Decompositions of graphs into 5-cycles and other small graphs

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Abstract

In this paper we consider the problem of finding the smallest number q such that any graph G of order n admits a decomposition into edge disjoint copies of a fixed graph H and single edges with at most q elements. We solve the case when H is the 5-cycle, the 5-cycle with a chord and any connected non-bipartite non-complete graph of order 4.

1 Introduction

Let G be a simple graph with vertex set V and edge set E. The number of vertices of a graph is its *order*. The *degree of a vertex* v is the number of edges that contain v and will be denoted by $\deg_G v$ or simply by $\deg v$. For $A \subseteq V$, $\deg(v, A)$ denotes the number of neighbors of v in the set A. The set of neighbors of v is denoted by $N_G(v)$ or briefly by $N(v)$ if it is clear which graph is being considered. Let $\overline{N}_G(v) = V - (N_G(v) \cup \{v\})$. The *complete bipartite graph* with parts of size m and n will be denoted by $K_{m,n}$ and the *cycle on n vertices* will be denoted by C_n. The *chromatic number* of G is denoted by $\chi(G)$.

Let $\mathscr{H}$ be a family of graphs. An *$\mathscr{H}$-decomposition* of G is a set of subgraphs $G_1, \ldots, G_t$ such that any edge of G is an edge of exactly one of $G_1, \ldots, G_t$ and all $G_1, \ldots, G_t \in \mathscr{H}$. Let $\phi(G, \mathscr{H})$ denote the minimum size of an $\mathscr{H}$-decomposition of G. The main problem related to $\mathscr{H}$-decompositions is the one of finding the smallest number $\phi(n, \mathscr{H})$ such that every graph G of order n admits an $\mathscr{H}$-decomposition with

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at most $\phi(n, \mathscr{H})$ elements. Here we address this problem for the special case where $\mathscr{H}$ consists of a fixed graph H and the single edge graph.

Let H be a graph with m edges and let $\mathrm{ex}(n, H)$ denote the maximum number of edges that a graph of order n can have without containing a copy of H. Then

$$\mathrm{ex}(n, H) \leq \phi(n, \mathscr{H}) \leq \frac{1}{m}\left(\binom{n}{2} - \mathrm{ex}(n, H)\right) + \mathrm{ex}(n, H).$$

Moreover, for the complete graph on n vertices, K_n, we have $\phi(K_n, \mathscr{H}) \geq \frac{1}{m}\binom{n}{2}$.

A theorem of Kövari, Sós and Turán [6] asserts that for the complete bipartite graph $K_{m,m}$, $\mathrm{ex}(n, K_{m,m}) = o(n^2)$. Therefore the decomposition problem into any fixed bipartite graph and singles edges is asymptotically solved and we have the following theorem.

Theorem 1.1. *Let H be a bipartite graph with m edges. Then*

$$\phi(n, \mathscr{H}) = \left(\frac{1}{m} + o(1)\right)\binom{n}{2}.$$

Suppose now, that H is a graph with chromatic number r, where $r \geq 3$.

The unique complete r-partite graph on n vertices whose partition sets differ in size by at most 1 is called the *Turán graph*; we denoted it by $T_r(n)$ and its number of edges by $t_r(n)$. Then $\phi(n, \mathscr{H}) \geq t_{r-1}(n) \geq \left(1 - \frac{1}{r-1}\right)\binom{n}{2}$, since $T_{r-1}(n)$ does not contain any copy of H. In fact we believe that this result is asymptotically correct. We conjecture the following.

Conjecture 1. *Let H be a graph with $\chi(H) \geq 3$. Then*

$$\phi(n, \mathscr{H}) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

Erdös, Goodman and Pósa [4] showed that the edges of any graph on n vertices can be decomposed into at most $\lfloor n^2/4 \rfloor$ triangles and single edges. Later Bollobás [1] generalized this result by showing that a graph of order n can be decomposed into at most $t_{r-1}(n)$ edge disjoint cliques of order r ($r \geq 3$) and edges.

In this paper we will prove similar results to the ones obtained by Erdös, Goodman and Pósa and by Bollobás for some special cases of graphs H of order 4 and 5 with chromatic number 3, namely C_5, C_5 with a chord and the two connected non-bipartite non-complete graphs on 4 vertices. The ideas involved in the proofs were inspired by the ideas developed by Erdös, Goodman and Pósa [4] and Bollobás [1].

2 Decompositions

Let $\mathscr{H}$ consist of a fixed graph H and the single edge graph. In this section we will study $\mathscr{H}$-decompositions for some fixed H. In all cases considered here the exact value of the function $\phi(n, \mathscr{H})$ will also be obtained.

The first case that we consider is $H = C_5$. In this case we can prove that any graph of order n, where $n \geq 6$, can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ copies of C_5 and single edges. Furthermore, the graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ shows that this result is, in fact, best possible. In the special case where our graph has order $n = 5$ we can find a graph with no copy of C_5 having 7 edges. In a similar way will also show that the above claim still holds if instead of C_5 we take H to be C_5 with a chord. This section will be concluded with similar results for the case where H is any connected non-bipartite non-complete graph on 4 vertices.

Theorem 2.2. *Any graph of order n, with $n \geq 6$, can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ copies of C_5 and single edges. Moreover, the bound is tight for $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Proof. This is by induction on the number of vertices in a graph. By inspection, and using Harary's [5] atlas of all graphs of order at most 6, we can see that the result holds for $n = 6$. Assume that it is true for all graphs of order less than n and note that for any positive integer n

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor .$$

Let G be a graph of order n, where $n \geq 7$, and let v be a vertex of minimum degree. If $\deg v \leq \lfloor \frac{n}{2} \rfloor$ then going from $G - v$ to G we only need to use the edges joining v to the other vertices of G and there are at most $\lfloor \frac{n}{2} \rfloor$ of these, so the induction hypothesis implies the result.

Assume that $\deg v > \lfloor \frac{n}{2} \rfloor$ and let $\deg v = d + m$ where $d = \lfloor \frac{n}{2} \rfloor$ and $m \geq 1$. Suppose that there are m edge disjoint C_5's containing v, so the $d + m$ edges incident with v can be decomposed into at most $m + (d + m - 2m) = d$ edge disjoint C_5's and edges, so the induction hypothesis implies the result.

To complete the proof, it remains to show that we can always find m edge disjoint C_5's containing vertex v.

Assume first that G is not the complete graph and let $x \in N(v)$ and $y \in \overline{N}(v)$. We have

$$\begin{aligned} \deg(x, N(v)) &\geq 2m - 1 \\ \deg(y, N(v)) &\geq 2m + 1. \end{aligned} \tag{2.1}$$

Let $x_1, \ldots x_m, z_1, \ldots, z_{m+1} \in N(y) \cap N(v)$ and let

$$X = \{x_1, \ldots x_m\} \text{ and } Y = N(v) - X.$$

Using (2.1) it is easy to see that $G[X, Y]$ has an X-perfect matching. Let $M = \{x_i, v_i\}_{i=1,\ldots,m}$ be an X-perfect matching such that $|\{v_1, \ldots, v_m\} \cap \{z_1, \ldots, z_{m+1}\}|$ is minimized. If $\{v_1, \ldots, v_m\} \cap \{z_1, \ldots, z_{m+1}\} = \emptyset$, then v, v_i, x_i, y, z_i, v, where $i = 1, \ldots, m$, are m edge disjoint C_5's containing v, and we are done.

Assume that $|\{v_1, \ldots, v_m\} \cap \{z_1, \ldots, z_{m+1}\}| = k$, for some $1 \leq k \leq m$, so say $v_i = z_i$ for $i = 1, \ldots, k$. As before, v, v_i, x_i, y, z_i, v, for $i = k+1, \ldots, m$, are $m - k$ edge disjoint C_5's containing v; hence it remains to show that we can find k other edge disjoint C_5's containing v.

Our choice of M implies that, for $i = 1, \ldots, k$, $N(x_i) \cap N(v) \subseteq N(y) \cup V = V \cup X \cup Z$, where

$$V = \{v_{k+1}, \ldots, v_m\} \text{ and } Z = \{z_1, \ldots, z_{m+1}\}.$$

(a) If $k = 1$ then $v, z_1, x_1, y, z_{m+1}, v$ is a 5-cycle and we are done.
(b) If $k = 2, 3$ then for $i = 1, 2$ we have $\deg(x_i; X \cup \{z_3, \ldots, z_{m+1}\} \cup V) \geq 2m - 3$ and $|(X - \{x_i\}) \cup \{z_3, \ldots, z_{m+1}\} \cup V| = 3m - 2 - k$. Then x_1 is adjacent to x_2 or they must have a common neighbor, say a, in $(X - \{x_1, x_2\}) \cup \{z_3, \ldots, z_{m+1}\} \cup V$. Figure 1 shows that we can always find k edge disjoint C_5's containing v.

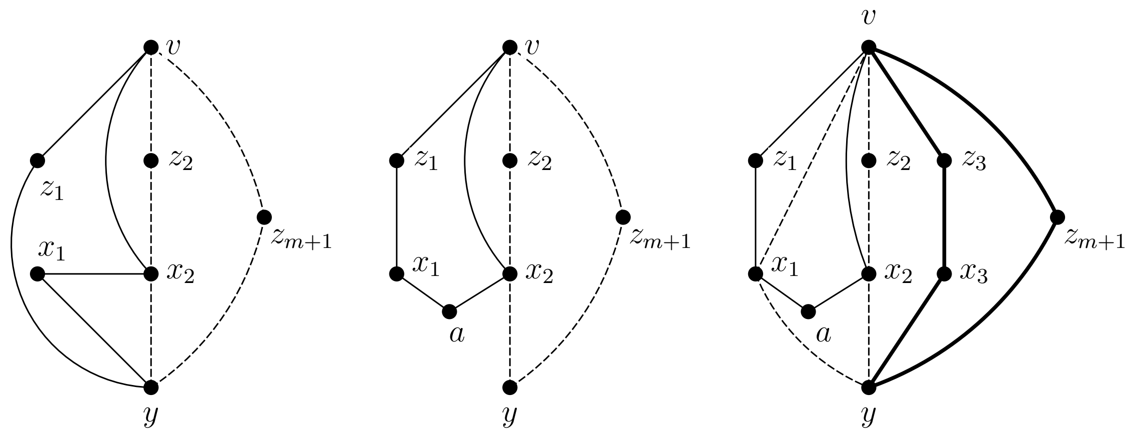


Figure 1: Case $k = 2, 3$

(c) Let $k \geq 4$ and let

$$X' = X - \{x_1, x_2, x_3\} \text{ and } Z' = Z - \{z_1, z_2, z_3\}.$$

For $k = 4$ and $i = 1, 2, 3$ we have $\deg(x_i; V \cup X' \cup Z') \geq 2m - 6$ and $|V \cup X' \cup Z'| = 3m - 9$. Then there exist $a, b \in V \cup X' \cup Z'$ with $a \neq b$ such that a is adjacent to x_1 and x_2 and b is adjacent to x_1 and x_3 or a is adjacent to x_1 and x_2 and b is adjacent to x_2 and x_3.

Assume that $k \geq 5$. Then for $i = 1, 2, 3$, $\deg(x_i, V \cup Z') \geq m - 3$, and $|V \cup Z'| = 2m - k - 2$. Thus there exist $a, b \in V \cup Z'$ with $a \neq b$ such that a is adjacent to x_1 and x_2 and b is adjacent to x_1 and x_3 or a is adjacent to x_2 and x_3 and b is adjacent to x_1 and x_3. Without loss of generality assume the first case holds in both situations (the second follows from symmetry). Then Figure 2 shows that we can always find three edge disjoint C_5's containing vertex v.

We repeat this procedure for every triple x_i, x_{i+1}, x_{i+2}, where $i \equiv 1 \pmod 3$, $i + 2 \leq k$ and $Z' = Z - \{z_i, z_{i+1}, z_{i+2}\}$.

If $k \equiv 0 \pmod 3$ then we are done, since we can find k edge disjoint C_5's containing v.

If $k \equiv 1 \pmod 3$ then we can find $k - 1$ C_5's as before that with $v, z_k, x_k, y, z_{m+1}, v$ form the required number of C_5's needed.

If $k \equiv 2 \pmod 3$ then x_{k-1} and x_k have a common neighbor in $V \cup (Z - \{z_{k-1}, z_k\})$, say a. Therefore, the $k - 2$ C_5's found so far, together with $v, z_{k-1}, x_{k-1}, a, x_k, v$ and $v, z_k, x_k, y, z_{m+1}, v$, give the required number of C_5's needed.

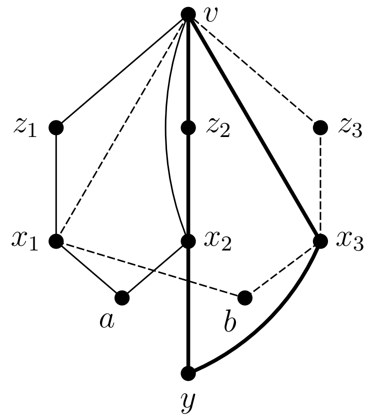


Figure 2: Case $k \geq 4$

Now suppose that $G = K_n$ and let vertices v and y be fixed. An argument similar to the one described in case (c) gives the required number of edge disjoint C_5's incident with v. Alternatively, using [7] we can find the exact number of edge disjoint C_5's in K_n and then see that the theorem holds. $\square$

Suppose that instead of a 5-cycle we consider decompositions of graphs into copies of H and single edges, where H is a 5-cycle with a chord. Using the same argument we can prove the following result.

Theorem 2.3. *Any graph of order n, with $n \geq 6$, can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ copies of H and single edges. This bound is best possible for $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Proof. We proceed as in the proof of Theorem 2.2 and will only describe the steps that are different.

If $\{v_1, \ldots, v_m\} \cap \{z_1, \ldots, z_{m+1}\} = \emptyset$, then v, v_i, x_i, y, z_i, v, where $i = 1, \ldots, m$, induce m edge disjoint copies of H containing v, and we are done.

Assume that $|\{v_1, \ldots, v_m\} \cap \{z_1, \ldots, z_{m+1}\}| = k$, for some $1 \leq k \leq m$, say $v_i = z_i$ for $i = 1, \ldots, k$. As before, v, v_i, x_i, y, z_i, v, for $i = k+1, \ldots, m$, induce $m - k$ edge disjoint copies of H containing v. For every triple x_i, x_{i+1}, x_{i+2} where $i \equiv 1 \pmod 3$ and $i+2 \leq k$, Figure 3 shows that we can always find two edge disjoint copies of H. So in total we have $2\lfloor \frac{k}{3} \rfloor$ copies of H.

Therefore, for $k \equiv 0 \pmod 3$ v is in at least $m - k + 2\lfloor \frac{k}{3} \rfloor$ edge disjoint copies of H, so we are left with at most $d + m - 3(m - k + 2\lfloor \frac{k}{3} \rfloor)$ single edges incident with v. Consequently, the edges incident with v can be decomposed with at most $m - k + 2\left\lfloor \frac{k}{3} \right\rfloor + d + m - 3\left(m - k + 2\left\lfloor \frac{k}{3} \right\rfloor\right) < d$ edge disjoint copies of H and single edges. Let $k \equiv 1, 2 \pmod 3$ and assume $m \geq 2$. The vertices $v, z_k, x_k, y, z_{m+1}, v$ induce another copy of H. So, in total, the $d + m$ edges incident with v can be decomposed into at most $m - k + 2\left\lfloor \frac{k}{3} \right\rfloor + 1 + d + m - 3\left(m - k + 2\left\lfloor \frac{k}{3} \right\rfloor + 1\right) \leq d$ edge disjoint copies of H and edges. If $m = 1$ then we can easily find a copy of H and the proof is complete. $\square$

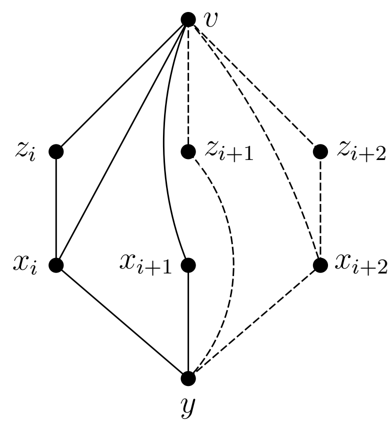
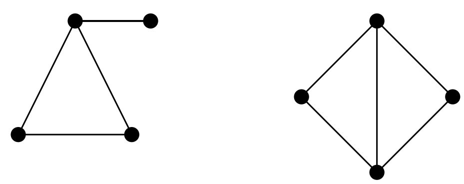


Figure 3: 2 copies of H

We conclude with the following result on decompositions of graphs into connected non-bipartite non-complete graphs of order 4 and single edges. Let H be one of the following graphs.



Theorem 2.4. *Any graph of order n, with $n \geq 4$, can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ copies of H and single edges. Furthermore, the bound is sharp for $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

To prove the theorem we will need the following result .

Theorem 2.5. *[2] Let G be a graph of order n with minimum degree k. Then G contains a path of length k.*

Proof of Theorem 2.4. We proceed by induction on the number of vertices. The result clearly holds for every graph with 4 vertices. Let G be a graph of order n, where $n \geq 5$, and let v be a vertex of minimum degree. If $\deg v \leq \lfloor \frac{n}{2} \rfloor$ then the result follows by induction as before. Suppose that $\deg v > \lfloor \frac{n}{2} \rfloor$ and let $\deg v = d + m$ where $d = \lfloor \frac{n}{2} \rfloor$ and $m \geq 1$.

Assume first that $m \geq 2$ and let $G_v := G[N(v)]$. Since $\deg_{G_v} x \geq 2m - 1$ for every vertex of G_v, Theorem 2.5 implies that G_v contains a path of length $2m-1$, say P. Then every 3 vertices of P give rise to one copy of H, so the edges incident with v can be decomposed into at most $\lfloor \frac{2m}{3} \rfloor + (d+m-3\lfloor \frac{2m}{3} \rfloor) \leq d$ edge disjoint copies of H and single edges, so the result follows by induction.

To complete the proof it remains to show that for $m = 1$ we can always find a copy of H containing vertex v. If we can find a path of length 2 in $N(v)$ then we are done. If not then $N(v)$ contains only independent edges. Hence all vertices in $N(v)$ must be adjacent to all vertices in $\overline{N}(v)$. Let $\{a, b\}$ be an independent edge in $N(v)$ and let $y \in \overline{N}(v)$; then the vertices v, a, b, y induce a copy of H and we are done. $\square$

Remark: The graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ shows that the number $\lfloor \frac{n^2}{4} \rfloor$ mentioned in previous theorems is best possible. So $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal graph for these decompositions. However, we do not know if it is the only one.

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