# Minimum Weight $H$-Decompositions of Graphs: The Bipartite Case

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**Abstract**

Given graphs $G$ and $H$ and a positive number $b$, a *weighted* $(H, b)$*-decomposition* of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $H$-subgraph. We assign a weight of $b$ to each $H$-subgraph in the decomposition and a weight of 1 to single edges. The total weight of the decomposition is the sum of the weights of all elements in the decomposition. Let $\phi(n, H, b)$ be the the smallest number such that any graph $G$ of order $n$ admits an $(H, b)$-decomposition with weight at most $\phi(n, H, b)$. The value of the function $\phi(n, H, b)$ when $b = 1$ was determined, for large $n$, by Pikhurko and Sousa [*Minimum $H$-Decompositions of Graphs*, Journal of Combinatorial Theory, B, **97** (2007), 1041–1055.] Here we determine the asymptotic value of $\phi(n, H, b)$ for any fixed bipartite graph $H$ and any value of $b$ as $n$ tends to infinity.

## 1 Introduction

Let $G$ and $H$ be two graphs and $b$ a positive number. A *weighted* $(H, b)$*-decomposition* of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $H$-subgraph, i.e., a graph isomorphic to $H$. We allow partitions only, that is, every edge of $G$ appears in precisely one part. We assign a weight of $b$ to each $H$-subgraph in the decomposition and a weight of 1 to single edges. The total weight of the decomposition is the sum of the weights of all elements in the decomposition. Let $\phi(G, H, b)$ be the smallest possible weight in an $(H, b)$-decomposition of $G$.

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Let $e(H)$ denote the number of edges in the graph $H$. If $b \geq e(H)$ we have $\phi(G,H,b) = e(G)$. In the case when $0 < b < e(H)$ and $H$ is a fixed graph we can easily see that $\phi(G,H,b) = e(G) - p_H(G)(e(H) - b)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint $H$-subgraphs that can be packed into $G$. Building upon a body of previous research, Dor and Tarsi [5] showed that if $H$ has a component with at least 3 edges then the problem of checking whether an input graph $G$ admits a partition into $H$-subgraphs is NP-complete. Thus, it is NP-hard to compute the function $\phi(G,H,b)$ for such $H$.

Our goal is to study the function

$$\phi(n,H,b) = \max\{\phi(G,H,b) \mid v(G) = n\},$$

which is the smallest number such that any graph $G$ with $n$ vertices admits an $(H,b)$-decomposition with weight at most $\phi(n,H,b)$.

Pikhurko and Sousa [11] considered the case $b = 1$ and proved the following results for large $n$.

**Theorem 1.1.** *Let $H$ be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n,H,1) = t_{r-1}(n) + o(n^2),$$

*where $t_r(n)$, called the* Túran number*, is the maximum number of edges of an $r$-partite graph on $n$ vertices.*

For a non-empty graph $H$, let $\gcd(H)$ denote the greatest common divisor of the degrees of $H$. For example, $\gcd(K_{6,4}) = 2$ while for any tree $T$ with at least 2 vertices we have $\gcd(T) = 1$.

**Theorem 1.2.** *Let $H$ be a bipartite graph with $m$ edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.*

*If $d = 1$, then if $\binom{n}{2} \equiv m - 1 \pmod{m}$,*

$$\phi(n,H,1) = \phi(n,K_n,1) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + m - 1,$$

*otherwise,*

$$\phi(n,H,1) = \phi(n,K_n^*,1) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + m - 2$$

*where $K_n^*$ denotes any graph obtained from $K_n$ after deleting at most $m-1$ edges in order to have $e(K_n^*) \equiv m - 1 \pmod{m}$. Furthermore, if $G$ is extremal then $G$ is either $K_n$ or $K_n^*$.*

*If $d \geq 2$, then*

$$\phi(n,H,1) = \frac{nd}{2m}\left(\left\lfloor \frac{n}{d} \right\rfloor - 1\right) + \frac{1}{2}n(d-1) + O(1).$$

*Moreover, there is a procedure with running time polynomial in $\log n$ which determines $\phi(n,H,1)$ and describes a family $\mathcal{D}$ of $n$-sequences such that a graph $G$ of order $n$ satisfies $\phi(G,H,1) = \phi(n,H,1)$ if and only if the degree sequence of $G$ belongs to $\mathcal{D}$. (It will be the case that $|\mathcal{D}| = O(1)$ and each sequence in $\mathcal{D}$ has $n - O(1)$ equal entries, so $\mathcal{D}$ can be described using $O(\log n)$ bits.)*

Our goal in this paper is to find the value of the function $\phi(n, H, b)$ for any fixed bipartite graph $H$ and $b \neq 1$.

## 2 The bipartite case

Let $H$ be any fixed bipartite graph. We start this section with an easy Lemma.

**Lemma 2.1.** *Let $H$ be a bipartite graph with $m$ edges and let $b \geq m$ be a constant. Then,*

$$\phi(n, H, b) = \binom{n}{2}.$$

*Proof.* Since $b \geq m = e(H)$, we clearly have $\phi(n, G, b) = e(G) \leq \binom{n}{2}$ for all graphs $G$ of order $n$. Therefore $\phi(n, H, b) \leq \binom{n}{2}$. To prove the lower bound observe that $\phi(n, K_n, b) \geq \frac{b}{m}\binom{n}{2} \geq \binom{n}{2}$.

□

Recall that for a non-empty graph $H$, $\gcd(H)$ denotes the greatest common divisor of the degrees of $H$. We will prove the following result.

**Theorem 2.2.** *Let $H$ be a bipartite graph with $m$ edges, let $d = \gcd(H)$ and $0 < b < m$ with $b \neq 1$ a constant. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.*

*If $d = 1$, then*

$$\phi(n, H, b) = b\frac{n(n-1)}{2m} + O(1). \tag{2.1}$$

*If $d \geq 2$, let $n - 1 = qd + r$ where $0 \leq r \leq d - 1$ is an integer.*

*If $r \neq 0$ and $d - 1 \leq \frac{bd}{m} + r$, then*

$$\phi(n, H, b) = \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(r - \frac{br}{m}\right) + O(1). \tag{2.2}$$

*If $r \neq 0$ and $d - 1 \geq \frac{bd}{m} + r$, then*

$$\phi(n, H, b) = \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d - 1 - \frac{br}{m} - \frac{bd}{m}\right) + O(1). \tag{2.3}$$

*If $r = 0$ and $\frac{b}{m} < 1 - \frac{5d^2}{5d^3-2}$, then*

$$\phi(n, H, b) = \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d - 1 - \frac{bd}{m}\right) + O(1). \tag{2.4}$$

*If $r = 0$ and $1 - \frac{5d^2}{5d^3-2} \leq \frac{b}{m} \leq 1 - \frac{1}{d}$, then*

$$\frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d - 1 - \frac{bd}{m}\right) - \frac{1}{2} \leq \phi(n, H, b) \leq \frac{b}{m}\binom{n}{2} + \frac{m-b}{5md^2}n. \tag{2.5}$$

*If* $r = 0$ *and* $\frac{b}{m} \geq 1 - \frac{1}{d}$*, then*

$$\frac{b}{m}\binom{n}{2} \leq \phi(n, H, b) \leq \frac{b}{m}\binom{n}{2} + \frac{m-b}{5md^2}n. \tag{2.6}$$

Before we start the proof, we provide some auxiliary results. We start with the following result appearing in Pikhurko and Sousa [11, Theorem 3.1].

**Lemma 2.3.** *For any bipartite graph $H$ with bipartition $(V_1, V_2)$ and any $A \subset V_1$ with $a \geq 1$ elements, there are integers $C$ and $n_0$ such that the following holds. In any graph $G$ of order $n \geq n_0$ with minimum degree $\delta(G) \geq \frac{2}{3}n$ there is a family of edge disjoint copies of $H$ such that the vertex subsets corresponding to $A \subset V(H)$ are disjoint and cover all but at most $C$ vertices of $G$. One can additionally ensure that each vertex of $G$ belongs to at most $3(v(H))^2$ copies of $H$.*

The following results appearing in Alon, Caro and Yuster [1, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [8], are crucial to the proof of our result.

**Lemma 2.4.** *For any non-empty graph $H$ with $m$ edges, there are $\gamma > 0$ and $N_0$ such that the following holds. Let $d = \gcd(H)$. Let $G$ be a graph of order $n \geq N_0$ and of minimum degree $\delta(G) \geq (1-\gamma)n$.*

*If $d = 1$, then*

$$p_H(G) = \left\lfloor \frac{e(G)}{m} \right\rfloor. \tag{2.7}$$

*If $d \geq 2$, let $\alpha_u = d\lfloor \frac{\deg(u)}{d} \rfloor$ for $u \in V(G)$ and let $X$ consist of all vertices whose degree is not divisible by $d$. If $|X| \geq \frac{n}{10d^3}$, then*

$$p_H(G) = \Big\lfloor \frac{1}{2m} \sum_{u \in V(G)} \alpha_u \Big\rfloor. \tag{2.8}$$

*If $|X| < \frac{n}{10d^3}$, then*

$$p_H(G) \geq \frac{1}{m}\left(e(G) - \frac{n}{5d^2}\right). \tag{2.9}$$

$\square$

*Proof of Theorem 2.2.* Given $H$, let $\gamma(H)$ and $N_0$ be given by Lemma 2.4. Assume that $\gamma \leq \gamma(H)$ is sufficiently small and that $n_0 \geq N_0$ is sufficiently large to satisfy all the inequalities we will encounter. Let $n \geq n_0$ and let $G$ be any graph of order $n$ with $\phi(G, H, b) = \phi(n, H, b)$. We will follow the proof of Pikhurko and Sousa [11, Theorem 1.4], thus only the main results will be stated.

Let $G_n = G$. Repeat the following at most $\lfloor n/\log n \rfloor$ times: If the current graph $G_i$ has a vertex $x_i$ of degree at most $(1-\gamma/2)i$, let $G_{i-1} = G_i - x_i$ and decrease $i$ by 1. Suppose we stopped after $s$ repetitions. Pikhurko and Sousa proved that $s < \lfloor n/\log n \rfloor$ and the graph $G_{n-s}$ has $\delta(G_{n-s}) \geq (1-\gamma/2)(n-s)$.

Let $\alpha = 2\gamma$. We will have another pass over the vertices $x_n, \dots, x_{n-s+1}$, each time decomposing the edges incident to $x_i$ by $H$-subgraphs and single edges. It will be the case that each time we remove the edges incident to the current vertex $x_i$, the degree of any other vertex drops by at most $3h^4$, where $h = v(H)$. Here is a formal description. Initially, let $G'_n = G$ and $i = n$. If in the current graph $G'_i$ we have $\deg_{G'_i}(x_i) \le \alpha n$, then we remove all $G'_i$-edges incident to $x_i$ as single edges and let $G'_{i-1} = G'_i - x_i$.

Suppose that $\deg_{G'_i}(x_i) > \alpha n$. Then, the set $X_i = \{y \in V(G_{n-s}) : x_i y \in E(G'_i)\}$, has at least $\alpha n - s + 1$ vertices. The minimum degree of $G[X_i]$ is

$$\delta(G[X_i]) \ge |X_i| - s - \frac{\gamma n}{2} - s \times 3h^4 \ge \frac{2}{3}|X_i|.$$

Let $y \in V(H)$, $A = N_H(y)$ and $a = |A|$. By Lemma 2.3 there is a constant $C$ such that all but at most $C$ vertices of $G[X_i]$ can be covered by edge disjoint copies of $H - y$ each of them having vertex disjoint sets $A$. Therefore, all but at most $C$ edges between $x_i$ and $X_i$ can be decomposed into copies of $H$. All other edges incident to $x_i$ are removed as single edges. Let $G'_{i-1}$ consist of the remaining edges of $G'_i - x_i$ (that is, those edges that do not belong to an $H$-subgraph of the above $x_i$-decomposition). This finishes the description of the case $\deg_{G'_i}(x_i) > \alpha n$.

Consider the sets $S = \{x_n, \dots, x_{n-s+1}\}$, $S_1 = \{x_i \in S : \deg_{G'_i}(x_i) \le \alpha n\}$, and $S_2 = S \backslash S_1$. Let their sizes be $s$, $s_1$ and $s_2$ respectively, so $s = s_1 + s_2$.

Let $F$ be the graph with vertex set $V(G_{n-s}) \cup S_2$, consisting of the edges coming from the removed $H$-subgraphs when we processed the vertices in $S_2$. We have

$$\phi(G, H, b) \le \phi(G'_{n-s}, H, b) + b\frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2}. \tag{2.10}$$

We know that $\phi(G'_{n-s}, H, b) = e(G'_{n-s}) - p_H(G'_{n-s})(m - b)$. The last statement of Lemma 2.3 guarantees that $\delta(G'_{n-s}) \ge (1 - \gamma)(n - s)$. Thus, $p_H(G'_{n-s})$ can be estimated using Lemma 2.4.

Consider first the case $d = 1$. Using the inequalities $e(F) \le (1 - \gamma/2)s_2 n$ and $\alpha \le b(2 - \gamma)/2m$, we obtain

$$\begin{aligned}
\phi(G, H, b) &\le \phi(G'_{n-s}, H, b) + b\frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2} \\
&\le e(G'_{n-s}) - p_H(G'_{n-s})(m - b) + b\frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2} \\
&\le e(G'_{n-s}) - \left\lfloor \frac{e(G'_{n-s})}{m} \right\rfloor (m - b) + b\frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2} \\
&\le \left( \frac{b}{m}\binom{n-s}{2} + m - b \right) + b\frac{2-\gamma}{2m} s_2 n + b\frac{2-\gamma}{2m} s_1 n + s_2 C + \binom{s}{2} \\
&\le \frac{b}{m}\binom{n-s}{2} + b\frac{2-\gamma}{2m} sn + s_2 C + \binom{s}{2} + m - b \\
&\le \frac{b}{m}\binom{n}{2} - b\frac{(n-1)s}{m} + b\frac{s(s-1)}{2m} + b\frac{2-\gamma}{2m} sn + s_2 C + \binom{s}{2} + m - b.
\end{aligned}$$

If $S \neq \emptyset$ then in order to prove that $\phi(G,H,b) < \frac{b}{m}\binom{n}{2} \leq \phi(H,K_n,b)$ and hence a contradiction to our assumption on $G$, it suffices to show that

$$b\frac{s}{m} + b\frac{s(s-1)}{2m} + \binom{s}{2} + s_2C + m - b < \left(\frac{b}{m} - \frac{b(2-\gamma)}{2m}\right)ns.$$

But this last inequality holds since we have $s < \frac{n}{\log n}$ and $n$ is sufficiently large. Thus, $S = \emptyset$ and

$$\begin{aligned}\phi(G,H,b) &= e(G) - (m-b)\left\lfloor \frac{e(G)}{m} \right\rfloor \\ &\leq b\frac{e(G)}{m} + (m-b) \\ &\leq b\frac{n(n-1)}{2m} + (m-b),\end{aligned} \tag{2.11}$$

giving us the upper bound. To prove the lower bound we consider the complete graph on $n$ vertices and we obtain

$$\phi(K_n,H,b) = e(K_n) - (m-b)\left\lfloor \frac{e(K_n)}{m} \right\rfloor \geq b\frac{n(n-1)}{2m}. \tag{2.12}$$

Consider the case $d \geq 2$ and let $n-1 = qd + r$ with $0 \leq r \leq d-1$ an integer. To prove the lower bounds we consider the complete graph of order $n \geq n_0$ and a graph $L$ of order $n \geq n_0$, which is (almost) $(qd-1)$-regular (except at most one vertex of degree $qd-2$). (Such a graph $L$ exists, which can be seen either directly or from Erdős and Gallai's result [6].) We have,

$$\begin{aligned}\phi(K_n,H,b) &= e(K_n) - p_H(K_n)(m-b) \\ &\geq \binom{n}{2} - \frac{1}{2} - \frac{ndq}{2m}(m-b) \\ &\geq \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(r - \frac{br}{m}\right) - \frac{1}{2},\end{aligned} \tag{2.13}$$

and,

$$\begin{aligned}\phi(L,H,b) &= e(L) - p_H(L)(m-b) \\ &\geq \frac{1}{2}n(qd-1) - \frac{1}{2} - \frac{nd(q-1)}{2m}(m-b) \\ &\geq \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d - 1 - \frac{br}{m} - \frac{bd}{m}\right) - \frac{1}{2},\end{aligned} \tag{2.14}$$

giving the required lower bounds in view of $q = \frac{n-1-r}{d}$.

We will now prove the upper bounds.

Assume first that (2.9) holds. Then, by (2.10)

$$\begin{aligned}
&\phi(G,H,b)\\
&\quad\le \phi(G'_{n-s},H,b)+b\frac{e(F)}{m}+s_1\alpha n+s_2C+\binom{s}{2}\\
&\quad\le e(G'_{n-s})-p_H(G'_{n-s})(m-b)+b\frac{e(F)}{m}+s_1\alpha n+s_2C+\binom{s}{2}\\
&\quad\le e(G'_{n-s})-\frac{m-b}{m}\left(e(G'_{n-s})-\frac{n-s}{5d^2}\right)+b\frac{e(F)}{m}+s_1\alpha n+s_2C+\binom{s}{2}\\
&\quad\le \frac{b}{m}\binom{n-s}{2}+\frac{b(2-\gamma)}{2m}s_2n+\frac{(m-b)(n-s)}{5md^2}+\frac{b(2-\gamma)}{2m}s_1n+s_2C+\binom{s}{2}\\
&\quad\le \frac{b}{m}\binom{n}{2}-\frac{b(n-1)s}{m}+\frac{bs(s-1)}{2m}+\frac{(m-b)(n-s)}{5md^2}+\frac{b(2-\gamma)}{2m}sn+s_2C+\binom{s}{2}.
\end{aligned}$$

For $s>\frac{2(m-b)}{5\gamma d^2b}$ we have $\frac{b}{m}-\frac{b(2-\gamma)}{2m}-\frac{m-b}{5md^2s}>0$. Thus, for $n$ sufficiently large

$$\frac{bs}{m}+\frac{bs(s-1)}{2m}-\frac{(m-b)s}{5md^2}+s_2C+\binom{s}{2}<\left(\frac{b}{m}-\frac{b(2-\gamma)}{2m}-\frac{m-b}{5md^2s}\right)ns.$$

That is, $\phi(G,H,b)<\frac{b}{m}\binom{n}{2}\le\phi(K_n,H,b)$ which contradicts the optimality of $G$. Otherwise, $s$ is bounded by a constant independent of $n$ and the coefficient of $sn$ is $-\frac{b}{m}+\frac{b(2-\gamma)}{2m}<0$. Thus, for the case $r\neq 0$, to obtain the contradiction $\phi(G,H,b)<\phi(K_n,H,b)$ it suffices to show that

$$\frac{b}{m}\binom{n}{2}+\frac{m-b}{5md^2}n<\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(r-\frac{br}{m}\right),$$

that is,

$$\frac{1}{5d^2}<\frac{1}{2}r,$$

which holds since $d\ge 2$ and $r\ge 1$.

If $r=0$ and $\frac{b}{m}<1-\frac{5d^2}{5d^3-2}$, to obtain the contradiction $\phi(G,H,b)<\phi(L,H,b)$ it suffices to show that

$$\frac{b}{m}\binom{n}{2}+\frac{m-b}{5md^2}n<\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(d-1-\frac{bd}{m}\right),$$

which holds since $\frac{b}{m}<1-\frac{5d^2}{5d^3-2}$. Otherwise, we have

$$\phi(G,H,b)<\frac{b}{m}\binom{n}{2}+\frac{m-b}{5md^2}n,$$

which is the upper bound stated in (2.5) and (2.6).

Finally, assume that (2.8) holds. It follows that $p_H(G)$ and thus $\phi(G,H,b)$, depends only on the degree sequence $d_1,\dots,d_n$ of $G$. Namely, the packing number $\ell = p_H(G)$ equals $\lfloor \frac{1}{2m}\sum_{i=1}^n r_i \rfloor$, where $r_i = d\lfloor d_i/d \rfloor$ is the largest multiple of $d$ not exceeding $d_i$.

Thus, it is enough for us to prove the upper bounds in (2.3) and (2.4) on $\phi_{\max}$, the maximum of

$$\phi(d_1,\dots,d_n) = \frac{1}{2}\sum_{i=1}^n d_i - (m-b)\left\lfloor \frac{1}{2m}\sum_{i=1}^n \left\lfloor \frac{d_i}{d}\right\rfloor d \right\rfloor, \tag{2.15}$$

over all (not necessarily graphical) sequences $d_1,\dots,d_n$ of integers with $0 \le d_i \le n-1$.

Let $d_1,\dots,d_n$ be an optimal sequence attaining the value $\phi_{\max}$. For $i=1,\dots,n$ let $d_i = q_i d + r_i$ with $0 \le r_i \le d-1$. Then, $\ell = \left\lfloor \frac{(q_1+\cdots+q_n)d}{2m} \right\rfloor$.

Recall that $n-1 = qd + r$ with $1 \le r \le d-1$. Define $R = qd-1$ to be the maximum integer which is at most $n-1$ and is congruent to $d-1$ modulo $d$. Let $C_1 = \{i \in [n] : r_i = d-1 \text{ and } d_i < R\}$ and $C_2 = \{i \in [n] : d_i = n-1\}$ if $n-1 \ne R$ and $C_2 = \emptyset$ otherwise.

Since $d_1,\dots,d_n$ is an optimal sequence, we have that if $r_i \ne d-1$ then $d_i = n-1$ for all $i \in [n]$. Also, $|C_1| \le \frac{2m}{d} - 1$ and $|C_2| \le 2m-1$. We have

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^n d_i &= \frac{1}{2}(n - |C_1 \cup C_2|)R + \frac{1}{2}\sum_{i\in C_1} d_i + \frac{1}{2}|C_2|(n-1)\\
&\le \frac{1}{2}nd(q-1) + \frac{1}{2}n(d-1) - \frac{d}{2}\sum_{i\in C_1}(q-1-q_i) + O(1),\\
\ell &\ge \left(\frac{1}{2m}\sum_{i=1}^n \left\lfloor \frac{d_i}{d}\right\rfloor d\right) - 1\\
&\ge \frac{1}{2m}nd(q-1) - \frac{d}{2m}\sum_{i\in C_1}(q-1-q_i) + O(1).
\end{aligned}$$

These estimates give us the required upper bound in (2.3) and (2.4).

$$\begin{aligned}
\phi_{\max} = \frac{1}{2}\sum_{i=1}^n d_i - (m-b)\ell &\le \frac{b}{2m}nd(q-1) + \frac{1}{2}n(d-1) + O(1)\\
&\le \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d - 1 - \frac{br}{m} - \frac{bd}{m}\right) + O(1).
\end{aligned} \tag{2.16}$$

The upper bound in (2.2) follows from the fact that

$$\frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(d-1-\frac{br}{m}-\frac{bd}{m}\right) \le \frac{b}{m}\binom{n}{2} + \frac{1}{2}n\left(r - \frac{br}{m}\right),$$

in view of $d - 1 \le \frac{bd}{m} + r$.

To finish the proof it remains to obtain a contradiction if $S \neq \emptyset$ holds. Let $\bar{d}_1, \ldots, \bar{d}_n$ be the degree sequence of the graph with vertex set $V(G)$ and edge set $E(G'_{n-s}) \cup E(F)$. Consider the new sequence of integers

$$d'_i = \begin{cases} \bar{d}_i, & \text{if } x_i \notin S, \\ \bar{d}_i + \left\lceil \frac{(1-3\gamma)}{m} n \right\rceil m, & \text{if } x_i \in S_1, \\ \bar{d}_i + \left\lceil \frac{\gamma}{4m} n \right\rceil m, & \text{if } x_i \in S_2. \end{cases}$$

Each $d'_i$ lies between 0 and $n-1$, so $\phi(d'_1, \ldots, d'_n) \leq \phi_{\max}$. We obtain

$$\begin{array}{rcl} \phi(G, H, b) & \leq & \phi(\bar{d}_1, \ldots, \bar{d}_n) + s_1 \alpha n + s_2 C + \binom{s}{2} \\ & < & \phi(d'_1, \ldots, d'_n) - \dfrac{b(1-3\gamma)}{2m} s_1 n - \dfrac{b\gamma}{8m} s_2 n + s_1 \alpha n + s_2 C + \dbinom{s}{2} \\ & < & \phi(d'_1, \ldots, d'_n) - \left( \dfrac{b(1-3\gamma)}{2m} - 2\gamma \right) s_1 n - \dfrac{b\gamma}{8m} s_2 n + s_2 C + \dbinom{s}{2} \\ & < & \phi(d'_1, \ldots, d'_n) - \dfrac{b\gamma}{8m} s_1 n - \dfrac{b\gamma}{8m} s_2 n + s_2 C + \dbinom{s}{2} \\ & \leq & \phi_{\max} - \dfrac{b\gamma}{10m} sn, \end{array}$$

which contradicts the already established facts that $\phi(n, H, b)$ is at most $\phi(G, H, b)$ by the optimality of $G$ and is at least $\phi_{\max}$ by (2.16). $\square$

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