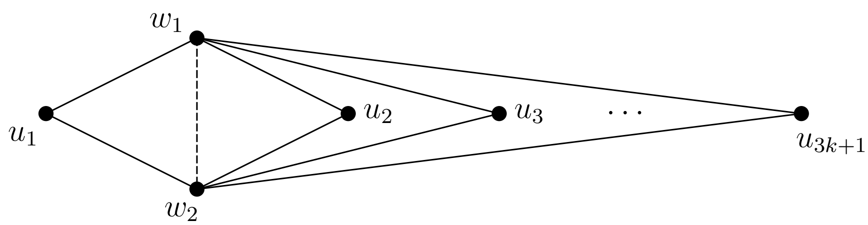


Figure 2: An illustration of the graph $G[N(v)]$ as in Lemma 4.5

Proof. Let $N(v) = \{u_1, \ldots, u_{3k+1}, w_1, w_2\}$, $\overline{N}(v) = \{y_1, \ldots, y_{3k+1}\}$ and $\deg_{G[N(v)]} u_i = 2$ for all $i = 1, \ldots, 3k+1$.

We first consider the case $3k+1$ even. We start our 3-fs decomposition by removing the 3-fs graphs with edge sets y_iu_i, y_iw_1, u_iw_1, y_iw_2, y_iu_{i+1}, $u_{i+1}w_2$, and $y_{i+1}u_i$, $y_{i+1}w_2$, u_iw_2, $y_{i+1}w_1$, $y_{i+1}u_{i+1}$, $u_{i+1}w_1$, where $1 \leq i \leq 3k+1$ and i is odd (See Figure 3). We remove $3k+1$ 3-fs graphs. After this first step the edges left are incident with the vertices v, $y_1, \ldots, y_{3k+1}$ and we might also have the edge w_1w_2. Furthermore, in the graph left v has degree $3k+3$ and the vertices $y_1, \ldots, y_{3k+1}$ have degree $3k-1$. Therefore, the edges incident with v and the edge w_1w_2 if it exists, can be decomposed using at most $k+1$ 3-fs graphs and for $1 \leq i \leq 3k+1$ the edges incident with y_i can be decomposed using at most k 3-fs graphs.

Therefore,

$$\phi_3(G) \leq 3k+1+k+1+(3k+1)k = 3k^2+5k+2.$$

Suppose that $3k+1$ is odd. We repeat the procedure described before to decompose all the edges incident $y_1, \ldots, y_{3k}$. In total we remove $3k+3k^2$ 3-fs graphs. If the edge w_1w_2 exists then the vertices w_1, w_2 and u_{3k+1} induce a triangle, so the edges incident with v plus the above triangle can be decomposed into $k+1$ 3-fs graphs. The edges left are incident with y_{3k+1} and can be decomposed using $k+1$ 3-fs graphs. If the edge

w_1w_2 is not in the graph then the edge w_1u_{3k+1} and the edges incident with v can be decomposed into $k+1$ 3-fs graphs and the edge w_2u_{3k+1} and the edges incident with y_{3k+1} can be decomposed into $k+1$ 3-fs graphs and this completes the proof. $\square$

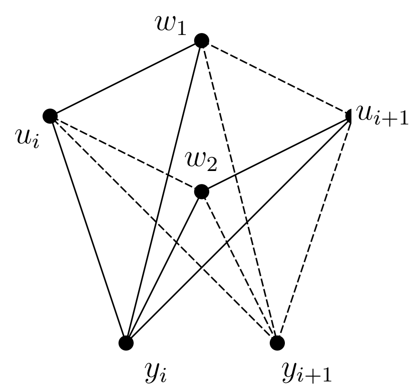


Figure 3: Decomposition used in the proof of Lemma 4.5

We are now able to prove Theorem 4.1.

Proof of Theorem 4.1. By induction on the number of vertices in a graph. By inspection, and using Harary's [4] atlas of all graphs of order at most 5, we can see that the result holds for all graphs with at most 5 vertices. Let G be a graph having n vertices, where $n \geq 6$. Let v be a vertex of minimum degree and let $\delta := \delta(G)$.

Let $n = 6k + r$, with $0 \leq r \leq 5$ and $k \geq 1$. The result follows directly from (2.1) or (2.2) and the induction hypothesis for all values of n except for $n = 6k + 3$ and $\delta \in \{3k+1, 3k+2\}$ and for $n = 6k+5$ and $\delta \in \{3k+1, 3k+2, 3k+3, 3k+4\}$.

Consider the case $n = 6k + 3$ and $\delta = 3k + 1$. If $G[N(v)]$ has an edge, say e, then the edges incident with v and the edge e form at most k 3-fs graphs, hence

$$\phi_3(G) \leq \phi_3(G - v - e) + k \leq \left\lceil \frac{(n-1)^2}{12} \right\rceil + k = \left\lceil \frac{n^2-1}{12} \right\rceil.$$

Assume that $G[N(v)]$ has no edges and let $u \in N(v)$. Then $\deg u \leq 3k + 2$ and we have

$$\phi_3(G) \leq \phi_3(G - \{v, u\}) + \left\lceil \frac{\deg v - 1}{3} \right\rceil + \left\lceil \frac{\deg u}{3} \right\rceil = \left\lceil \frac{n^2-1}{12} \right\rceil.$$

Let $n = 6k+3$ and $\delta = 3k+2$. Then $\deg_{G[N(v)]}(u) \geq 1$ for all $u \in N(v)$. If $\deg_{G[N(v)]}(u) \geq 2$, for all $u \in N(v)$, then using Lemma 2.2 we can decompose the edges incident with v and some edges in $G[\overline{N}(v)]$ into at most k 3-fs graphs and the result follows by induction. Suppose that $\deg_{G[N(v)]}(u) = 1$ for some $u \in N(v)$. In this case observe that $\deg u = 3k+2$ and let $w \in N(v)$ be adjacent to u. Consider the 3-fs graph, F, with edges vu, vw, uw, vx and vy for some $x, y \in N(v) - \{u, w\}$. Observe that such x, y exist since we assumed that $k \geq 1$. Let G_1 be the graph obtained after deleting the edges of F. Then $\deg_{G_1}(v) = 3k-2$ and $\deg_{G_1}(u) = 3k$. Therefore,

$$\phi_3(G) \leq \phi_3(G_1 - \{v, u\}) + \left\lceil \frac{\deg_{G_1}(v)}{3} \right\rceil + \left\lceil \frac{\deg_{G_1}(u)}{3} \right\rceil + 1 = \left\lceil \frac{n^2-1}{12} \right\rceil .$$

From now and until the end of the proof let $n = 6k+5$. Observe that

$$\left\lceil \frac{n^2-1}{12} \right\rceil = \left\lceil \frac{(n-1)^2}{12} \right\rceil + k = \left\lceil \frac{(n-2)^2-1}{12} \right\rceil + 2k + 1.$$

Let $\delta = 3k+1$. If $G[N(v)]$ has an edge e then the e and the edges incident with v form at most k 3-fs graphs and we are done. Assume that $G[N(v)]$ has no edges and let $u \in N(v)$. Then $\deg u \leq 3k+4$. By Lemma 4.2 we can assume that G is not bipartite, so $G[\overline{N}(v)]$ must have at least one edge. Therefore the edges incident with v and u form at most $2k+1$ 3-fs graphs and the result holds.

Let $\delta = 3k+2$. If $\deg_{G[N(v)]}(u) \geq 2$, for all $u \in N(v)$, then using Lemma 2.2 we can decompose the edges incident with v into at most k 3-fs graphs. Suppose first that exists $u \in N(v)$ such that $\deg_{G[N(v)]}(u) = 1$, then $\deg u \leq 3k+4$. Therefore, the edges incident with v and u can be decomposed into at most $2k+1$ 3-fs graphs.

Now suppose that for all $u \in N(v)$ we have either $\deg_{G[N(v)]}(u) = 0$ or $\deg_{G[N(v)]}(u) \geq 2$. If the latter condition happens then it is not hard to see that $N(v)$ must contain at least 2 independent edges or a triangle. Thus, the edges incident with v can be decomposed into at most k 3-fs graphs and the result follows by induction. Therefore it remains to consider the case when $\deg_{G[N(v)]}(u) = 0$ for all $u \in N(v)$.

(a) If $\deg u = 3k+3$ for all $u \in N(v)$ then $G[N(v), \overline{N}(v)]$ is a complete bipartite graph. Let $u \in N(v)$. By Lemma 4.2 we can assume that G is not bipartite, that is,

$G[\overline{N}(v)]$ must have at least one edge. If $G[\overline{N}(v)]$ has 2 independent edges or a triangle then the edges incident with v and u can be decomposed into at most $k+1$ and k 3-fs graphs respectively and the result follows by induction. If $G[\overline{N}(v)]$ has at most one independent edge and no triangles, then $G[\overline{N}(v)]$ is a star and our graph is as in Lemma 4.3 so the result holds.

(b) Suppose now that exists $u \in N(v)$ such that $\deg u = 3k+2$. Then $\exists\, y \in \overline{N}(v)$ such that u is not adjacent to y. If $G[N(u)]$ has an edge then the edges incident with v and u can be decomposed into at most $2k+1$ 3-fs graphs. If $G[N(u)]$ has no edges then all edges in $G[\overline{N}(v)]$ are incident with y, i.e., $G[\overline{N}(v)]$ is a star with center y. Assume first that $G[\overline{N}(v)]$ has exactly one edge, say yy'. Then y and y' must have at least one common neighbor in $N(v)$, say u'. Remove the 3-fs graph with edges $u'y, u'y', yy', u'y_1, u'y_2$, for some $y_1, y_2 \in \overline{N}(v) - \{y, y'\}$. The graph left is bipartite, hence it suffices to remove all 3-fs graphs with centers in $N(v)$. Let $u_1, \ldots, u_{3k}$ be the remaining vertices of $N(v)$. Observe that $\deg u_i = 3k+3$, for $i = 1, \ldots, 3k$. Then,

$$\begin{aligned}
\phi_3(G) &\leq 1 + \left\lceil \frac{\deg u}{3} \right\rceil + \left\lceil \frac{\deg u' - 4}{3} \right\rceil + 3k \left\lceil \frac{\deg u_1}{3} \right\rceil \\
&\leq 1 + k + 1 + k + 3k(k+1) = \left\lceil \frac{n^2 - 1}{12} \right\rceil .
\end{aligned}$$

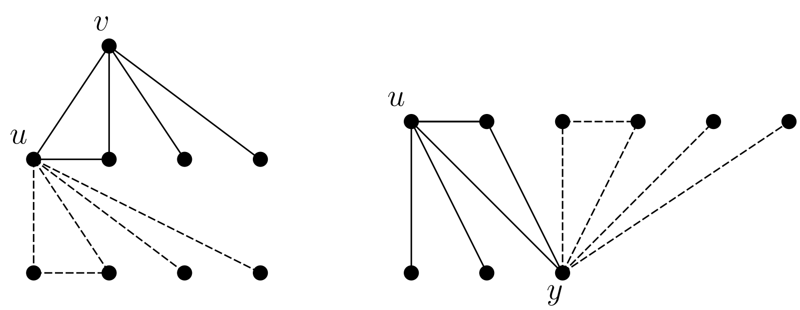
Now, assume that $G[\overline{N}(v)]$ has at least 2 edges. If $G[N(u')]$ has an edge for some $u' \in N(v)$, say yy_1, we remove the 3-fs graph with edges $u'y, u'y_1, yy_1, y_1u, y_1u''$, for some $u'' \in N(v) - \{u, u'\}$. Then,

$$\begin{aligned}
\phi_3(G) &\leq \phi_3(G - \{v, u, u', y_1\}) + \\
&\quad + \left\lceil \frac{\deg v}{3} \right\rceil + \left\lceil \frac{\deg u - 2}{3} \right\rceil + \left\lceil \frac{\deg u' - 3}{3} \right\rceil + \left\lceil \frac{\deg y_1 - 4}{3} \right\rceil + 1 \\
&\leq \left\lceil \frac{(n-4)^2 - 1}{12} \right\rceil + \left\lceil \frac{3k+2}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3k-1}{3} \right\rceil + 1 \\
&\leq 3k^2 + k + 4k + 2 = \left\lceil \frac{n^2 - 1}{12} \right\rceil .
\end{aligned}$$

If $N(u')$ has no edge for all $u' \in N(v)$ then $G \subseteq K_{3k+3,3k+2}$ and the result follows from Lemma 4.2.

Let $\delta = 3k+3$. Then $\deg_{G[N(v)]}(u) \geq 1$, for all $u \in N(v)$. If $\deg_{G[N(v)]}(u) \geq 3$, for all $u \in N(v)$, then by Lemma 2.2 we can decompose the edges incident with v and some other edges in $G[N(v)]$ into at most k 3-fs.

(a) Assume first that exists $u \in N(v)$ such that $\deg_{G[N(v)]} u = 1$, then $\deg u = 3k+3$. If $G[\overline{N}(v)]$ has an edge, say e, then the edge e and the edges incident with v and u can be decomposed into at most $2k+1$ 3-fs graphs (see Figure 4(i)). If $G[\overline{N}(v)]$ has no edges then $G[N(v), \overline{N}(v)]$ is a complete bipartite graph. So, if $G[N(v)]$ has 2 independent edges then the edges incident with u and y, for some $y \in \overline{N}(v)$ can be decomposed into at most $2k+1$ 3-fs graphs and the result follows (see Figure 4(ii)). If $G[N(v)]$ has only one independent edge then $G[N(v)] = K_{1,3k+2}$ and Lemma 4.4 applies.



(i) (ii)

Figure 4: $\delta = 3k+3$ case (a)

(b) Assume that for all $u \in N(v), \deg_{G[N(v)]} u \geq 2$ and that exists $u \in N(v)$ such that $\deg_{G[N(v)]} u = 2$. Let u be adjacent to w_1 and w_2. Observe that $G[N(v)]$ has at least 2 independent edges since $\deg_{G[N(v)]} u \geq 2$ for all $u \in N(v)$. If there exist 3 independent edges in $G[N(v)]$ then the 3 independent edges and the edges incident with v can be decomposed into k 3-fs graphs. So assume that $G[N(v)]$ has exactly 2 independent edges. In this case $G[N(v)]$ looks like Figure 2, unless $k = 1$ and $N(v)$ consists of 2 vertex disjoint triangles and in this case the result holds. Assume first that w_1 and w_2 are adjacent. If exists $u' \in N(v) - \{w_1, w_2\}$ such that $\deg u' = 3k+3$ then the edges incident with v and u' form at most $2k+1$ 3-fs graphs and we are done. Assume that $\deg u' = 3k+4$ for all $u' \in N(v) - \{w_1, w_2\}$. If $G[\overline{N}(v)]$ has no edges then Lemma 4.5 applies. If $G[\overline{N}(v)]$ has an edge then we are able to decompose the edges incident with v and u' into at most $2k+1$ 3-fs graphs.

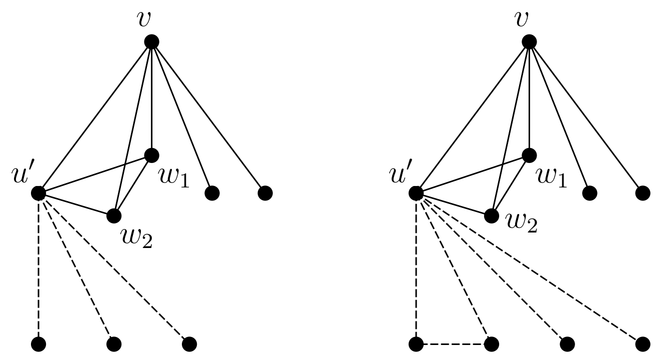


Figure 5: $\delta = 3k + 3$ case (b) and w_1 adjacent to w_2

Now suppose that w_1 and w_2 are not adjacent. Then w_2 must have at least one neighbor in $\overline{N}(v)$. If w_2 has exactly one neighbor in $\overline{N}(v)$, say y then $\deg w_2 = 3k + 3$. Furthermore, y must be adjacent to some $u_1 \in N(v) - \{w_1, w_2\}$. Thus we can decompose the edges incident with v and w_2 using at most $2k + 1$ 3-fs graphs. (see Figure 6(i))

Suppose w_2 has at least two neighbors in $\overline{N}(v)$ and let y and y' be two of them. If exists $u_1 \in N(v)$ such that $\deg u_1 = 3k + 3$ then u_1 is adjacent to at least one of y or y', say y and we apply induction to $G - \{v, u_1\}$. (see Figure 6(ii)) Assume that $\deg x = 3k + 4$ for all $x \in N(v) - \{w_1, w_2\}$ and let $u_1 \in N(v)$. If $\overline{N}(v)$ has no edges then the result follows from Lemma 4.5. Let $\overline{N}(v)$ have at least one edge, say e. If $e = \{y, y'\}$ then there exits a K_4 incident with u_1, otherwise there exist two triangles incident with u_1. In both cases we can decompose the edges incident with v and u_1, using at most $2k + 1$ 3-fs graphs.

Let $\delta = 3k + 4$. Then $\deg_{G[N(v)]}(u) \geq 3$, for all $u \in N(v)$.

(a) If $\deg_{G[N(v)]}(u) \geq 4$ for all $u \in N(v)$ the result follows by Lemma 2.2 and the induction hypothesis.

(b) Suppose that exists $u \in N(v)$ such that $\deg_{G[N(v)]}(u) = 3$. Then $\deg_G(u) = 3k + 4$. Observe that u is adjacent to all elements of $\overline{N}(v)$. Let u be adjacent to u_1, u_2, u_3 in $N(v)$. Because of degree constraints, u_3 is adjacent to both u_1 and u_2 or has a neighbor in $\overline{N}(v)$, say w. We have to consider three distinct cases.

(i) $G[\overline{N}(v)]$ has an edge, say xy and u_3 is adjacent to u_1 or u_2, say u_2;

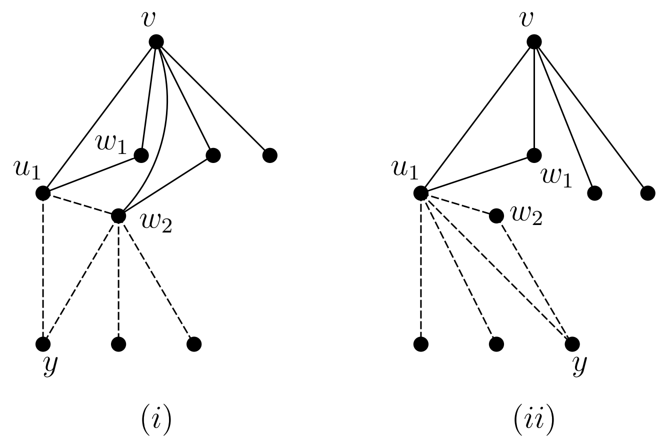


Figure 6: $\delta = 3k + 3$ case (b) and w_1 not adjacent to w_2

(ii) $G[\overline{N}(v)]$ has an edge, say xy and u_3 is not adjacent to u_1 and u_2. In this case u_3 has a neighbor in $\overline{N}(v) - \{x, y\}$, say w;

(iii) $G[\overline{N}(v)]$ has no edges. In this case observe that all vertices in $\overline{N}(v)$ are adjacent to all vertices in $N(v)$.

In all these cases Figure 7 shows that we can always decompose the edges incident with v and u using at most $2k + 1$ 3-fs graphs. This completes the proof. □

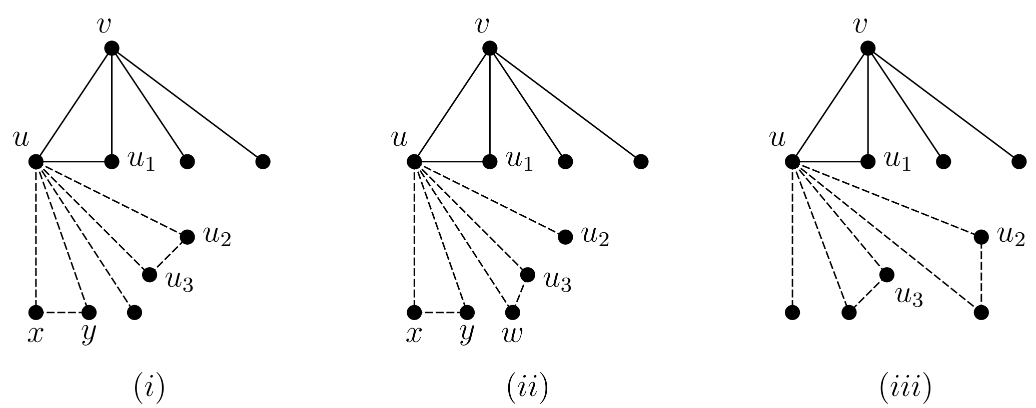


Figure 7: case $\delta = 3k + 4$

Open questions: It remains an interesting problem to better estimate the function $\phi_t(n)$ for a fixed $t \geq 4$, now known to satisfy $\frac{n^2}{4t} \leq \phi_t(n) \leq \frac{n^2}{4t} + \frac{n}{4t} + n$. The first open instance of this problem is the case $t = 4$ and in this case we conjecture that $\phi_4(n)$ equals $\left\lceil \frac{n^2}{16} \right\rceil$ if n is even and $\left\lceil \frac{n^2-1}{16} \right\rceil$ if n is odd.

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