4-Cycle Decompositions of Graphs

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ABSTRACT

In this paper we consider the problem of finding the smallest number $\phi$ such that any graph $G$ of order $n$ admits a decomposition into edge disjoint copies of $C_4$ and single edges with at most $\phi$ elements. We solve this problem for $n$ sufficiently large.

Keywords: Graph Decomposition; 4-Cycle Packing; Graph Packing

1. Introduction

All graphs in this paper are finite, undirected and simple. For notation and terminology not discussed here the reader is referred to [1].

Given two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $H$-subgraph, i.e., a graph isomorphic to $H$. We allow partitions only, that is, every edge of $G$ appears in precisely one part. Let $\phi(G)$ be the smallest possible number of parts in an $H$-decomposition of $G$. For non-empty $H$, let $p_H(G)$ be the maximum number of pairwise edge-disjoint $H$-subgraphs that can be packed into $G$ and $e(G)$ the number of edges in $G$. It is easy to see that

$$\phi(G) = e(G) - p_H(G)(e(H) - 1).$$

(1.1)

Here we study the function

$$\phi_H(n) = \max \left\{ \phi(G) \mid \nu(G) = n \right\},$$

which is the smallest number, such that, any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi_H(n)$ elements.

The function $\phi_H(n)$ was first studied by Erdős, Goodman and Pósa [2], who proved that $\phi_K_r(n) = t_r(n)$, where $K_r$ denotes the complete graph (clique) of order $r$ and $t_r(n)$ is the maximum size of an $r$-partite graph on $n$ vertices. A decade later, this result was extended by Bollobás [3], who proved that

$$\phi_{K_2}(n) = t_{r-1}(n), \quad \text{for all } n \geq r \geq 3.$$

Recently, Pikhurko and Sousa [4] studied $\phi_H(n)$ for arbitrary graphs $H$.

**Theorem 1.1.** (See Theorem 1.1 from [4]) Let $H$ be any fixed graph of chromatic number $r \geq 3$. Then,

$$\phi_H(n) = t_{r-1}(n) + o(n^2).$$

Let $ex(n,H)$ denote the maximum number of edges in a graph of order $n$, that does not contain $H$ as a subgraph. Recall that $ex(n,K_2) = t_{r-1}(n)$. Pikhurko and Sousa [4] also made the following conjecture.

**Conjecture 1.** For any graph $H$ with chromatic number at least 3, there is $n_0 = n_0(H)$ such that $\phi_H(n) = ex(n,H)$, for all $n \geq n_0$.

The exact value of the function $\phi_H(n)$ is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order $r \geq 4 \ (n \geq r)$ [5] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [6,7]. Later, Özkahya and Person [8] determined it for all edge-critical graphs with chromatic number $r \geq 3$ and $n$ sufficiently large. They proved the following result.

**Theorem 1.2.** ([8]) Let $H$ be any edge-critical graph with chromatic number $r \geq 3$. Then, there exists $n_0$ such that $\phi_H(n) = ex(n,H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\phi_H(n)$ is the Turán graph $T_{r-1}(n)$.

Recently, Allen, Böttcher and Person [9] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1.

The case when $H$ is a bipartite graph has been less studied. Pikhurko and Sousa [4] determined $\phi_H(n)$ for any fixed bipartite graph with an $O(1)$ additive error. For a non-empty graph $H$, let $gcd(H)$ denote the greatest common divisor of the degrees of $H$. For example, $gcd(K_{6,6}) = 2$ while for any tree $T$ with at least 2 vertices we have $gcd(T) = 1$. They proved the following result.
Theorem 1.3. (See Theorem 1.3 from [4]) Let $H$ be a bipartite graph with $m$ edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold:

1. If $d = 1$, then $\phi_t(n) = \lfloor n(n-1) / 2m \rfloor + C$, where $C = m-1$ or $C = m-2$.
2. If $d \geq 2$, then

$$\phi_t(n) = \frac{nd}{2m} \left[ n + \frac{n}{d} - 1 \right] + \frac{1}{2} n(n-1) + O(1).$$

Moreover, there is a procedure running in polynomial in $\log n$ time which determines $\phi_t(n)$ and describes a family $\mathcal{D}$ of $n$-sequences such that a graph $G$ of order $n$ satisfies $\phi_t(G) = \phi_t(n)$ if and only if the degree sequence of $G$ belongs to $\mathcal{D}$. (It will be the case that $|\mathcal{D}| = O(1)$ and each sequence in $\mathcal{D}$ has $n - O(1)$ equal entries, so $\mathcal{D}$ can be described using $O(\log n)$ bits.)

Here we will determine the exact value of $\phi_t(n)$ for $n$ sufficiently large.

Theorem 1.4. There is $n_0 = n_0(C_t)$ such that for all $n \geq n_0$ the following statements hold:

1. If $n$ is even then $\phi_t(n) = \frac{n^2}{8} + \frac{n}{4} + 1$.
2. If $n = 1 \mod 8$ then $\phi_t(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{14}{2}$.
3. If $n = 3 \mod 8$ then $\phi_t(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}$.
4. If $n = 5 \mod 8$ then $\phi_t(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}$.
5. If $n = 7 \mod 8$ then $\phi_t(n) = \frac{n^2}{8} + \frac{n}{8} + 2$.

2. Proof of Theorem 1.4

In this section we will prove Theorem 1.4, but first we need to introduce the tools. We start with the following easy result about $H$-decompositions.

Lemma 5. (Lemma 1.3) For any non-empty graph $H$ with $m$ edges and any integer $n$, we have

$$\phi_t(n) \leq \frac{1}{m} \left[ \left( \frac{n}{2} \right) \right] + \frac{m-1}{m} \text{ex}(n,H).$$

In particular, if $H$ is a fixed bipartite graph with $m$ edges and $n \to \infty$, then

$$\phi_t(n) = \left( \frac{1}{m} + O(1) \right) \left[ \frac{n}{2} \right].$$

The following result is the well known Erdős-Gallai theorem that gives a necessary and sufficient condition for a finite sequence to be the degree sequence of a simple graph.

Theorem 2.6. (Erdős-Gallai Theorem [10]) Let $0 \leq d_1 \leq \cdots \leq d_n$ be a sequence of integers. There is a graph with degree sequence $d_1, \cdots, d_n$ if and only if

1. $d_1 + \cdots + d_n$ is even;
2. For each $1 \leq k \leq n$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=1}^{n} \min\{d_i, k\}.$$  (2.3)

The following results appearing in Alon, Caro and Yuster [11, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [12] are essential to the proof of Theorem 1.4.

Lemma 2.7. For any non-empty graph $H$ with $m$ edges, there are $\gamma > 0$ and $N_0$ such that the following holds. Let $d = \gcd(H)$, $G$ be a graph of order $n \geq N_0$ and of minimum degree $\delta(G) \geq (1 - \gamma)n$. If $d = 1$, then

$$p_H(G) = \left[ \frac{e(G)}{m} \right].$$

If $d \geq 2$, let $\alpha_u = d \left[ \deg(u) / d \right]$ for $u \in V(G)$ and let $X$ consist of all vertices whose degree is not divisible by $d$. If $|X| \geq n / 10d^2$, then

$$p_H(G) = \left[ \frac{1}{2m} \sum_{u \in V(G)} \alpha_u \right].$$

If $|X| < n / 10d^2$, then

$$p_H(G) \geq \frac{1}{m} \left( e(G) - \frac{n}{5d^2} \right).$$

One can extract the following result from the proof of Theorem 1.2 from [4].

Lemma 2.8. Let $H$ be a bipartite graph with $m$ edges and let $\gcd(H) = d \geq 2$. Then, there is $n_0 = n_0(H)$ such that if $G$ is a graph of order $n \geq n_0$ with $\phi_t(G) = \phi_t(n)$ then the following holds:

1. Let $d_1, \cdots, d_n$ be the degree sequence of $G$, then

$$\phi_t(G) = \left[ \frac{1}{2m} \sum_{i=1}^{n} (d_i - (m-1)) \right] + \frac{1}{m} \sum_{i=1}^{n} \frac{\alpha_i}{d_i}.$$  (2.7)

2. Let $n = dq + r$ with $0 \leq r \leq d - 1$ and $d_i = qd_i + r_i$ with $0 \leq r_i \leq d - 1$. Then, for $1 \leq i \leq n$ exactly one of the following holds:

(a) $d_i = qd_i - 1$;
(b) $i \in C_i = \{i \in [n] \mid r_i = d - 1 \}$ and $d_i < qd - 1$;
(c) $i \in C_i = \{i \in [n] \mid d_i = n - 1 \}$ if $n - 1 \neq R$ and
result from [4] (Lemma 3.1) states that there is a constant \( C \), such that all but at most \( C \) vertices of \( G[X_i] \) can be covered by edge disjoint copies of \( H - y \) each of them having vertex disjoint sets \( A \). Therefore, all but at most \( C \) edges between \( x_i \) and \( x_i \) can be decomposed into copies of \( H \). All other edges incident to \( x_i \) are removed as single edges. Let \( G_{i}^{r} \) consist of the remaining edges of \( G_r - x_i \) (that is, those edges that do not belong to an \( H \)-subgraph of the above \( x_i \)-decomposition). This finishes the description of the case \( \deg_{G_i}(x_i) > an \).

Consider the sets \( S = \{ x_{n}, \ldots, x_{n+1} \} \), \( S_1 = \{ x_i \in S | \deg_{G_i}(x_i) \leq an \} \), and \( S_2 = S/S_1 \). Let their sizes be \( s \), \( s_1 \), and \( s_2 \) respectively, so \( s = s_1 + s_2 \).

For \( F \) be the graph with vertex set \( V(G_{n+1}) \cup S_2 \), consisting of the edges coming from the removed \( H \)-subgraphs when we processed the vertices in \( S_2 \). We have

\[
\phi_H(G) \leq \phi_H(G_{n+1}) + \frac{e(F)}{m} + \frac{s_1 an + s_2C}{2}.
\]

We know that \( \phi_H(G_{n+1}) = p_H(G_{n+1}(m-1)) \), furthermore, \( \delta(G_{n+1}) \geq (1-\gamma)(n-s) \). Thus, \( p_H(G_{n+1}) \) can be estimated using Lemma 2.7.

If (2.6) holds, some calculations show that there exits a graph \( G^* \) such that \( \phi_H(G) < \phi_H(G^*) \), which contradicts the optimality of \( G \).

Therefore, (2.5) must hold. It follows that \( p_H(G) \) and thus \( \phi_H(G) \), depends only on the degree sequence \( d_1, \ldots, d_n \) of \( G \). Namely, the packing number

\[
e = p_H(G) \text{ equals } \left[ \frac{1}{2m} \sum_{i=1}^{n} r_i \right], \text{ where } r_i = d_i/d
\]

is the largest multiple of \( d \) not exceeding \( d_i \).

Therefore,

\[
\phi_H(G) = 1 + \frac{1}{2m} \sum_{i=1}^{n} d_i \left( m - 1 \right) \left[ \frac{1}{2m} \sum_{i=1}^{n} \frac{d_i}{d} \right],
\]

where \( d_1, \ldots, d_n \) is the degree sequence of \( G \).

To conclude the proof we need to estimate the values that the degrees of \( G \) can attain. To do that we need to prove an upper bound on \( \phi_H(G) \) by estimating \( \phi_{\max} \), the maximum of

\[
\phi(d_1, \ldots, d_n) = \frac{1}{2} \sum_{i=1}^{n} d_i \left( m - 1 \right) \left[ \frac{1}{2m} \sum_{i=1}^{n} \frac{d_i}{d} \right],
\]

over all (not necessarily graphical) sequences \( d_1, \ldots, d_n \) of integers with \( 0 \leq d_i \leq n - 1 \).

Let \( d_1, \ldots, d_n \) be an optimal sequence attaining the value \( \phi_{\max} \). For \( i = 1, \ldots, n \) let \( d_i = q_id + r_i \) with

\[
0 \leq r_i \leq d_i - 1. \text{ Then, } e = \left( \frac{q_1 \pm \cdots \pm q_n}{2m} \right).
\]
Let \( n = qd + r \) with \( 0 \leq r \leq d - 1 \) and \( q = \lfloor n/d \rfloor \). Define \( R = qd - 1 \) to be the maximum integer which is at most \( n - 1 \) and is congruent to \( d - 1 \) modulo \( d \). Let \( C_1 = \{ i \in [n] | r_i = d - 1 \text{ and } d_i \neq R \} \) and \( C_2 = \{ i \in [n] | d_i = n - 1 \} \) if \( n - 1 \neq R \) and \( C_2 = \emptyset \) otherwise.

Since \( d_1, \ldots, d_n \) is an optimal sequence, we have that if \( r_i \neq d - 1 \) then \( d_i = n - 1 \) for all \( i \in [n] \). To conclude the proof it remains to show that \( |C_1| \leq \frac{2m}{d} - 1 \) and \( |C_2| \leq 2m - 1 \). Suppose first that \( |C_1| \geq \frac{2m}{d} - 1 \).

Consider the new sequence of integers

\[
d'_i = \begin{cases} 
d_i + d, & \text{if } i \in C_1, \\
d_i, & \text{if } i \notin C_1.
\end{cases}
\]

Then, \( \ell' = \ell + 1 \) and \( \phi' = \phi_{\max} + 1 \) which contradicts our assumption on \( \phi_{\max} \).

Now suppose that \( |C_2| \geq 2m \) and consider the new sequence of integers \( d'_1, \ldots, d'_n \) obtained from \( d_1, \ldots, d_n \) by replacing \( 2m \) values of \( n - 1 \) by \( R \). Then, \( \ell' = \ell - d \) and \( \phi' \geq \phi_{\max} + m - d > \phi_{\max} \), which contradicts our assumption on \( \phi_{\max} \) and the proof is concluded.

We now have all the tools needed to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( n \) be given by Lemma 2.8.

Let \( G \) be a graph of order \( n \geq n_0 \) with \( \phi_{\max}(G) = \phi_{\max}(n) \) and degree sequence \( d_1, \ldots, d_n \). For \( i = 1, \ldots, n \) let \( d_i = 2q_i + r_i \) with \( 0 \leq r_i \leq 1 \). Let \( R = 2\lfloor n/2 \rfloor - 1 + 1 \) and let the sets \( C_1 \) and \( C_2 \) be as in Lemma 2.8.

Let \( n = 2q + r \) with \( 0 \leq r \leq 1 \) and \( q = \lfloor n/2 \rfloor \). From (2.7) we obtain

\[
\phi_{\max}(n) = n(q - 1) + \frac{n}{2} \left[ C_2 \right] r - \sum_{i \in C_1} (q - 1 - q_i) \tag{2.12}
\]

\[
- \beta - 3 \left[ \frac{\beta}{4} \right] \leq 1.
\]

**Proof.** Routine calculations show that for \( \beta = 1 \) we have \( - \beta - 3 \left[ \frac{\beta}{4} \right] \leq 1 \). Suppose \( \beta = 1 \). Then \( C_1 \) has exactly one element, thus the sequence \( (d_i)_{i=1, \ldots, n} \) has exactly one element equal to \( n - 3 \) and all the others equal to \( n - 1 \). But this is not a degree sequence of a graph since condition (2.3) of Theorem 2.6 does not hold for \( k = n - 2 \).

Therefore, using the estimate of Claim 1 in (2.13) it follows that

\[
\phi_{\max}(n) \leq n^2 + n + 1.
\]

To prove the lower bound consider the graph \( L \) obtained from \( K_n \) after the deletion of the edges of a \( C_4 \). Using (1.1) and (2.5) we show that

\[
\phi_{\max}(L_n) = n^2 + n + 1.
\]

We now consider the case when \( n \) is an odd number.

**Case 1:** Let \( n = 8t + 1 \) and \( q = 4t \).

From (2.12) we obtain

\[
\phi_{\max}(n) = n \left( n - 3 \right) + \frac{n}{2} \left( 8t^2 - t \right) + \frac{1}{2} \left[ \alpha - \beta - 3 \left[ \frac{\alpha - \beta - 1}{4} \right] \right].
\]

**Claim 2.** Let \( d_1, \ldots, d_n \) be the degree sequence of a graph. Then,

\[
\frac{1}{2} \left[ \alpha - \beta - 3 \left[ \frac{\alpha - \beta - 1}{4} \right] \right] \leq \frac{5}{2}.
\]

**Proof.** Routine calculations show that the result follows if \( \alpha \neq 0 \) or \( \beta \neq 0 \). If \( \alpha = 0 \) and \( \beta = 0 \) then \( d_i = n - 2 \) for all \( i \leq n \). This is not a degree sequence of a graph since \( \sum_{i \leq n} d_i \) is not even.

Therefore, using the estimate of Claim 2 in (2.14) we prove that

\[
\phi_{\max}(n) \leq n^2 + n + 14.
\]

As for the lower bound consider the graph \( L' \) with all vertices of degree \( n - 2 \) except one of degree \( n - 3 \). Using (1.1) and (2.5) we show that

\[
\phi_{\max}(L') = n^2 + n + 14.
\]

**Case 2:** Let \( n = 8t + 3 \) and \( q = 4t + 1 \).

From (2.12) we obtain...
\[ \phi_{C_4}(n) = \frac{n}{2} (n-3) + \frac{n}{2} (8t^2 + 3t) + \frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta}{4} \right\rfloor. \] (2.15)

**Claim 3.** Let \( d_1, \cdots, d_n \) be the degree sequence of a graph. Then,
\[
\frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta}{4} \right\rfloor \leq \frac{3}{2}.
\]

**Proof.** It follows from routine calculations for all values of \( \alpha \) and \( \beta \) except when \( \alpha = 0 \) and \( \beta = 1 \). Suppose that \( \alpha = 0 \) and \( \beta = 1 \). Then \( C_2 = \emptyset \) and \( C_1 \) has exactly one element, thus the sequence \( (d_i)_{i=1,\cdots,n} \) has exactly one element equal to \( n-2 \) and all the others equal to \( n-1 \). But this is not a degree sequence of a graph since \( \sum d_i \) is not even. \( \square \)

Therefore, using the estimate of Claim 3 in (2.15) we prove that
\[
\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.
\]

As for the lower bound consider the graph \( L \) with degree sequence \( d_1 = d_2 = n-4, \ d_3 = \cdots = d_{n-1} = n-2 \) and \( d_n = n-1 \) (the existence of \( L \) can be proved directly or by Erdös-Gallai theorem, Theorem 2.6). Using (1.1) and (2.5) we have
\[
(\phi_{C_4}(L) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.
\]

**Case 3:** Let \( n = 8t + 5 \) and \( q = 4t + 2 \).

From (2.12) we obtain
\[
\phi_{C_4}(n) = \frac{n}{2} (n-3) + \frac{n}{2} (8t^2 + 7t) + \frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor. \] (2.16)

**Claim 4.** Let \( d_1, \cdots, d_n \) be the degree sequence of a graph. Then,
\[
\frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor \leq \frac{5}{2}.
\]

**Proof.** Routine calculations show that
\[
\alpha/2 - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor \leq -5/2 \quad \text{for all values of } \alpha \text{ and } \beta \text{ except for } \alpha = 2 \text{ and } \beta = 0 \text{ or } \alpha = 0 \text{ and } \beta = 2 \.
\]

Suppose first that \( \alpha = 2 \) and \( \beta = 0 \). Then the sequence \( (d_i)_{i=1,\cdots,n} \) has two elements equal to \( n-1 \) and all the others equal to \( n-2 \). This is not a degree sequence of a graph since \( \sum d_i \) is not even. \( \square \)

Suppose now that \( \alpha = 0 \) and \( \beta = 2 \). If \( |C_1| = 2 \) then the sequence has two elements equal to \( n-4 \) and all the others equal to \( n-2 \) and this is not a degree sequence of a graph since \( \sum d_i \) is not even. Finally, if \( |C_1| = 1 \) then we have one element equal to \( n-6 \) and all the others equal to \( n-2 \). Again, this is not a degree sequence of a graph since \( \sum d_i \) is not even. \( \square \)

Therefore, using the estimate of Claim 4 in (2.16) we prove that
\[
\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.
\]

As for the lower bound consider the graph \( K_n-I \) obtained from \( K_n \) by deleting the edges of a maximum matching. Using (1.1) and (2.5) we show that
\[
\phi_{C_4}(K_n-I) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.
\]

**Case 4:** Let \( n = 8t + 7 \) and \( q = 4t + 3 \).

From (2.12) we obtain
\[
\phi_{C_4}(n) = \frac{n}{2} (n-3) + \frac{n}{2} (8t^2 + 7t) + \frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 14}{4} \right\rfloor. \] (2.17)

**Claim 5.** Let \( d_1, \cdots, d_n \) be the degree sequence of a graph. Then,
\[
\frac{1}{2} \alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 14}{4} \right\rfloor \leq -\frac{17}{2}.
\]

**Proof.** It follows directly from simple calculations. \( \square \)

Therefore, using the estimate of Claim 5 in (2.17) we prove that
\[
\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + 2.
\]

Furthermore, using (1.1) and (2.5) we have
\[
\phi_{C_4}(K_n-I) = \frac{n^2}{8} + \frac{n}{8} + 2,
\]
so the equality follows and the proof is now complete. \( \square \)

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