Monochromatic K_r-Decompositions of Graphs

Henry Liu*

Centro de Matemática e Aplicações

Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa

Quinta da Torre, 2829-516 Caparica, Portugal

h.liu@fct.unl.pt

Teresa Sousa*

Departamento de Matemática and Centro de Matemática e Aplicações

Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa

Quinta da Torre, 2829-516 Caparica, Portugal

tmjs@fct.unl.pt

June 3, 2013

Abstract

Given graphs G and H, and a colouring of the edges of G with k colours, a *monochromatic H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a monochromatic graph isomorphic to H. Let $\phi_k(n, H)$ be the smallest number ϕ such that any graph G of order n and any colouring of its edges with k colours, admits a monochromatic H-decomposition with at most ϕ parts. Here we study the function $\phi_k(n, K_r)$ for $k \geq 2$ and $r \geq 3$.

*This work was partially supported by FCT - Fundação para a Ciência e a Tecnologia through Projects PTDC/MAT/113207/2009 and PEst-OE/MAT/UI0297/2011 (CMA).

1 Introduction

All graphs in this paper are finite, undirected and simple. For notation and terminology not discussed here the reader is referred to [3].

Given two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms an H-subgraph, i.e., a graph isomorphic to H. We allow partitions only, that is, every edge of G appears in precisely one part. Let $\phi(G, H)$ be the smallest possible number of parts in an H-decomposition of G. It is easy to see that, for non-empty H, $\phi(G, H) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint H-subgraphs that can be packed into G and $e(G)$ denotes the number of edges in G. Building upon a body of previous research, Dor and Tarsi [4] showed that if H has a component with at least 3 edges, then the problem of checking whether an input graph G is perfectly decomposable into H-subgraphs is NP-complete. Hence, it is NP-hard to compute the function $\phi(G, H)$ for such H. Therefore, the aim is to study the function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number such that any graph G of order n admits an H-decomposition with at most $\phi(n, H)$ parts, where $v(G)$ denotes the number of vertices in G.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [5], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n, K_3) = t_2(n)$, where K_s denotes the complete graph of order s and we often refer to K_3 as a *triangle*, and $t_{r-1}(n)$ denotes the number of edges in the Turán graph of order n, $T_{r-1}(n)$, which is the unique complete $(r-1)$-partite graph on n vertices where every partition class has either $\lfloor \frac{n}{r-1} \rfloor$ or $\lceil \frac{n}{r-1} \rceil$ vertices. *Turán's Theorem* [19] states that $T_{r-1}(n)$ is the unique graph on n vertices that has the maximum number of edges and contains no complete subgraph of order r. A decade later, the result of Erdős, Goodman and Pósa was extended by Bollobás [1], who proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

General graphs H were only considered recently by Pikhurko and Sousa [15] who proved the following result.

Theorem 1.1. [15] *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

However, the exact value of the function $\phi(n, H)$ is far from being known. A graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the *chromatic number* of H. For $r \geq 4$, a *clique-extension of order* r is a connected graph that consists of a K_{r-1} plus another vertex adjacent to at most $r - 2$ vertices of K_{r-1}. Sousa determined the value of $\phi(n, H)$ for a few special edge-critical graphs, namely for clique-extensions of order $r \geq 4$ (for $n \geq r$) [17] and the cycles of length 5 (for $n \geq 6$) and 7 (for $n \geq 10$) [16, 18]. Later, Özkahya and Person [14] determined it for all edge-critical graphs with chromatic number $r \geq 3$ and n sufficiently large. Let $\mathrm{ex}(n, H)$ denote the maximum number of edges in a graph of order n, that does not contain H as a subgraph. Recall that $\mathrm{ex}(n, K_r) = t_{r-1}(n)$. They proved the following result.

Theorem 1.2. [14] *Let H be any edge-critical graph with chromatic number $r \geq 3$. Then, there exists n_0 such that $\phi(n, H) = \mathrm{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\phi(n, H)$ is the Turán graph $T_{r-1}(n)$.*

The case when H is a bipartite graph has been less studied. Pikhurko and Sousa [15] determined $\phi(n, H)$ for any fixed bipartite graph with an $O(1)$ additive error. For a non-empty graph H, let $\gcd(H)$ denote the greatest common divisor of the degrees of H. For example, $\gcd(K_{6,4}) = 2$, while for any tree T with at least 2 vertices we have $\gcd(T) = 1$. They proved the following result.

Theorem 1.3. [15] *Let H be a bipartite graph with m edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.*

(a) If $d = 1$, then $\phi(n, H) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + C$, where $C = m - 1$ or $C = m - 2$.

(b) If $d \geq 2$, then $\phi(n, H) = \frac{nd}{2m}\left(\left\lfloor \frac{n}{d} \right\rfloor - 1\right) + \frac{1}{2}n(d-1) + O(1)$.

Here, our aim is to consider a coloured version of the H-decomposition problem. We define the problem more precisely.

A *k-edge-colouring* of a graph G is a function $c : E(G) \to \{1, \ldots, k\}$. We think of c as a colouring of the edges of G, where each edge is given one of k possible colours. Given a fixed graph H, a graph G of order n and a k-edge-colouring of the edges of G, a *monochromatic H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or a monochromatic copy of H. Let $\phi_k(G, H)$ be the smallest number such that, for any k-edge-colouring of G, there exists a monochromatic H-decomposition of G with at most $\phi_k(G, H)$ elements. Our aim is to study the function

$$\phi_k(n, H) = \max\{\phi_k(G, H) \mid v(G) = n\},$$

which is the smallest number such that, any k-edge-coloured graph of order n admits a monochromatic H-decomposition with at most $\phi_k(n, H)$ elements.

Here our goal is to study the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 3$. Our results involve the Ramsey numbers and the Turán numbers. Recall that for $r \geq 3$ and $k \geq 2$, the *Ramsey number for K_r*, denoted by $R_k(r)$, is the smallest value of s for which every k-edge-colouring of K_s contains a monochromatic K_r. The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite for all $r \geq 3$ and $k \geq 2$. In fact, for the Ramsey numbers $R_k(r)$, only three of them are currently known. In 1955, Greenwood and Gleason [8] were the first to determine $R_2(3) = 6$, $R_3(3) = 17$ and $R_2(4) = 18$.

In this work we will also consider 'blow-up' versions of k-edge-colourings. A more precise definition of a 'blow-up' is as follows. For $s \geq 2$, let G be an s-partite graph with partition classes $V_1, \ldots, V_s$, let f be a k-edge-colouring of G, and let f' be a k-edge-colouring of K_s. We say that f, or G, is a *blow-up* of f' if the vertices of K_s can be labelled $v_1, \ldots, v_s$ such that, for all $x \in V_i$ and $y \in V_j$ with $1 \leq i \neq j \leq s$, we have $f(xy) = f'(v_i v_j)$. We can easily prove a lower bound on the value of $\phi_k(n, K_r)$ for all $r \geq 3$ and $k \geq 2$.

Lemma 1.4. *Let $r \geq 3$, $k \geq 2$ and $n \geq R_k(r)$. Then,*

$$\phi_k(n, K_r) \geq t_{R_k(r)-1}(n).$$

Proof. By the definition of $R_k(r)$, there exists a k-edge-colouring f' of the complete graph $K_{R_k(r)-1}$ with no monochromatic K_r. Now, consider the Turán graph

$T_{R_k(r)-1}(n)$ with a k-edge-colouring f which is a blow-up of f'. Then the graph $T_{R_k(r)-1}(n)$ with the k-edge-colouring f has no monochromatic K_r and thus $\phi_k(n, K_r) \geq \phi_k(T_{R_k(r)-1}(n), K_r) = t_{R_k(r)-1}(n)$. $\square$

Hence, the construction in Lemma 1.4 shows that we cannot be guaranteed to be able to find a monochromatic K_r-decomposition of any k-edge-coloured graph on n vertices, with less than $t_{R_k(r)-1}(n)$ elements. In fact, we will prove that the value of $t_{R_k(r)-1}(n)$ is asymptotically correct for $k \geq 4$ and $r = 3$ (see Theorem 1.5 below), and exact for $k = 2, 3$ and $r = 3$ (see Theorem 1.7) and for $k \geq 2$ and $r \geq 4$ (see Theorem 1.9), with n sufficiently large in both cases.

Theorem 1.5. *For all $k \geq 2$, we have*

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2). \tag{1.1}$$

In particular, it is known that $R_2(3) = 6$ and $R_3(3) = 17$. Indeed, for two colours, it is easy to see that the only 2-edge-colouring of K_5 not containing a monochromatic triangle is the one where each colour induces a cycle of length 5, as shown in Figure 1. Let f_2 denote this 2-edge-colouring of K_5.

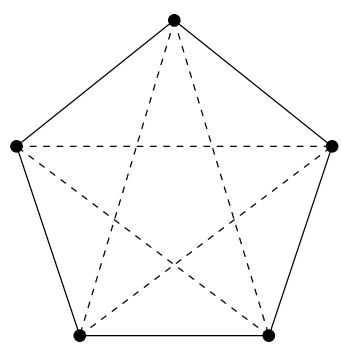

Figure 1: The 2-edge-colouring f_2 of K_5

For three colours, the Ramsey number $R_3(3) = 17$ was first determined, in 1955, by Greenwood and Gleason [8]. Later, in 1968, Kalbfleisch and Stanton [12] considered the structures of all possible 3-edge-colourings of K_{16} not containing a monochromatic

triangle. Their result is stated in terms of the *Clebsch graph*, which is a well-known 5-regular, Hamiltonian, triangle-free graph on 16 vertices and 40 edges.

Theorem 1.6. [12] *There exist exactly two different* 3*-edge-colourings of* K_{16} *with no monochromatic triangle. In each case, each colour class induces the Clebsch graph.*

Let f_3 and f_3' be the two 3-edge-colourings of K_{16} as in Theorem 1.6. Consequently, we can improve the upper bound in (1.1) for the cases $k = 2, 3$, as follows.

Theorem 1.7. *Let* $k = 2, 3$*. There is an* n_0 *such that, for all* $n \geq n_0$*, we have*

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n). \tag{1.2}$$

That is, $\phi_2(n, K_3) = t_5(n)$ *and* $\phi_3(n, K_3) = t_{16}(n)$*.*

Moreover, the only k*-edge-coloured graph* G *with* $\phi_k(G, K_3) = \phi_k(n, K_3)$ *is* $G = T_{R_k(3)-1}(n)$*, and* G *is a blow-up of the* 2*-edge-colouring* f_2 *for* $k = 2$*, or of the* 3*-edge-colouring* f_3 *or* f_3' *for* $k = 3$*.*

For monochromatic K_3-decompositions we make the following conjecture.

Conjecture 1.8. *Let* $k \geq 4$*. Then* $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ *for* $n \geq R_k(3)$*.*

For larger cliques and n sufficiently large we are able to find the value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$. We recall that the Ramsey number $R_2(4) = 18$ is also well-known.

Theorem 1.9. *Let* $r \geq 4$*,* $k \geq 2$*. There is an* $n_0 = n_0(r, k)$ *such that, for all* $n \geq n_0$*, we have*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n). \tag{1.3}$$

In particular, $\phi_2(n, K_4) = t_{17}(n)$*.*

Moreover, the only graph attaining $\phi_k(n, K_r)$ *is the Turán graph* $T_{R_k(r)-1}(n)$*.*

In Section 2 we will prove Theorem 1.5 and Theorem 1.7, and in Section 3 we will prove Theorem 1.9.

2 Monochromatic K_3-decompositions

In this section we will prove Theorems 1.5 and 1.7. Before presenting the proofs, we need to introduce the tools and prove some auxiliary results.

Firstly, we recall the following version of the Erdős-Stone-Simonovits Theorem [7, 6] (see also [2], Ch. VI.4, Theorem 4.2).

Theorem 2.1 (Erdős-Stone-Simonovits Theorem [7, 6])**.** *Let $r \geq 3$, and G be a graph on n vertices, with $\frac{1}{2}(1 - \frac{1}{r-1} + o(1))n^2$ edges that does not contain a K_r. Then G contains an $(r-1)$-subgraph G' with the following properties.*

(a) Each partition class of G' has $(\frac{1}{r-1} + o(1))n$ vertices.

(b) $e(G') = \frac{1}{2}(1 - \frac{1}{r-1} + o(1))n^2$.

(c) The minimum degree of G' is $(1 - \frac{1}{r-1} + o(1))n$.

Using Theorem 2.1, we will prove an edge-coloured version of the Erdős-Stone-Simonovits Theorem in the case when we forbid monochromatic triangles.

Proposition 2.2. *Let $k = 2, 3$ and let G be a graph on n vertices with $\frac{1}{2}(1 - \frac{1}{R_k(3)-1} + o(1))n^2$ edges. Suppose that G is k-edge-coloured so that there is no monochromatic copy of K_3. Then G contains an $(R_k(3)-1)$-partite subgraph G' such that the following properties hold.*

(i) Each partition class of G' has $(\frac{1}{R_k(3)-1} + o(1))n$ vertices.

(ii) $e(G') = \frac{1}{2}(1 - \frac{1}{R_k(3)-1} + o(1))n^2$.

(iii) The minimum degree of G' is $(1 - \frac{1}{R_k(3)-1} + o(1))n$.

(iv) The k-edge-colouring on G' is a blow-up of the 2-edge-colouring f_2 on K_5 if $k = 2$, or of the 3-edge-colouring f_3 or f_3' on K_{16} if $k = 3$.

Proof. Throughout, let $k = 2$ or $k = 3$. We note that G does not contain a $K_{R_k(3)}$, otherwise, a k-edge-coloured $K_{R_k(3)}$ would contain a monochromatic copy of K_3. By Theorem 2.1, with $r = R_k(3)$, G contains an $(R_k(3) - 1)$-partite subgraph G' with

partition classes $V_1, \ldots, V_{R_k(3)-1}$, where $|V_i| = (\frac{1}{R_k(3)-1} + o(1))n$ for every $1 \leq i \leq R_k(3) - 1$, $e(G') = \frac{1}{2}(1 - \frac{1}{R_k(3)-1} + o(1))n^2$, and the minimum degree of G' is $(1 - \frac{1}{R_k(3)-1} + o(1))n$. We claim that G' is the required subgraph. G' satisfies properties (i), (ii) and (iii), and in fact, property (iii) implies the following.

(iii*) For every $1 \leq i \neq j \leq R_k(3) - 1$ and $v \in V_i$, there are $(\frac{1}{R_k(3)-1} + o(1))n$ edges between v and V_j.

It remains to prove that G' also satisfies property (iv).

Claim 1. *Let $1 \leq p \leq R_k(3) - 1$ and $\ell_1, \ldots, \ell_p \in \{1, \ldots, R_k(3) - 1\}$ be all distinct. For every $1 \leq q \leq p$, let $U_q \subset V_{\ell_q}$ be such that $|U_q| \geq (c + o(1))n$, for some constant $c > 0$. Then, there exist vertices $x_1, \ldots, x_p$ with $x_q \in U_q$ for every $1 \leq q \leq p$, such that $x_1, \ldots, x_p$ form a copy of K_p in G'.*

Proof. We apply property (iii*) repeatedly. Let $x_1 \in U_1$. For every $2 \leq q \leq p$, let $U'_q \subset U_q$ be the neighbours of x_1 in U_q, so that $|U'_q| \geq (c + o(1))n$. Let $x_2 \in U'_2$. For every $3 \leq q \leq p$, let $U''_q \subset U'_q$ be the neighbours of x_2 in U'_q, so that $|U''_q| \geq (c+o(1))n$. Let $x_3 \in U''_3$. Repeating this procedure successively, we obtain the vertices $x_1, \ldots, x_p$, with $x_q \in U_q$ for every $1 \leq q \leq p$, which are suitable for the claim. □

Claim 2. *For every $1 \leq i \neq j \leq R_k(3) - 1$ and $u \in V_i$, all edges between u and V_j have the same colour in G'.*

Proof. For the sake of simplicity, assume that the edges of G are k-coloured with colours red, blue if $k = 2$ and red, blue and green if $k = 3$. Suppose that there exist $v, w \in V_j$ such that uv is red and uw is blue. We show that this implies that there is a monochromatic copy of K_3, which will be a contradiction.

For $k = 2$, by property (iii*), we can assume, without loss of generality, that there exist $\ell_1, \ell_2 \in \{1, \ldots, 5\} \setminus \{i, j\}$ such that there are at least $(\frac{1}{10} + o(1))n$ red edges between u and each of V_{ℓ_1} and V_{ℓ_2}. Let $U_1 \subset V_{\ell_1}$ and $U_2 \subset V_{\ell_2}$ be the red neighbours of u in $V_{\ell_1} \cup V_{\ell_2}$, and let $U'_1 \subset U_1$ and $U'_2 \subset U_2$ be the neighbours of v in $U_1 \cup U_2$. Note that $|U'_1|, |U'_2| \geq (\frac{1}{10} + o(1))n$. By Claim 1 with $p = 2$, there are vertices $x_1 \in U'_1$ and $x_2 \in U'_2$ so that x_1, x_2 form a K_2 in G' and therefore u, v, x_1, x_2 form a K_4 in G'. But

then, either we have a red K_3 using u and two of v, x_1, x_2, or we have a blue K_3 on v, x_1, x_2.

Similarly for $k = 3$, either there exist $\ell_1, \ldots, \ell_5 \in \{1, \ldots, 16\} \setminus \{i, j\}$ such that there are at least $(\frac{1}{48} + o(1))n$ edges between u and each of $V_{\ell_1}, \ldots, V_{\ell_5}$ with all edges red or all edges blue; or there exist $\ell'_1, \ldots, \ell'_6 \in \{1, \ldots, 16\} \setminus \{i, j\}$ such that there are at least $(\frac{1}{48} + o(1))n$ green edges between u and each of $V_{\ell'_1}, \ldots, V_{\ell'_6}$. It suffices to consider the former, where the edges involved are red. Let $U_1 \subset V_{\ell_1}, \ldots, U_5 \subset V_{\ell_5}$ be the red neighbours of u in $V_{\ell_1} \cup \cdots \cup V_{\ell_5}$, and $U'_1 \subset U_1, \ldots, U'_5 \subset U_5$ be the neighbours of v in $U_1 \cup \cdots \cup U_5$. Note that $|U'_1|, \ldots, |U'_5| \geq (\frac{1}{48} + o(1))n$. By Claim 1 with $p = 5$, there are vertices $x_1 \in U'_1, \ldots, x_5 \in U'_5$ such that $x_1, \ldots, x_5$ form a K_5 in G' and therefore $u, v, x_1, \ldots, x_5$ form a K_7 in G'. If there is a red edge among $v, x_1, \ldots, x_5$, then we have a red K_3, using u and the red edge. Otherwise, $v, x_1, \ldots, x_5$ form a 2-edge-coloured K_6, using blue and green, and hence there is a blue K_3 or a green K_3. □

We are now able to conclude the proof of the proposition. By Claim 2 and property (iii*), we see that for every $1 \leq i \neq j \leq R_k(3) - 1$, every edge between V_i and V_j must have the same colour in G'. Otherwise, for some $i \neq j$, we have $u, u' \in V_i$ such that, there are $(\frac{1}{R_k(3)-1} + o(1))n$ edges in one colour between u and V_j, and $(\frac{1}{R_k(3)-1} + o(1))n$ edges in another colour between u' and V_j. But then, there exists $v \in V_j$ such that there are two edges of different colours from v to V_i, contradicting Claim 2.

Finally, if some three of the V_i are such that the edges in G' that they induce have the same colour, then by Claim 1, we have a monochromatic K_3. It follows that the colouring on G' must be a blow-up of f_2 if $k = 2$, or of f_3 or f'_3 if $k = 3$. Therefore, property (iv) holds, and we are done. □

The ideas used in the proof of Proposition 2.2 also enable us to deduce the following corollary.

Corollary 2.3. *For $k = 2, 3$, every k-edge-colouring of the Turán graph $T_{R_k(3)-1}(n)$ without a monochromatic copy of K_3 is a blow-up of the 2-edge-colouring f_2 of K_5 if $k = 2$, or of the 3-edge-colouring f_3 or f'_3 of K_{16} if $k = 3$.* □

We now consider a long-standing conjecture of Tuza, which concerns the relation-

ship between the minimum number of edges needed to cover all triangles in a graph G and the maximum number of edge-disjoint triangles in G. For $r \geq 3$, a K_r-*cover* in a graph is a set of edges meeting all copies of K_r, that is, the removal of a K_r-cover results in a K_r-free graph. A K_r-*packing* in a graph is a set of pairwise edge-disjoint copies of K_r. The K_r-*covering number* of a graph G, denoted by $\tau_r(G)$, is the minimum size of a K_r-cover of G and the K_r-*packing number* of G, denoted by $\nu_r(G)$, is the maximum size of a K_r-packing of G.

One can easily observe that

$$\nu_3(G) \leq \tau_3(G) \leq 3\nu_3(G). \tag{2.1}$$

In 1981, Tuza [20] conjectured that the second inequality of (2.1) is not optimal.

Conjecture 2.4. [20] *For every graph G, we have $\tau_3(G) \leq 2\nu_3(G)$.*

Conjecture 2.4 remains open, and many partial results have been proved. By using results of Krivelevich [13], and Haxell and Rödl [11], Yuster [21] proved the following theorem with states that, asymptotically, Tuza's conjecture holds, and he also extended the result to larger cliques.

Theorem 2.5. [13, 11, 21] *Let G be a graph on n vertices. Then,*

(i) $\tau_3(G) \leq 2\nu_3(G) + o(n^2)$;

(ii) $\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G) + o(n^2)$, *for* $r \geq 4$.

Next, we recall the following result of Győri [9, 10] about the existence of edge-disjoint copies of K_r in graphs on n vertices with more than $t_{r-1}(n)$ edges.

Theorem 2.6. [9, 10] *Let $r \geq 3$, and G be a graph on n vertices, with $e(G) = t_{r-1}(n) + m$, where $m = o(n^2)$. Then G contains $m - O(\frac{m^2}{n^2}) = (1 - o(1))m$ edge-disjoint copies of K_r.*

Finally, we recall that the *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum number of colours needed to colour the edges of G such that no two adjacent edges have the same colour. The chromatic index of complete graphs is well-known and we have the following result.

Theorem 2.7 ([3], Ch. V.2)**.** *Let $s \geq 2$. We have*

$$\chi'(K_s) = \begin{cases} s & \text{if } s \text{ is odd,} \\ s-1 & \text{if } s \text{ is even.} \end{cases}$$

Hence, for any graph G on n vertices, the edge set of G can be partitioned into at most n matchings.

We are now able to prove Theorems 1.5 and 1.7.

Proof of Theorem 1.5. The lower bound of (1.1) was proved in Lemma 1.4. We have to prove the upper bound. Let $k \geq 2$ be fixed, let $\varepsilon > 0$ be arbitrary and let n_0 be sufficiently large. Let G be a graph on $n \geq n_0$ vertices with its edges k-coloured with colours $1, \ldots, k$. We will show that G admits a monochromatic K_3-decomposition with at most $t_{R_k(3)-1}(n) + \varepsilon n^2$ parts.

Let $e(G) = t_{R_k(3)-1}(n) + \varepsilon n^2 + m$, where m is an integer. If $m \leq 0$ then G can be decomposed into single edges and we are done.

Suppose that $m > 0$. Observe that it suffices to show that we can find at least $\frac{m}{2}$ edge-disjoint monochromatic copies of K_3, since then G admits a monochromatic K_3-decomposition with at most $e(G) - 2 \cdot \frac{m}{2} = t_{R_k(3)-1}(n) + \varepsilon n^2$ parts, as required. Therefore, and in order to get a contradiction, assume that the maximum number of edge-disjoint monochromatic copies of K_3 in our graph G is at most $\frac{m}{2}$. For $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices, containing all the edges in colour i. Our assumption implies that $\sum_{i=1}^{k} \nu_3(G_i) \leq \frac{m}{2}$. By Theorem 2.5, we have $\tau_3(G_i) \leq 2\nu_3(G_i) + \frac{\varepsilon}{2k} n^2$ for every $1 \leq i \leq k$. Therefore, we have

$$\sum_{i=1}^{k} \tau_3(G_i) \leq \sum_{i=1}^{k} \left(2\nu_3(G_i) + \frac{\varepsilon}{2k} n^2\right) \leq m + \frac{\varepsilon}{2} n^2.$$

That is, by deleting at most $m + \frac{\varepsilon}{2} n^2$ edges from G, we obtain a subgraph $F \subset G$ which does not contain a monochromatic copy of K_3. On the other hand, we have $e(F) \geq t_{R_k(3)-1}(n) + \frac{\varepsilon}{2} n^2 > t_{R_k(3)-1}(n)$. Turán's Theorem implies that F must contain $K_{R_k(3)}$ as a subgraph and hence F contains a monochromatic copy of K_3. We have a contradiction, and the upper bound of Theorem 1.5 follows. $\square$

Proof of Theorem 1.7. The lower bound of (1.2) was proved in Lemma 1.4. It remains to prove the upper bound. Throughout, let $k \in \{2, 3\}$. Let n_0 be sufficiently large to satisfy all the inequalities we will encounter. Let $n \geq n_0$ and let G be a k-edge-coloured graph on n vertices. For the sake of simplicity we assume that the colours used are always red and blue if $k = 2$ and red, blue and green if $k = 3$. We want to show that G admits a monochromatic K_3-decomposition with at most $t_{R_k(3)-1}(n)$ parts, with equality if and only if $G = T_{R_k(3)-1}(n)$, and G is a blow-up of the 2-edge-colouring f_2 (for $k = 2$), or of the 3-edge-colouring f_3 or f_3' (for $k = 3$). Let $e(G) = t_{R_k(3)-1}(n) + m$, where m is an integer.

If $m < 0$, we can decompose G into single edges and there is nothing to prove.

If $m = 0$ and G contains a monochromatic copy of K_3, then G admits a monochromatic K_3-decomposition with at most $t_{R_k(3)-1}(n) - 2$ parts and we are done. Otherwise, if G does not contain a monochromatic K_3, then G does not contain a copy of $K_{R_k(3)}$. Thus, Turán's Theorem implies that $G = T_{R_k(3)-1}(n)$. By Corollary 2.3, G is a blow-up of f_2 (for $k = 2$), or of f_3 or f_3' (for $k = 3$). In each case, the only monochromatic K_3-decomposition of G has exactly $t_{R_k(3)-1}(n)$ parts, each part being a single edge.

Now, let $m > 0$. As before, it suffices to prove that G contains more than $\frac{m}{2}$ edge-disjoint monochromatic copies of K_3.

If $m = o(n^2)$, then by Theorem 2.6 with $r = R_k(3)$, G contains $(1 - o(1))m > \frac{m}{2}$ edge-disjoint copies of $K_{R_k(3)}$. Since each $K_{R_k(3)}$ contains a monochromatic copy of K_3, this implies that G contains more than $\frac{m}{2}$ edge-disjoint monochromatic copies of K_3.

Finally, assume that $m \geq Cn^2$, for some constant $C > 0$. In order to get a contradiction, suppose that the maximum number of edge-disjoint monochromatic copies of K_3 in G is at most $\frac{m}{2}$. Let G_1 and G_2 be the subgraphs of G on n vertices, containing all the red and blue edges, and in addition for $k = 3$, let G_3 be the analogous green subgraph. By Theorem 2.5, our assumption implies that

$$\sum_{p=1}^{k} \tau_3(G_p) \leq \sum_{p=1}^{k} 2\nu_3(G_p) + o(n^2) \leq m + o(n^2).$$

That is, by deleting at most $m + o(n^2)$ edges from G, we obtain a subgraph

$F \subset G$ which does not contain a monochromatic copy of K_3. Note that we must delete precisely $m + o(n^2)$ edges, otherwise we would have $e(F) > t_{R_k(3)-1}(n)$, so Turán's Theorem implies that F contains $K_{R_k(3)}$ as a subgraph, which contains a monochromatic copy of K_3, a contradiction.

Hence, we have $e(F) = t_{R_k(3)-1}(n) + o(n^2) = \frac{1}{2}(1 - \frac{1}{R_k(3)-1} + o(1))n^2$. By Proposition 2.2, F contains an $(R_k(3) - 1)$-partite subgraph F' with partitition classes $V_1, \ldots, V_{R_k(3)-1}$ such that

(a) $|V_i| = (\frac{1}{R_k(3)-1} + o(1))n$ for every $1 \leq i \leq R_k(3) - 1$;

(b) $e(F') = \frac{1}{2}(1 - \frac{1}{R_k(3)-1} + o(1))n^2$;

(c) The k-edge-colouring on F' is a blow-up of the 2-edge-colouring f_2 on K_5 if $k = 2$, and of the 3-edge-colouring f_3 or f_3' on K_{16} if $k = 3$.

Note that properties (a) and (b) imply:

(d) For every $1 \leq i \neq j \leq R_k(3) - 1$, there are $|V_i||V_j| - o(n^2)$ edges between V_i and V_j in F'.

Now, restore the deleted edges which lie inside $V_1, \ldots, V_{R_k(3)-1}$ to obtain the subgraph $G' \subset G$, and note that there are $m - o(n^2)$ such edges. Let $W_1, \ldots, W_{R_k(3)-1}$ be a relabelling of $V_1, \ldots, V_{R_k(3)-1}$ such that for every $1 \leq i \leq R_k(3) - 1$, all edges between W_i and W_{i+1} in F' are red (indices taken cyclically here and throughout). This can clearly be done, since by property (c) and Theorem 1.6 (for $k = 3$), F' is a blow-up of f_2 (for $k = 2$), and of f_3 or f_3' (for $k = 3$), so that in each case, each colour contains a Hamilton cycle in $K_{R_k(3)-1}$. Let $1 \leq i \leq R_k(3) - 1$. By Theorem 2.7, we can partition the restored red edges in W_i into at most $|W_i|$ matchings. By property (a), we can disregard $o(n)$ matchings to get t remaining matchings $M_1, \ldots, M_t$ in W_i, where $t \leq |W_{i+1}|$. For each matching M_j, we associate M_j with a unique vertex $x_j \in W_{i+1}$, so that $x_j \neq x_{j'}$ for different matchings $M_j, M_{j'}$. Let $E_{i,1} = M_1 \cup \cdots \cup M_t$. By property (d), $E_{i,1}$ and $\{x_1, \ldots, x_t\}$ induce $|E_{i,1}| - o(n^2)$ red copies of K_3, where each red K_3 has one edge in some M_j, and the third vertex is x_j. Applying this procedure for every $1 \leq i \leq R_k(3) - 1$, we have $\sum_{i=1}^{R_k(3)-1} |E_{i,1}| - o(n^2)$ red copies of K_3. These red copies of K_3 are edge-disjoint, since if T and T' are distinct such

copies, then T has two vertices in W_ℓ and one vertex in $W_{\ell+1}$, and T' has two vertices in $W_{\ell'}$ and one vertex in $W_{\ell'+1}$, for some $1 \leq \ell, \ell' \leq R_k(3) - 1$. Clearly, T and T' are edge-disjoint if $\ell \neq \ell'$, or if $\ell = \ell'$ with T and T' not sharing a vertex in W_ℓ. If $\ell = \ell'$ with T and T' sharing a vertex in W_ℓ, then their vertices in $W_{\ell+1}$ are distinct, so that T and T' are again edge-disjoint.

We repeat this whole procedure for the blue edges, and also for the green edges when $k = 3$, where on each occasion, we use a similar but different relabelling of $V_1, \ldots, V_{R_k(3)-1}$. For the blue edges, let $E_{i,2}$ be the similarly obtained sets of blue edges, and in addition, for $k = 3$ and the green edges, let $E_{i,3}$ be the similarly obtained sets of green edges. It follows that we have

$$\sum_{p=1}^{k} \sum_{i=1}^{R_k(3)-1} |E_{i,p}| - o(n^2) = (m - o(n^2) - o(n^2)) - o(n^2) > \frac{m}{2}$$

edge-disjoint monochromatic copies of K_3 in $G' \subset G$. This is a contradiction and the proof is completed. $\square$

3 Monochromatic K_r-decompositions

In this section we will study monochromatic K_r-decompositions for larger cliques and we will prove Theorem 1.9. Throughout this section we fix $k \geq 2$ and $r \geq 4$.

Proof of Theorem 1.9. The lower bound of (1.3) was proved in Lemma 1.4. Let us now prove the upper bound. Let $n_0 = n_0(r, k)$ be sufficiently large to satisfy all the inequalities we will encounter. Let $n \geq n_0$ and let G be any k-edge-coloured graph on n vertices. For the sake of simplicity, let $R = R_k(r)$. We will show that $\phi_k(G, K_r) \leq t_{R-1}(n)$ with equality if and only if $G = T_{R-1}(n)$.

Let $e(G) = t_{R-1}(n) + m$, where m is an integer. If $m < 0$, we can decompose G into single edges and there is nothing to prove. If $m = 0$ and G contains a monochromatic copy of K_r then G admits a monochromatic K_r-decomposition with at most $t_{R-1}(n) - \binom{r}{2} + 1$ parts and we are done. If G does not contain a monochromatic K_r, then the definition of the Ramsey number implies that G does not contain a copy of K_R. Therefore, $G = T_{R-1}(n)$ by Turán's Theorem. Now, let $m > 0$ and let ℓ be the

maximum number of edge-disjoint monochromatic copies of K_r in G. If $\ell > \frac{m}{\binom{r}{2}-1}$, then

$$\phi_k(G, K_r) \leq \ell + e(G) - \binom{r}{2}\ell < t_{R-1}(n).$$

Therefore, it suffices to show that $\ell > \frac{m}{\binom{r}{2}-1}$.

Consider first the case $m = o(n^2)$. By Theorem 2.6 with $r = R$, the graph G contains $(1-o(1))m$ edge-disjoint copies of K_R. Since each K_R contains a monochromatic copy of K_r, this implies that $\ell > \frac{m}{\binom{r}{2}-1}$ and we are done.

Finally, assume that $m \geq Cn^2$, for some constant $C > 0$. In order to get a contradiction, suppose that $\ell \leq \frac{m}{\binom{r}{2}-1}$. For $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices that contains all edges coloured with colour i. By Theorem 2.5, our assumption implies that

$$\begin{aligned}
\sum_{i=1}^{k} \tau_r(G_i) &\leq \sum_{i=1}^{k} \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G_i) + o(n^2) \\
&\leq \left\lfloor \frac{r^2}{4} \right\rfloor \ell + o(n^2) \\
&\leq \left\lfloor \frac{r^2}{4} \right\rfloor \frac{m}{\binom{r}{2} - 1} + o(n^2) \\
&\leq \frac{4}{5}m + o(n^2), \text{ since } r \geq 4.
\end{aligned}$$

That is, by deleting at most $\frac{4}{5}m + o(n^2)$ edges from G, we obtain a subgraph G' that does not contain a monochromatic copy of K_r. But

$$e(G') \geq e(G) - \frac{4}{5}m - o(n^2) \geq t_{R-1}(n) + \frac{1}{5}m - o(n^2) > t_{R-1}(n).$$

Therefore, Turán's Theorem implies that G' must contain a copy of K_R which contains a monochromatic copy of K_r. This is a contradiction and our proof is complete.

□

4 Acknowledgments

The authors acknowledge the support from FCT - Fundação para a Ciência e a Tecnologia (Portugal), through the Projects PTDC/MAT/113207/2009 and PEst-

OE/MAT/UI0297/2011 (CMA). The authors would like to thank the referees for helpful suggestions which improved the presentation of the paper.

References

[1] B. Bollobás. On complete subgraphs of different orders. *Math. Proc. Cambridge Philos. Soc.*, 79(1):19–24, 1976.

[2] B. Bollobás. *Extremal Graph Theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[3] B. Bollobás. *Modern Graph Theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[4] D. Dor and M. Tarsi. Graph decomposition is NP-complete: a complete proof of Holyer's conjecture. *SIAM J. Comput.*, 26(4):1166–1187, 1997.

[5] P. Erdős, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canad. J. Math.*, 18:106–112, 1966.

[6] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966.

[7] P. Erdös and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[8] R. E. Greenwood and A. M. Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, 7:1–7, 1955.

[9] E. Győri. On the number of edge-disjoint triangles in graphs of given size. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 267–276. North-Holland, Amsterdam, 1988.

[10] E. Győri. On the number of edge disjoint cliques in graphs of given size. *Combinatorica*, 11(3):231–243, 1991.

[11] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[12] J. G. Kalbfleisch and R. G. Stanton. On the maximal triangle-free edge-chromatic graphs in three colors. *J. Combin. Theory*, 5:9–20, 1968.

[13] M. Krivelevich. On a conjecture of Tuza about packing and covering of triangles. *Discrete Math.*, 142(1-3):281–286, 1995.

[14] L. Özkahya and Y. Person. Minimum H-decompositions of graphs: edge-critical case. *J. Combin. Theory Ser. B*, 102(3):715–725, 2012.

[15] O. Pikhurko and T. Sousa. Minimum H-decompositions of graphs. *J. Combin. Theory Ser. B*, 97(6):1041–1055, 2007.

[16] T. Sousa. Decompositions of graphs into 5-cycles and other small graphs. *Electron. J. Combin.*, 12:Research Paper 49, 7 pp. (electronic), 2005.

[17] T. Sousa. Decompositions of graphs into a given clique-extension. *Ars Combin.*, 100:465–472, 2011.

[18] T. Sousa. Decompositions of graphs into cycles of length seven and single edges. *Ars Combin.*, to appear.

[19] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[20] Zs. Tuza. In *Finite and Infinite Sets*, volume 37 of *Colloquia Mathematica Societatis János Bolyai*, page 888. North-Holland Publishing Co., Amsterdam, 1984.

[21] R. Yuster. Dense graphs with a large triangle cover have a large triangle packing. *Combin. Probab. Comput.*, 21:952–962, 2012.