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4-Cycle Decompositions of Graphs

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ABSTRACT

In this paper we consider the problem of finding the smallest number ϕ such that any graph G of order n admits a decomposition into edge disjoint copies of C_4 and single edges with at most ϕ elements. We solve this problem for n sufficiently large.

Keywords: Graph Decomposition; 4-Cycle Packing; Graph Packing

1. Introduction

All graphs in this paper are finite, undirected and simple. For notation and terminology not discussed here the reader is referred to [1].

Given two graphs G and H, an H*-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms an H-subgraph, *i.e.*, a graph isomorphic to H. We allow partitions only, that is, every edge of G appears in precisely one part. Let $\phi_H(G)$ be the smallest possible number of parts in an H-decomposition of G. For non-empty H, let $p_H(G)$ be the maximum number of pairwise edge-disjoint H-subgraphs that can be packed into G and $e(G)$ the number of edges in G. It is easy to see that

$$\phi_H(G) = e(G) - p_H(G)\big(e(H)-1\big). \tag{1.1}$$

Here we study the function

$$\phi_H(n) = \max\left\{\phi_H(G) \middle| v(G) = n\right\},$$

which is the smallest number, such that, any graph G of order n admits an H-decomposition with at most $\phi_H(n)$ elements.

The function $\phi_H(n)$ was first studied by Erdös, Goodman and Pósa [2], who proved that $\phi_{K_3}(n) = t_2(n)$, where K_r denotes the complete graph (clique) of order r and $t_r(n)$ is the maximum size of an r-partite graph on n vertices. A decade later, this result was extended by Bollobás [3], who proved that

$$\phi_{K_r}(n) = t_{r-1}(n), \quad \text{for all } n \ge r \ge 3.$$

Recently, Pikhurko and Sousa [4] studied $\phi_H(n)$ for arbitrary graphs H.

Theorem 1.1. (See Theorem 1.1 from [4]) *Let H be any fixed graph of chromatic number $r \ge 3$. Then,*

$$\phi_H(n) = t_{r-1}(n) + o(n^2).$$

Let $\mathrm{ex}(n,H)$ denote the maximum number of edges in a graph of order n, that does not contain H as a subgraph. Recall that $\mathrm{ex}(n,K_r) = t_{r-1}(n)$. Pikhurko and Sousa [4] also made the following conjecture.

Conjecture 1. *For any graph H with chromatic number at least* 3, *there is $n_0 = n_0(H)$ such that $\phi_H(n) = \mathrm{ex}(n,H)$, for all $n \ge n_0$.*

The exact value of the function $\phi_H(n)$ is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order $r \ge 4$ ($n \ge r$) [5] and the cycles of length 5 ($n \ge 6$) and 7 ($n \ge 10$) [6,7]. Later, Özkahya and Person [8] determined it for all edge-critical graphs with chromatic number $r \ge 3$ and n sufficiently large. They proved the following result.

Theorem 1.2. ([8]) *Let H be any edge-critical graph with chromatic number $r \ge 3$. Then, there exists n_0 such that $\phi_H(n) = \mathrm{ex}(n,H)$, for all $n \ge n_0$. Moreover, the only graph attaining $\phi_H(n)$ is the Turán graph $T_{r-1}(n)$.*

Recently, Allen, Böttcher and Person [9] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1.

The case when H is a bipartite graph has been less studied. Pikhurko and Sousa [4] determined $\phi_H(n)$ for any fixed bipartite graph with an $O(1)$ additive error. For a non-empty graph H, let $\gcd(H)$ denote the greatest common divisor of the degrees of H. For example, $\gcd(K_{6,4}) = 2$ while for any tree T with at least 2 vertices we have $\gcd(T) = 1$. They proved the following result.

Theorem 1.3. (See Theorem 1.3 from [4]) *Let* H *be a bipartite graph with* m *edges and let* $d=\gcd(H)$. *Then there is* $n_0=n_0(H)$ *such that for all* $n\ge n_0$ *the following statements hold.*

(1) If $d=1$, then $\phi_H(n)=\left\lfloor\frac{n(n-1)}{2m}\right\rfloor+C$, where $C=m-1$ or $C=m-2$.

(2) If $d\ge 2$, then
$\phi_H(n)=\frac{nd}{2m}\left(\left\lfloor\frac{n}{d}\right\rfloor-1\right)+\frac{1}{2}n(d-1)+O(1)$.

Moreover, there is a procedure running in polynomial in $\log n$ time which determines $\phi_H(n)$ and describes a family $\mathcal{D}$ of n-sequences such that a graph G of order n satisfies $\phi_H(G)=\phi_H(n)$ if and only if the degree sequence of G belongs to $\mathcal{D}$. (It will be the case that $|\mathcal{D}|=O(1)$ and each sequence in $\mathcal{D}$ has $n-O(1)$ equal entries, so $\mathcal{D}$ can be described using $O(\log n)$ bits.)

Here we will determine the exact value of $\phi_{C_4}(n)$ for n sufficiently large.

Theorem 1.4. *There is* $n_0=n_0(C_4)$ *such that for all* $n\ge n_0$ *the following statements hold.*

(1) If n is even then $\phi_{C_4}(n)=\frac{n^2}{8}+\frac{n}{4}+1.$

(2) If $n\equiv 1(\mathrm{mod}\,8)$ then $\phi_{C_4}(n)=\frac{n^2}{8}+\frac{n}{8}+\frac{14}{8}.$

(3) If $n\equiv 3(\mathrm{mod}\,8)$ then $\phi_{C_4}(n)=\frac{n^2}{8}+\frac{n}{8}+\frac{3}{2}.$

(4) If $n\equiv 5(\mathrm{mod}\,8)$ then $\phi_{C_4}(n)=\frac{n^2}{8}+\frac{n}{8}+\frac{10}{8}.$

(5) If $n\equiv 7(\mathrm{mod}\,8)$ then $\phi_{C_4}(n)=\frac{n^2}{8}+\frac{n}{8}+2.$

2. Proof of Theorem 1.4

In this section we will prove Theorem 1.4, but first we need to introduce the tools. We start with the following easy result about H-decompositions.

Lemma 5. (Lemma 1.3) *For any non-empty graph* H *with* m *edges and any integer* n, *we have*

$$\phi_H(n)\le\frac{1}{m}\binom{n}{2}+\frac{m-1}{m}\mathrm{ex}(n,H). \tag{2.1}$$

In particular, if H is a fixed bipartite graph with m edges and $n\to\infty$, then

$$\phi_H(n)=\left(\frac{1}{m}+O(1)\right)\binom{n}{2}. \tag{2.2}$$

The following result is the well known Erdös-Gallai theorem that gives a necessary and sufficient condition for a finite sequence to be the degree sequence of a simple graph.

Theorem 2.6. (Erdős-Gallai Theorem [10]) *Let* $0\le d_1\le\cdots\le d_n$ *be a sequence of integers. There is a graph with degree sequence* $d_1,\cdots,d_n$ *if and only if*

(1) $d_1+\cdots+d_n$ is even;

(2) for each $1\le k\le n$

$$\sum_{i=n-k+1}^{n}d_i\le k(k-1)+\sum_{i=1}^{n-k}\min\{d_i,k\}. \tag{2.3}$$

The following results appearing in Alon, Caro and Yuster [11, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [12] are essential to the proof of Theorem 1.4.

Lemma 2.7. *For any non-empty graph* H *with* m *edges, there are* $\gamma>0$ *and* N_0 *such that the following holds. Let* $d=\gcd(H)$. *Let* G *be a graph of order* $n\ge N_0$ *and of minimum degree* $\delta(G)\ge(1-\gamma)n$.

If $d=1$, then

$$p_H(G)=\left\lfloor\frac{e(G)}{m}\right\rfloor. \tag{2.4}$$

If $d\ge 2$, let $\alpha_u=d\left\lfloor\frac{\deg(u)}{d}\right\rfloor$ for $u\in V(G)$ and let X consist of all vertices whose degree is not divisible by d. If $|X|\ge\frac{n}{10d^3}$, then

$$p_H(G)=\left\lfloor\frac{1}{2m}\sum_{u\in V(G)}\alpha_u\right\rfloor. \tag{2.5}$$

If $|X|<\frac{n}{10d^3}$, then

$$p_H(G)\ge\frac{1}{m}\left(e(G)-\frac{n}{5d^2}\right). \tag{2.6}$$

□

One can extract the following result from the proof of Theorem 1.2 from [4].

Lemma 2.8. *Let* H *be a bipartite graph with* m *edges and let* $\gcd(H)=d\ge 2$. *Then, there is* $n_0=n_0(H)$ *such that if* G *is a graph of order* $n\ge n_0$ *with* $\phi_H(G)=\phi_H(n)$ *then the following holds:*

(1) Let $d_1,\cdots,d_n$ be the degree sequence of G, then

$$\phi_H(G)=\frac{1}{2}\sum_{i=1}^{n}d_i-(m-1)\left\lfloor\frac{1}{2m}\sum_{i=1}^{n}\left\lfloor\frac{d_i}{d}\right\rfloor d\right\rfloor. \tag{2.7}$$

(2) Let $n=qd+r$ with $0\le r\le d-1$ and $d_i=q_id+r_i$ with $0\le r_i\le d-1$. Then, for $1\le i\le n$ exactly one of the following holds:

(a) $d_i=qd-1$;

(b) $i\in C_1=\{i\in[n]\,|\,r_i=d-1 \text{ and } d_i<qd-1\}$;

(c) $i\in C_2=\{i\in[n]\,|\,d_i=n-1\}$ if $n-1\ne R$ and

$C_2 = \varnothing$ otherwise.

Furthermore, $|C_1| \le \frac{2m}{d} - 1$ and $|C_2| \le 2m - 1$.

In the following we briefly sketch the proof of Lemma 2.8 by giving the argument form [4]. We refrain from doing all the calculations.

Sketch of the proof of Lemma 2.8. Let $\gamma(C_4)$ and N_0 be given by Lemma 2.7. Assume that γ is sufficiently small and that $n_0 \ge N_0$ is sufficiently large to satisfy all the inequalities we will encounter. Let $n \ge n_0$ and let G be any graph of order n with $\phi_{C_4}(G) = \phi_{C_4}(n)$.

Let $G_n = G$. Repeat the following at most $\lfloor n/\log n \rfloor$ times: If the current graph G_i has a vertex x_i of degree at most $(1-\gamma/2)i$, let $G_{i-1} = G_i - x_i$ and decrease i by 1.

Suppose we stopped after s repetitions. Then, either $\delta(G_{n-s}) \ge (1-\gamma/2)(n-s)$ or $s = \lfloor n/\log n \rfloor$. Let us show that the later cannot happen. Otherwise, we have

$$e(G) \le \binom{n-s}{2} + \left(1 - \frac{\gamma}{2}\right) \sum_{i=n-s+1}^{n} i < \binom{n}{2} - \frac{\gamma n^2}{4 \log n}. \quad (2.8)$$

Let t satisfy $K_{t,t} \supset H$. Using the fact that $\mathrm{ex}(n, K_{t,t}) = O(n^{2-1/t})$, (2.1), and (2.8) we obtain

$$\phi_H(G) \le \frac{1}{m}\left(\binom{n}{2} - \frac{\gamma}{4}\frac{n^2}{\log n}\right) + \frac{m-1}{m} c n^{2-1/t} < \frac{1}{m}\binom{n}{2} \le \phi_H(K_n),$$

which contradicts our assumption on G. Therefore, $s < \lfloor n/\log n \rfloor$ and we have $\delta(G_{n-s}) \ge (1-\gamma/2)(n-s)$.

Let $\alpha = 2\gamma$. We will have another pass over the vertices $x_n, \cdots, x_{n-s+1}$, each time decomposing the edges incident to x_i by H-subgraphs and single edges. It will be the case that each time we remove the edges incident to the current vertex x_i, the degree of any other vertex drops by at most $3h^4$, where $h = v(H)$. Here is a formal description. Initially, let $G'_n = G$ and $i = n$. If in the current graph $G_{i'}$ we have $\deg_{G_{i'}}(x_i) \le \alpha n$, then we remove all $G_{i'}$-edges incident to x_i as single edges and let $G'_{i-1} = G_{i'} - x_i$.

Suppose that $\deg_{G_{i'}}(x_i) > \alpha n$. Then, the set

$$X_i = \left\{ y \in V(G_{n-s}) \,\middle|\, x_i y \in E(G_{i'}) \right\},$$

has at least $\alpha n - s + 1$ vertices. The minimum degree of $G[X_i]$ is

$$\delta(G[X_i]) \ge |X_i| - s - \frac{\gamma n}{2} - s \times 3h^4 \ge \frac{2}{3}|X_i|.$$

Let $y \in V(H)$, $A = \Gamma_H(y)$ and $a = |A|$. Another result from [4] (Lemma 3.1) states that there is a constant C, such that, all but at most C vertices of $G[X_i]$ can be covered by edge disjoint copies of $H - y$ each of them having vertex disjoint sets A. Therefore, all but at most C edges between x_i and X_i can be decomposed into copies of H. All other edges incident to x_i are removed as single edges. Let G'_{i-1} consist of the remaining edges of $G_{i'} - x_i$ (that is, those edges that do not belong to an H-subgraph of the above x_i-decomposition). This finishes the description of the case $\deg_{G_{i'}}(x_i) > \alpha n$.

Consider the sets $S = \{x_n, \cdots, x_{n-s+1}\}$, $S_1 = \left\{ x_i \in S \,\middle|\, \deg_{G_{i'}}(x_i) \le \alpha n \right\}$, and $S_2 = S/S_1$. Let their sizes be s, s_1, and s_2 respectively, so $s = s_1 + s_2$.

Let F be the graph with vertex set $V(G_{n-s}) \cup S_2$, consisting of the edges coming from the removed H-subgraphs when we processed the vertices in S_2. We have

$$\phi_H(G) \le \phi_H(G'_{n-s}) + \frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2}. \quad (2.9)$$

We know that $\phi_H(G'_{n-s}) = e(G'_{n-s}) - p_H(G'_{n-s}(m-1))$, furthermore, $\delta(G'_{n-s}) \ge (1-\gamma)(n-s)$. Thus, $p_H(G'_{n-s})$ can be estimated using Lemma 2.7.

If (2.6) holds, some calculations show that there exits a graph G^* such that $\phi_H(G) < \phi_H(G^*)$, which contradicts the optimality go G.

Therefore, (2.5) must hold. It follows that $p_H(G)$ and thus $\phi_H(G)$, depends only on the degree sequence $d_1, \cdots, d_n$ of G. Namely, the packing number $\ell = p_H(G)$ equals $\left\lfloor \frac{1}{2m} \sum_{i=1}^{n} r_i \right\rfloor$, where $r_i = d \lfloor d_i/d \rfloor$ is the largest multiple of d not exceeding d_i.

Therefore,

$$\phi_H(G) = \frac{1}{2}\sum_{i=1}^{n} d_i - (m-1)\left\lfloor \frac{1}{2m} \sum_{i=1}^{n} \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor, \quad (2.10)$$

where $d_1, \cdots, d_n$ is the degree sequence of G.

To conclude the proof we need to estimate the values that the degrees of G can attain. To do that we need to prove an upper bound on $\phi_H(G)$ by estimating $\phi_{\max}$, the maximum of

$$\phi(d_1, \cdots, d_n) = \frac{1}{2}\sum_{i=1}^{n} d_i - (m-1)\left\lfloor \frac{1}{2m} \sum_{i=1}^{n} \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor, \quad (2.11)$$

over all (not necessarily graphical) sequences $d_1, \cdots, d_n$ of integers with $0 \le d_i \le n-1$.

Let $d_1, \cdots, d_n$ be an optimal sequence attaining the value $\phi_{\max}$. For $i = 1, \cdots, n$ let $d_i = q_i d + r_i$ with $0 \le r_i \le d-1$. Then, $\ell = \left\lfloor \frac{(q_1 + \cdots + q_n) d}{2m} \right\rfloor$.

Let $n = qd + r$ with $0 \le r \le d-1$ and $q = \lfloor n/d \rfloor$. Define $R = qd - 1$ to be the maximum integer which is at most $n-1$ and is congruent to $d-1$ modulo d. Let $C_1 = \{ i \in [n] | r_i = d-1 \text{ and } d_i < R \}$ and $C_2 = \{ i \in [n] | d_i = n-1 \}$ if $n-1 \ne R$ and $C_2 = \varnothing$ otherwise.

Since $d_1, \cdots, d_n$ is an optimal sequence, we have that if $r_i \ne d-1$ then $d_i = n-1$ for all $i \in [n]$. To conclude the proof it remains to show that $|C_1| \le \frac{2m}{d} - 1$ and $|C_2| \le 2m-1$. Suppose first that $|C_1| \ge \frac{2m}{d} =: t$. Consider the new sequence of integers

$$d_i^* = \begin{cases} d_i + d, & \text{if } i \in C_1, \\ d_i, & \text{if } i \notin C_1. \end{cases}$$

Then, $\ell^* = \ell + 1$ and $\phi^* = \phi_{\max} + 1$ which contradicts our assumption on $\phi_{\max}$.

Now suppose that $|C_2| \ge 2m$ and consider the new sequence of integers $d_1^*, \cdots, d_n^*$ obtained from $d_1, \cdots, d_n$ by replacing $2m$ values of $n-1$ by R. Then, $\ell^* = \ell - d$ and $\phi^* \ge \phi_{\max} + m - d > \phi_{\max}$, which contradicts our assumption on $\phi_{\max}$ and the proof is concluded. □

We now have all the tools needed to prove Theorem 1.4.

Proof of Theorem 1.4. Let n_0 be given by Lemma 2.8. Let G be a graph of order $n \ge n_0$ with $\phi_{C_4}(G) = \phi_{C_4}(n)$ and degree sequence $d_1, \cdots, d_n$. For $i = 1, \cdots, n$ let $d_i = 2q_i + r_i$ with $0 \le r_i \le 1$. Let, $R = 2(\lfloor n/2 \rfloor - 1) + 1$ and let the sets C_1 and C_2 be as in Lemma 2.8.

Let $n = 2q + r$ with $0 \le r \le 1$ and $q = \lfloor n/2 \rfloor$. From (2.7) we obtain

$$\begin{aligned} &\phi_{C_4}(n) \\ &= n(q-1) + \frac{n}{2} + \frac{1}{2}|C_2| r - \sum_{i \in C_1} (q - 1 - q_i) \\ &-3 \left\lfloor \frac{1}{4} n(q-1) + \frac{1}{4}|C_2| - \frac{1}{4} \sum_{i \in C_1} (q - 1 - q_i) \right\rfloor \end{aligned} \tag{2.12}$$

In what follows let $\alpha = |C_2|$ and $\beta = \sum_{i \in C_1} (q - 1 - q_i)$.

We consider first the case when n is even. Then $C_2 = \varnothing$ and we have

$$\begin{aligned} \phi_{C_4}(n) &= n(q-1) + \frac{n}{2} - \beta - 3 \left\lfloor \frac{1}{4} n(q-1) - \frac{\beta}{4} \right\rfloor \\ &= n(q-1) + \frac{n}{2} - \frac{3q(q-1)}{2} - \beta - 3 \left\lfloor -\frac{\beta}{4} \right\rfloor \end{aligned} \tag{2.13}$$

Claim 1. *Let $d_1, \cdots, d_n$ be the degree sequence of a graph. Then,*

$$-\beta - 3 \left\lfloor -\frac{\beta}{4} \right\rfloor \le 1.$$

Proof. Routine calculations show that for $\beta \ne 1$ we have $-\beta - 3 \left\lfloor -\frac{\beta}{4} \right\rfloor \le 1$. Suppose $\beta = 1$. Then C_1 has exactly one element, thus the sequence $(d_i)_{i=1,\cdots,n}$ has exactly one element equal to $n-3$ and all the others equal to $n-1$. But this is not a degree sequence of a graph since condition (2.3) of Theorem 2.6 does not hold for $k = n-2$. □

Therefore, using the estimate of Claim 1 in (2.13) it follows that

$$\phi_{C_4}(n) \le \frac{n^2}{8} + \frac{n}{4} + 1.$$

To prove the lower bound consider the graph L_5 obtained from K_n after the deletion of the edges of a C_5. Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L_5) = \frac{n^2}{8} + \frac{n}{4} + 1.$$

We now consider the case when n is an odd number.

Case 1: Let $n = 8t + 1$ and $q = 4t$.

From (2.12) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 - t) \\ &+ \frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta - 1}{4} \right\rfloor \end{aligned} \tag{2.14}$$

Claim 2. *Let $d_1, \cdots, d_n$. be the degree sequence of a graph. Then,*

$$\frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta - 1}{4} \right\rfloor \le \frac{5}{2}.$$

Proof. Routine calculations show that the result follows if $\alpha \ne 0$ or $\beta \ne 0$. If $\alpha = 0$ and $\beta = 0$ then $d_i = n-2$ for all $1 \le i \le n$. This is not a degree sequence of a graph since $\sum_{i=1}^n d_i$ is not even. □

Therefore, using the estimate of Claim 2 in (2.14) we prove that

$$\phi_{C_4}(n) \le \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}.$$

As for the lower bound consider the graph L^* with all vertices of degree $n-2$ except one of degree $n-3$. Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L^*) = \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}.$$

Case 2: Let $n = 8t + 3$ and $q = 4t + 1$.

From (2.12) we obtain

$$\phi_{C_4}(n)=\frac{n}{2}(n-3)+\frac{n}{2}-3\left(8t^2+3t\right)+\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta}{4}\right\rfloor. \tag{2.15}$$

Claim 3. *Let* $d_1,\cdots,d_n$. *be the degree sequence of a graph. Then,*

$$\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta}{4}\right\rfloor\le\frac{3}{2}.$$

Proof. It follows from routine calculations for all values of α and β except when $\alpha=0$ and $\beta=1$. Suppose that $\alpha=0$ and $\beta=1$. Then $C_2=\varnothing$ and C_1 has exactly one element, thus the sequence $(d_i)_{i=1,\cdots,n}$ has exactly one element equal to $n-2$ and all the others equal to $n-1$. But this is not a degree sequence of a graph since $\sum d_i$ is not even. □

Therefore, using the estimate of Claim 3 in (2.15) we prove that

$$\phi_{C_4}(n)\le\frac{n^2}{8}+\frac{n}{8}+\frac{3}{2}.$$

As for the lower bound consider the graph L with degree sequence $d_1=d_2=n-4$, $d_3=\cdots=d_{n-1}=n-2$ and $d_n=n-1$ (the existence of L can be proved directly or by Erdös-Gallai theorem, Theorem 2.6). Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L)=\frac{n^2}{8}+\frac{n}{8}+\frac{3}{2}.$$

Case 3: Let $n=8t+5$ and $q=4t+2$.
From (2.12) we obtain

$$\phi_{C_4}(n)=\frac{n}{2}(n-3)+\frac{n}{2}-3\left(8t^2+7t\right)+\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta+5}{4}\right\rfloor. \tag{2.16}$$

Claim 4. *Let* $d_1,\cdots,d_n$. *be the degree sequence of a graph. Then,*

$$\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta+5}{4}\right\rfloor\le-\frac{5}{2}.$$

Proof. Routine calculations show that $\alpha/2-\beta-3\left\lfloor\frac{\alpha-\beta+5}{4}\right\rfloor\le-5/2$ for all values of α and β except for $\alpha=2$ and $\beta=0$ or $\alpha=0$ and $\beta=2$.

Suppose first that $\alpha=2$ and $\beta=0$. Then the sequence $(d_i)_{i=1,\cdots,n}$ has two elements equal to $n-1$ and all the others equal to $n-2$. This is not a degree sequence of a graph since $\sum_{i=1}^{n}d_i$ is not even. □

Suppose now that $\alpha=0$ and $\beta=2$. If $|C_1|=2$ then the sequence has two elements equal to $n-4$ and all the others equal to $n-2$ and this is not a degree sequence of a graph since $\sum d_i$ is not even. Finally, if $|C_1|=1$ then we have one element equal to $n-6$ and all the others equal to $n-2$. Again, this is not a degree sequence of a graph since $\sum d_i$ is not even. □

Therefore, using the estimate of Claim 4 in (2.16) we prove that

$$\phi_{C_4}(n)\le\frac{n^2}{8}+\frac{n}{8}+\frac{10}{8}.$$

As for the lower bound consider the graph K_n-I obtained from K_n by deleting the edges of a maximum matching. Using (1.1) and (2.5) we show that

$$\phi_{C_4}(K_n-I)=\frac{n^2}{8}+\frac{n}{8}+\frac{10}{8}.$$

Case 4: Let $n=8t+7$ and $q=4t+3$.
From (2.12) we obtain

$$\phi_{C_4}(n)=\frac{n}{2}(n-3)+\frac{n}{2}-3\left(8t^2+11t\right)+\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta+14}{4}\right\rfloor. \tag{2.17}$$

Claim 5. *Let* $d_1,\cdots,d_n$. *be the degree sequence of a graph. Then,*

$$\frac{1}{2}\alpha-\beta-3\left\lfloor\frac{\alpha-\beta+14}{4}\right\rfloor\le-\frac{17}{2}.$$

Proof. It follows directly from simple calculations. □

Therefore, using the estimate of Claim 5 in (2.17) we prove that

$$\phi_{C_4}(n)\le\frac{n^2}{8}+\frac{n}{8}+2.$$

Furthermore, using (1.1) and (2.5) we have

$$\phi_{C_4}(K_n-I)=\frac{n^2}{8}+\frac{n}{8}+2,$$

so the equality follows and the proof is now complete. □

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