On the Operator and Essential Norms of Fourier Convolution Operators and Wiener-Hopf Operators with the Same Symbol

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Abstract Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$, $1 < p < \infty$, $1 \le q < \infty$, and Φ be a Young's function satisfying the (δ_2, Δ_2)-condition and $\lim_{x \to 0^+} \frac{\Phi(x)}{x} = 0$. Suppose $X(\Omega)$ is the Lorentz space $L^{p,q}(\Omega)$ or the Orlicz space $L^{\Phi}(\Omega)$. We show that if a is a Fourier multiplier on $X(\mathbb{R})$, $W^0(a)$ is the corresponding Fourier convolution operator on $X(\mathbb{R})$, and $W(a)$ is the corresponding Wiener-Hopf operator on $X(\mathbb{R}_+)$, then the operator and essential norms of $W^0(a)$ on $X(\mathbb{R})$ and of $W(a)$ on $X(\mathbb{R}_+)$ are all the same.

1 Introduction and main result

For a Banach space $\mathcal{X}$, let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$ and $\mathcal{K}(\mathcal{X})$ denote set of all compact linear operators on $\mathcal{X}$. The operator norm of an operator $A \in \mathcal{B}(\mathcal{X})$ is denoted by $\|A\|_{\mathcal{B}(\mathcal{X})}$ and its essential norm is defined by

$$\|A\|_{\mathcal{B}(\mathcal{X}),\mathrm{e}} := \inf_{K \in \mathcal{K}(\mathcal{X})} \|A + K\|_{\mathcal{B}(\mathcal{X})}.$$

For $f \in L^1(\mathbb{R})$, let Ff denote the Fourier transform

$$(Ff)(\xi) := \int_{\mathbb{R}} f(x) e^{ix\xi}\, dx, \qquad \xi \in \mathbb{R}.$$

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If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $Ff \in L^2(\mathbb{R})$ and $\|Ff\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\|f\|_{L^2(\mathbb{R})}$. Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the operator F extends to a bounded linear operator of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$, which will also be denoted by F. The inverse of F is given by $(F^{-1}f)(x) = (2\pi)^{-1}(Ff)(-x)$ for a.e. $x \in \mathbb{R}$.

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space - their technical definitions will be postponed to Section 2.1. The class of all Banach function spaces is very broad: it includes Lebesgue spaces, Orlicz spaces, Lorentz spaces, and variable Lebesgue spaces. For our purposes, the separability of $X(\mathbb{R})$ will be necessary so that $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see, e.g., [6, Lemma 2.2]).

Let $X(\mathbb{R})$ be a separable Banach function space. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the operator

$$W^0(a)f := F^{-1}(a \cdot Ff),$$

maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The operator $W^0(a)$ is called Fourier convolution operator and the function a will also be referred to as the symbol of $W^0(a)$. The set $\mathcal{M}_X$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_X} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

Recall that the Hardy-Littlewood maximal function Mf of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is defined by

$$(Mf)(x) := \sup_{\mathcal{I} \ni x} \frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} |f(t)|\, dt,$$

where the supremum is taken over all bounded intervals $\mathcal{I} \subset \mathbb{R}$ that contain x and $|\mathcal{I}|$ denotes the length of $\mathcal{I}$. The Hardy-Littlewood maximal operator M is defined by the rule $f \mapsto Mf$. If M is bounded on a separable Banach function space $X(\mathbb{R})$ or on its associate space $X'(\mathbb{R})$, then $\mathcal{M}_X$ is a Banach algebra (see [6, Corollary 2.4]).

Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$. In this paper, we will consider one of the following Banach function spaces: a Lorentz space $L^{p,q}(\Omega)$ with $1 < p < \infty$ and $1 \le q < \infty$ or an Orlicz space $L^\Phi(\Omega)$ with the Young's function Φ satisfying the (δ_2, Δ_2)-condition and

$$\lim_{x \to 0^+} \frac{\Phi(x)}{x} = 0. \tag{1}$$

We refer to [1, Chap. 4], [11, Chap. II], [12, Chaps. 4 and 8] and also to Sections 2.4 and 2.5 for the definitions and basic properties of these Banach function spaces.

Consider the restriction operator r_+ from $X(\mathbb{R})$ into $X(\mathbb{R}_+)$, along with the zero extension operator ℓ_+ from $X(\mathbb{R}_+)$ into $X(\mathbb{R})$. Given $a \in \mathcal{M}_X$, the Wiener-Hopf operator with symbol a is defined by the formula

$$W(a) := r_+ W^0(a) \ell_+.$$

The aim of this paper is to extend basic equalities for the operator and essential norms of the operators $W^0(a)$ and $W(a)$ known for the case of Lebesgue spaces

$L^p(\Omega)$ (see [2, Section 9.5(a)–(b)] and [5, Proposition 2.2]) to the setting of Lorentz and Orlicz spaces.

Our main result is the following.

Theorem 1 *Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$, $1 < p < \infty$, $1 \leq q < \infty$, and Φ be a Young's function satisfying the (δ_2, Δ_2)-condition and* (1)*. Suppose $X(\Omega)$ is the Lorentz space $L^{p,q}(\Omega)$ or the Orlicz space $L^\Phi(\Omega)$. For every $a \in \mathcal{M}_X$, we have*

$$\left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R})),\mathrm{e}} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}}.$$

The paper is organized as follows. In Section 2, we recall definitions of rearrangement-invariant Banach function spaces and their Boyd indices. Further, we give definitions and recall some properties of Lorentz spaces and Orlicz spaces, being the most prominent examples of rearrangement-invariant Banach function spaces. In Section 3, we recall the notion of limit operators and compute the limit operators of multiplication by the characteristic function of $\mathbb{R}_+$ and of compact operators on rearrangement-invariant Banach function spaces. Armed with the results of Section 3,we prove Theorem 1 in Section 4 following some ideas borrowed from the monographs by Duduchava [5] and Böttcher and Silbermann [2].

2 Preliminaries

2.1 Banach function spaces

Let $\mathbb{R}_+ := (0, \infty)$ and $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$. The set of all Lebesgue measurable extended complex-valued functions on Ω is denoted by $\mathfrak{M}(\Omega)$. The subset of functions in $\mathfrak{M}(\Omega)$ whose values lie in $[0, \infty]$ will be denoted as $\mathfrak{M}^+(\Omega)$. The Lebesgue measure of a measurable set $E \subseteq \Omega$ is denoted by $|E|$ and its characteristic function by χ_E. Following [1, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\Omega) \to [0, \infty]$, is called a Banach function norm if, for all $f, g, f_n \in \mathfrak{M}^+(\Omega)$, $n \in \mathbb{N}$, for all constants $c \geq 0$, and for all measurable subsets E of Ω, the following properties hold:

$(A1)$ $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(cf) = c\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$;

$(A2)$ $0 \leq g \leq f$ a.e $\Rightarrow \rho(g) \leq \rho(f)$ (lattice or ideal property);

$(A3)$ $0 \leq f_n \uparrow f$ a.e $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (Fatou property);

$(A4)$ $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$;

$(A5)$ $|E| < \infty \Rightarrow \displaystyle\int_E f(x)\,dx \leq C\rho(f)$

for some constant $C = C(E, \rho) \in (0, \infty)$ depending on E and ρ but independent of f. Identifying functions that only differ on a set of measure zero, one defines a Banach function space as

$$X(\Omega) := \{f \in \mathfrak{M}(\Omega) : \rho(|f|) < \infty\}.$$

For each $f \in X(\Omega)$, the norm of f is defined by

$$\|f\|_{X(\Omega)} := \rho(|f|).$$

Under the natural linear space operations, $(X(\Omega), \|\cdot\|_{X(\Omega)})$ becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\Omega)$ by

$$\rho'(g) := \sup\left\{\int_\Omega f(x)g(x)\,dx : f \in \mathfrak{M}^+(\Omega),\ \rho(f) \le 1\right\}, \quad g \in \mathfrak{M}^+(\Omega).$$

By [1, Chap. 1, Theorem 2.2], ρ' is itself a Banach function norm. The Banach function space $X'(\Omega)$ determined by ρ' is called the associate space of $X(\Omega)$.

2.2 Rearrangement-invariant Banach function spaces

Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$. Let $\mathfrak{M}_0(\Omega)$ and $\mathfrak{M}_0^+(\Omega)$ be the classes of a.e. finite functions in $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$, respectively. The distribution function μ_f of $f \in \mathfrak{M}_0(\Omega)$ is given by

$$\mu_f(\lambda) := |\{x \in \Omega : |f(x)| > \lambda\}|, \quad \lambda \ge 0.$$

Two functions $f, g \in \mathfrak{M}_0(\Omega)$ are said to be equimeasurable if $\mu_f = \mu_g$. The non-increasing rearrangement of $f \in \mathfrak{M}_0(\Omega)$ is the function defined by

$$f^*(t) := \inf\left\{\lambda \ge 0 : \mu_f(\lambda) \le t\right\}, \quad t \ge 0,$$

where we use the standard convention that $\inf \emptyset = +\infty$.

A Banach function norm $\rho : \mathfrak{M}^+(\Omega) \to [0, \infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\Omega)$ we have $\rho(f) = \rho(g)$. In this case, the Banach function space generated by ρ is said to be rearrangement-invariant.

2.3 Boyd indices

Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$ and let $X(\Omega)$ be a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm ρ. By [1, Chap. 2, Theorem 4.10], there is a unique rearrangement-invariant Banach function norm $\overline{\rho}$ over the half-line $\mathbb{R}_+$ equipped with the Lebesgue measure, defined by

$$\overline{\rho}(h) := \sup\left\{\int_{\mathbb{R}_+} g^*(t)h^*(t)\,dt : g \in \mathfrak{M}^+(\mathbb{R}),\ \rho'(g) \le 1\right\}, \quad h \in \mathfrak{M}_0^+(\mathbb{R}_+),$$

and such that $\rho(f) = \overline{\rho}(f^*)$ for all $f \in \mathfrak{M}_0^+(\mathbb{R})$. The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{R}_+)$.

For each $t > 0$, let D_t denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+)$ as

$$(D_t f)(x) := f(tx), \quad x \in \mathbb{R}_+.$$

With $X(\Omega)$ and $\overline{X}(\mathbb{R}_+)$ as above, let $h_X(t)$ denote the operator norm of $D_{1/t}$ as an operator on $\overline{X}(\mathbb{R}_+)$. By [1, Chap. 3, Proposition 5.11], for each $t > 0$, the operator D_t is bounded on $\overline{X}(\mathbb{R}_+)$ and the function h_X is increasing and submultiplicative on $(0, \infty)$.

The Boyd indices of $X(\Omega)$ are the numbers α_X and β_X defined by

$$\alpha_X := \sup_{0<t<1} \frac{\ln h_X(t)}{\ln t}, \qquad \beta_X := \inf_{1<t<\infty} \frac{\ln h_X(t)}{\ln t}. \tag{2}$$

By [1, Chap. 3, Proposition 5.13],

$$0 \le \alpha_X \le \beta_X \le 1.$$

The following result is a consequence of [1, Chap. 3, Proposition 5.13, Theorem 5.17].

Theorem 2 *Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space and $X'(\mathbb{R})$ be its associate space.*

(a) *The operator M is bounded on $X(\mathbb{R})$ if and only if $\beta_X < 1$.*
(b) *The operator M is bounded on $X'(\mathbb{R})$ if and only if $0 < \alpha_X$.*

2.4 Lorentz spaces

Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$. For $0 < p, q \le \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all functions $f \in \mathfrak{M}_0(\Omega)$ such that the quantity

$$\|f\|_{L^{p,q}(\Omega)} := \begin{cases} \left(\displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & q = \infty, \end{cases}$$

is finite, where

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)\, ds, \quad t > 0.$$

According to [1, Chap. 4, Theorem 4.6], if $1 < p < \infty$ and $1 \le q \le \infty$, then $(L^{p,q}(\Omega), \|\cdot\|_{L^{p,q}(\Omega)})$ is a rearrangement-invariant Banach function space, with Boyd indices

$$\alpha_{L^{p,q}} = \beta_{L^{p,q}} = p^{-1}.$$

By [1, Chap 4, Theorem 4.7], the associate space of $L^{p,q}(\Omega)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\Omega)$, where $1/p + 1/p' = 1/q + 1/q' = 1$.

These observations and Theorem 2 imply that if $1 < p < \infty$ and $1 \le q \le \infty$, then the Hardy-Littlewood maximal operator M is bounded on the Lorentz space $L^{p,q}(\mathbb{R})$ and on its associate space.

By [1, Chap. 4, Corollary 4.8] (see also [12, Corollary 8.5.4]), the space $L^{p,q}(\Omega)$ is separable provided $1 < p < \infty$ and $1 \le q < \infty$. As a result of this, it follows that $L^{p,q}(\Omega)$ is reflexive if $1 < p, q < \infty$ (see [12, Corollary 8.5.5]). Note that both Corollaries 8.5.4 and 8.5.5 in [12] contain the incorrect condition $1 \le q \le \infty$, which should be replaced by $1 \le q < \infty$ and $1 < q < \infty$, respectively.

Combining the above observations with Theorem 2 and [6, Corollary 2.4], we arrive at the following.

Lemma 1 *If $1 < p < \infty$ and $1 \le q < \infty$, then $\mathcal{M}_{L^{p,q}}$ is a Banach algebra.*

2.5 Orlicz spaces

Following [1, Chap. 4, Section 8] and [11, Chap. II, Section I], let $\varphi : [0,\infty) \to [0,\infty]$ be increasing and left-continuous with $\varphi(0) = 0$. Suppose that φ is neither identically zero nor identically infinite on $(0,\infty)$. Let ψ denote the left continuous inverse of φ defined by

$$\psi(v) := \inf\{u \ge 0 : \varphi(u) \ge v\}, \quad v \ge 0.$$

The function ψ has the same properties as φ: it is increasing, left-continuous, vanishes at the origin, and is neither identically zero nor identically infinite on $(0,\infty)$. It is easy to verify that φ is the left-continuous inverse of ψ:

$$\varphi(u) = \inf\{v \ge 0 : \psi(v) \ge u\}, \quad u \ge 0.$$

The functions

$$\Phi(u) := \int_0^u \varphi(t)\,dt, \quad u \ge 0, \qquad \Psi(v) := \int_0^v \psi(t)\,dt, \quad v \ge 0,$$

are called complementary Young's functions.

Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$. For a Young's function Φ, the Orlicz space $L^\Phi(\Omega)$ consists of all functions $f \in \mathfrak{M}_0(\Omega)$ such that

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty,$$

for some $\lambda = \lambda(f) > 0$. By [1, Chap. 4, Theorem 8.9], the space $L^\Phi(\Omega)$ endowed with the norm

$$\|f\|_{L^\Phi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

is a rearrangement-invariant Banach function space. By [1, Chap. 4, Theorem 8.14, Corollary 8.15], the associate space of $L^\Phi(\Omega)$ is, up to equivalence of norms, the Orlicz space $L^\Psi(\Omega)$. More precisely,

$$\|g\|_{L^\Psi(\Omega)} \le \|g\|_{(L^\Phi)'(\Omega)} \le 2\|g\|_{L^\Psi(\Omega)}, \quad g \in L^\Psi(\Omega).$$

Let Φ^{-1} be the right-continuous inverse of Φ defined by

$$\Phi^{-1}(t) := \sup\{s \ge 0 : \Phi(s) \le t\}, \quad t \ge 0.$$

Then

$$h_{L^\Phi}(t) = \sup_{s\in(0,\infty)} \frac{\Phi^{-1}(s)}{\Phi^{-1}(s/t)}, \quad t \in (0,\infty) \tag{3}$$

(see [3, Theorems 5.3 and 5.5] and also [4, Theorem], [1, Chap. 4, Theorem 8.18]). Note that we use the definition $h_{L^\Phi}(t) = \|D_{1/t}\|_{\mathcal{B}(\overline{L^\Phi}(\mathbb{R}_+))}$ adapted in [1, Chaps. 3–4], on the other, hand D_t is used instead of $D_{1/t}$ in [3, 4].

The function h_{L^Φ} has the same form for $L^\Phi(\mathbb{R})$ and $L^\Phi(\mathbb{R}_+)$. Therefore the Boyd indices of $L^\Phi(\mathbb{R})$ and of $L^\Phi(\mathbb{R}_+)$ are the same and are defined by (2) with h_{L^Φ} given by (3).

Following [11, Chap. II, Section 1, Definition 3], a Young's function Φ is said to satisfy the (δ_2, Δ_2)-condition if there is a constant $m \in (0,\infty)$ such that $\Phi(2u) \le m\Phi(u)$ for all $u \in (0,\infty)$.

The following statement is well-known.

Lemma 2 *Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$ let Φ be a Young's function. The following statements are equivalent:*

(a) the Orlicz space $L^\Phi(\Omega)$ is separable;
(b) Φ satisfies the (δ_2, Δ_2)-condition;
(c) $0 < \alpha_{L^\Phi}$.

Proof. (a) $\Leftrightarrow$ (b) was proved in [11, Chap. II, Section 3, Theorem 5]. (b) $\Leftrightarrow$ (c) follows from [3, Lemmas 3.5–3.6 and 5.9]. □

As a consequence of Theorem 2, Lemma 2 and [6, Corollary 2.4], we get the following.

Lemma 3 *If Φ is a Young's function satisfying the (δ_2, Δ_2)-condition, then $\mathcal{M}_{L^\Phi}$ is a Banach algebra.*

3 Limit operators

3.1 Definitions and elementary properties

Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{X})$ is said to converge strongly to some operator T if for every $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \|T_n x - Tx\|_{\mathcal{X}} = 0.$$

In this case, we write $T_n \to T$ and we refer to the operator T as the strong limit of the sequence $(T_n)_{n\in\mathbb{N}}$ which we will denote by

$$T := \operatorname*{s-lim}_{n\to\infty} T_n.$$

Let $\mathcal{X}$ be a Banach space, $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of isometries on $\mathcal{X}$. If the sequence $(U_n^{-1} T U_n)_{n\in\mathbb{N}}$ converges strongly to some operator, then we define

$$(T)_{\mathcal{U}} := \operatorname*{s-lim}_{n\to\infty} U_n^{-1} T U_n,$$

which will be referred to as the limit operator of T with respect to $\mathcal{U}$.

Let us mention elementary properties of limit operators.

Proposition 1 ([10, Proposition 3.4], [13, Proposition 1.2.2]) *Let $\mathcal{X}$ be a Banach space and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of isometries on $\mathcal{X}$.*

(i) *For all $T \in \mathcal{B}(\mathcal{X})$, if $(T)_{\mathcal{U}}$ exists, then*

$$\|(T)_{\mathcal{U}}\|_{\mathcal{B}(\mathcal{X})} \le \|T\|_{\mathcal{B}(\mathcal{X})}.$$

(ii) *For all $T, S \in \mathcal{B}(\mathcal{X})$ and $\alpha \in \mathbb{C}$, if $(T)_{\mathcal{U}}$ and $(S)_{\mathcal{U}}$ exist, then*

$$(T+S)_{\mathcal{U}} = (T)_{\mathcal{U}} + (S)_{\mathcal{U}}, \quad (\alpha T)_{\mathcal{U}} = \alpha (T)_{\mathcal{U}}, \quad (TS)_{\mathcal{U}} = (T)_{\mathcal{U}} (S)_{\mathcal{U}}.$$

3.2 Limit operators of the operator of multiplication by the characteristic function of the positive half-line

Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space. For $f \in X(\mathbb{R})$ and $h \in \mathbb{R}$, consider the translation operator T_h defined by

$$(T_h f)(x) := f(x+h), \qquad x \in \mathbb{R}. \tag{4}$$

Since the functions f and $T_h f$ are equimeasurable, the translation operator T_h is an isometry on $X(\mathbb{R})$.

Lemma 4 *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space. Consider a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $\mathcal{T} := (T_{h_n})_{n\in\mathbb{N}}$. If $h_n \to -\infty$ as $n \to \infty$, then the limit operator of $\chi_{\mathbb{R}_+} I$ with respect to $\mathcal{T}$ is equal to I.*

Proof. Let $f \in C_c^\infty(\mathbb{R})$. For all $g \in X'(\mathbb{R})$ with $\|g\|_{X'(\mathbb{R})} \le 1$, we have

$$\begin{aligned}\int_{\mathbb{R}} \left| \left[\left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) f \right] (x) g(x) \right| dx &= \int_{\mathbb{R}} \left| T_{-h_n} \chi_{\mathbb{R}_+}(x) - 1 \right| \, |f(x) g(x)| \, dx \\ &\le \left\| T_{-h_n} \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f)} \int_{\mathbb{R}} |f(x) g(x)| \, dx \\ &\le \left\| T_{-h_n} \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f)} \|f\|_{X''(\mathbb{R})},\end{aligned}$$

from which we conclude that

$$\left\| \left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) f \right\|_{X''(\mathbb{R})} \le \left\| T_{-h_n} \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f)} \|f\|_{X''(\mathbb{R})}.$$

Employing the Lorentz-Luxemburg theorem (see [1, Chap. 1, Theorem 2.7]), it follows that

$$\left\| \left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) f \right\|_{X(\mathbb{R})} \le \left\| T_{-h_n} \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f)} \|f\|_{X(\mathbb{R})}.$$

Observe that for all $n \in \mathbb{N}$,

$$\left\| T_{-h_n} \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f)} = \left\| \chi_{\mathbb{R}_+} - 1 \right\|_{L^\infty(\operatorname{supp} f - h_n)}.$$

The hypothesis that $h_n \to -\infty$ implies that there exists some $N \in \mathbb{N}$ such that for all $n > N$,

$$\operatorname{supp} f - h_n \subseteq \overline{\mathbb{R}_+},$$

and hence $\|\chi_{\mathbb{R}_+} - 1\|_{L^\infty(\operatorname{supp} f - h_n)} = 0$. Therefore, for all $n > N$,

$$\left\| \left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) f \right\|_{X(\mathbb{R})} = 0.$$

This establishes that

$$\lim_{n\to\infty} \left\| \left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) f \right\|_{X(\mathbb{R})} = 0,$$

for every $f \in C_c^\infty(\mathbb{R})$. On account of [7, Lemma 2.12(i)], the separability of $X(\mathbb{R})$ implies that the set $C_c^\infty(\mathbb{R})$ is dense in $X(\mathbb{R})$. Combining this with what we have just proved, [14, Lemma 1.4.1] yields that

$$\operatorname*{s-lim}_{n\to\infty} \left(T_{h_n}^{-1} \chi_{\mathbb{R}_+} I\, T_{h_n} - I \right) = 0,$$

i.e., $(\chi_{\mathbb{R}_+} I)_{\mathcal{T}} = I$. □

3.3 Limit operators of a compact operator

Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space. following [1, Chap. 2, Definition 5.1], for each finite value of $t \geq 0$, let E be a measurable subset of $\mathbb{R}$ of measure t and let

$$\varphi_X(t) := \|\chi_E\|_{X(\mathbb{R})}, \quad t \geq 0.$$

The function so defined is called the fundamental function of $X(\mathbb{R})$.

The following lemma is a one-dimensional version of [8, Lemma 7].

Lemma 5 *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space such that $\varphi_X(t)/t \to 0$ as $t \to \infty$. Consider a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $\mathcal{T} := (T_{h_n})_{n\in\mathbb{N}}$. If K is a compact operator on $X(\mathbb{R})$ and $h_n \to \pm\infty$ as $n \to \infty$, then the limit operator of K with respect to $\mathcal{T}$ is equal to the zero operator.*

Let us specify the above result to the case of Lorentz and Orlicz spaces.

Corollary 1 *Let $1 < p < \infty$, $1 \leq q < \infty$ and let Φ be a Young's function satisfying the (δ_2, Δ_2)-condition and (1). Suppose $X(\mathbb{R})$ is the Lorentz space $L^{p,q}(\mathbb{R})$ or the Orlicz space $L^\Phi(\mathbb{R})$. Consider a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $\mathcal{T} := (T_{h_n})_{n\in\mathbb{N}}$. If K is a compact operator on $X(\mathbb{R})$ and $h_n \to \pm\infty$ as $n \to \infty$, then the limit operator of K with respect to $\mathcal{T}$ is equal to the zero operator.*

Proof. It is well-known that $\varphi_{L^{p,q}}(t) = t^{1/p}$ (see, e.g., [12, Proposition 8.4.1]). Hence

$$\lim_{t\to\infty} \frac{\varphi_{L^{p,q}}(t)}{t} = \lim_{t\to\infty} t^{1/p-1} = 0.$$

So, the hypotheses of Lemma 5 are satisfied for $X(\mathbb{R}) = L^{p,q}(\mathbb{R})$.

By [1, Chap. 4, Lemma 8.17], the fundamental function of $L^\Phi(\mathbb{R})$ is given by

$$\varphi_{L^\Phi}(t) = \frac{1}{\Phi^{-1}(1/t)}, \quad 0 < t < \infty.$$

It follows from the (δ_2, Δ_2)-condition that $\Phi(u) \in (0, \infty)$ for all $u \in (0, \infty)$. Then Φ is continuous and strictly increasing on $[0, \infty)$. Hence $x = \Phi^{-1}(s)$ if and only if $s = \Phi(x)$ for $s \in (0, \infty)$ (see [1, Chap. 4, formulas (8.26)–(8.27)]). Therefore

$$\lim_{t\to\infty} \frac{\varphi_{L^\Phi}(t)}{t} = \lim_{t\to\infty} \frac{1}{t\Phi^{-1}(1/t)} = \lim_{s\to 0^+} \frac{s}{\Phi^{-1}(s)} = \lim_{x\to 0^+} \frac{\Phi(x)}{x} = 0.$$

Thus, the hypotheses of Lemma 5 are satisfied for $X(\mathbb{R}) = L^\Phi(\mathbb{R})$. It remains to apply Lemma 5. $\square$

4 Proof of the main result

4.1 Equality of the operator and essential norms of the Fourier convolution operator

In this section, we prove the main result (Theorem 1). The proof is divided into three steps. The first step is to establish the equality of the operator and the essential norms of the Fourier convolution operator $W^0(a)$. Since $W^0(a)$ is a translation-invariant operator, the desired result follows from a more general result obtained recently by the first author and Shargorodsky [9].

Theorem 3 *Let $1 < p < \infty$, $1 \le q < \infty$ and let Φ be a Young's function satisfying the (δ_2, Δ_2)-condition. Suppose $X(\mathbb{R})$ is the Lorentz space $L^{p,q}(\mathbb{R})$ or the Orlicz space $L^\Phi(\mathbb{R})$. If $a \in \mathcal{M}_X$, then*

$$\left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R})),\mathrm{e}} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

Proof. It follows from [1, Chap. 1, Corollary 5.6 and Theorem 3.11] that if $X(\mathbb{R})$ is separable, then the set of all simple compactly supported functions is dense in $X(\mathbb{R})$. Since the operator $W^0(a) \in \mathcal{B}(X(\mathbb{R}))$ is translation-invariant and the space $X(\mathbb{R})$ is rearrangement-invariant, and hence, translation-invariant, the desired result follows from [9, Theorem 5.1]. ◻

4.2 Equality of the operator norms of the Wiener-Hopf operator and the Fourier convolution operator

The second step is to prove the equality of operator norms of the Fourier convolution operator $W^0(a)$ and the Wiener-Hopf operator $W(a)$ with the same symbol following the idea by Duduchava [5].

Theorem 4 *Let $1 < p < \infty$, $1 \le q < \infty$ and let Φ be a Young's function satisfying the (δ_2, Δ_2)-condition. Suppose $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$ and $X(\Omega)$ is the Lorentz space $L^{p,q}(\Omega)$ or the Orlicz space $L^\Phi(\Omega)$. For all $a \in \mathcal{M}_X$, the operator norms of the Fourier convolution operator $W^0(a)$ and the Wiener-Hopf operator $W(a)$ coincide:*

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

Proof. The proof follows the argument presented in [5, Proposition 2.2]. Fix $a \in \mathcal{M}_X$. By exploiting the submultiplicative property of the operator norm, we find that

$$\begin{aligned} \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} &\le \|r_+\|_{\mathcal{B}(X(\mathbb{R}),X(\mathbb{R}_+))} \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} \|\ell_+\|_{\mathcal{B}(X(\mathbb{R}_+),X(\mathbb{R}))} \\ &\le \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}. \end{aligned} \tag{5}$$

The proof of the reverse inequality is more involved. First of all, observe that the product of the operator of extension by zero ℓ_+ and the restriction operator r_+ is nothing but the operator of multiplication by the characteristic function of $\mathbb{R}_+$:

$$\ell_+ r_+ = M(\chi_{\mathbb{R}_+}).$$

Fix $\varepsilon > 0$. Since the underlying rearrangement-invariant Banach function space $X(\mathbb{R})$ is separable, [7, Lemma 2.12(a)] yields that the subspace $C_c^\infty(\mathbb{R})$ is dense in $X(\mathbb{R})$. As a consequence of this, we have

$$\left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} = \sup_{\substack{f\in C_c^\infty(\mathbb{R})\\ \|f\|_{X(\mathbb{R})}=1}} \left\|W^0(a)f\right\|_{X(\mathbb{R})}.$$

Given this characterization, the notion of supremum assures the existence of some $f \in C_c^\infty(\mathbb{R})$ with $\|f\|_{X(\mathbb{R})} = 1$ such that

$$\left\|W^0(a)f\right\|_{X(\mathbb{R})} > \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} - \frac{\varepsilon}{2}.$$

The fact that f has compact support implies that there exists some $R > 0$ such that $\operatorname{supp} f \subset [-R, R]$. With this in mind, consider the family of functions defined by

$$f_h := r_+ T_h f$$

for each $h < 0$, where the translation operator T_h is defined by (4). It is clear that if $h < -R$, we have $\operatorname{supp} T_h f \subset \mathbb{R}_+$,

$$\|f_h\|_{X(\mathbb{R}_+)} = \|f\|_{X(\mathbb{R})} = 1,$$

and

$$\begin{aligned}
\|W(a)f_h\|_{X(\mathbb{R}_+)} &= \|\ell_+ W(a) f_h\|_{X(\mathbb{R})} = \left\|\ell_+ r_+ W^0(a)\ell_+ r_+ T_h f\right\|_{X(\mathbb{R})}\\
&= \left\|M(\chi_{\mathbb{R}_+})W^0(a)M(\chi_{\mathbb{R}_+})T_h f\right\|_{X(\mathbb{R})} = \left\|M(\chi_{\mathbb{R}_+})W^0(a)T_h f\right\|_{X(\mathbb{R})}\\
&= \left\|M(\chi_{\mathbb{R}_+})T_h W^0(a) f\right\|_{X(\mathbb{R})} = \left\|M(\chi_{\mathbb{R}_+ + h})W^0(a) f\right\|_{X(\mathbb{R})}\\
&= \left\|M(\chi_{(h,\infty)})W^0(a) f\right\|_{X(\mathbb{R})} = \left\|\chi_{(h,\infty)} \cdot W^0(a) f\right\|_{X(\mathbb{R})}.
\end{aligned}$$

Since $X(\mathbb{R})$ is separable, it follows from [1, Chap. 1, Corollary 5.6] and [1, Chap. 1, Proposition 3.6] that the Lebesgue dominated convergence theorem is true in $X(\mathbb{R})$. Therefore

$$\lim_{h\to-\infty} \left\|\chi_{(h,\infty)} \cdot W^0(a)f - W^0(a)f\right\|_{X(\mathbb{R})} = 0,$$

and hence

$$\lim_{h\to-\infty} \left\|\chi_{(h,\infty)} \cdot W^0(a)f\right\|_{X(\mathbb{R})} = \left\|W^0(a)f\right\|_{X(\mathbb{R})}.$$

As a consequence of this, for sufficiently large $-h$, we have

$$\left\|\chi_{(h,\infty)} \cdot W^0(a)f\right\|_{X(\mathbb{R})} > \left\|W^0(a)f\right\|_{X(\mathbb{R})} - \frac{\varepsilon}{2}.$$

Putting everything together, we get

$$\begin{aligned}\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} &\geq \|W(a)f_h\|_{X(\mathbb{R}_+)} = \|\chi_{(h,\infty)} \cdot W^0(a)f\|_{X(\mathbb{R})} \\ &> \left\|W^0(a)f\right\|_{X(\mathbb{R})} - \frac{\varepsilon}{2} > \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} - \varepsilon.\end{aligned}$$

Finally, since $\varepsilon > 0$ was considered arbitrary, we arrive at the conclusion that

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} \geq \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))},$$

which, combined with (5), proves the result. □

4.3 Equality of the essential norms of the Fourier convolution operator and the Wiener-Hopf operator

The final step of the proof Theorem 1 consists in establishing the equality of the essential norms of the Fourier convolution operator $W^0(a)$ and the Wiener-Hopf operator $W(a)$ with the same symbol.

Theorem 5 *Let $\Omega \in \{\mathbb{R}_+, \mathbb{R}\}$, $1 < p < \infty$, $1 \leq q < \infty$, and Φ be a Young's function satisfying the (δ_2, Δ_2)-condition and* (1)*. Suppose $X(\Omega)$ is the Lorentz space $L^{p,q}(\Omega)$ or the Orlicz space $L^\Phi(\Omega)$. For all $a \in \mathcal{M}_X$, the essential norms of the Fourier convolution operator $W^0(a)$ and the Wiener-Hopf operator $W(a)$ coincide:*

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R})),\mathrm{e}}. \tag{6}$$

Proof. Fix $a \in \mathcal{M}_X$. By Theorems 3 and 4, it follows that

$$\begin{aligned}\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}} &= \inf_{K\in\mathcal{K}(X(\mathbb{R}_+))} \|W(a)+K\|_{\mathcal{B}(X(\mathbb{R}_+))} \leq \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} \\ &= \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R})),\mathrm{e}}.\end{aligned} \tag{7}$$

The proof of the reverse inequality is based on the argument presented in [2, Proposition 4.1]. Let us start by noting that for all $K \in \mathcal{K}(X(\mathbb{R}_+))$,

$$\begin{aligned}\|W(a)+K\|_{\mathcal{B}(X(\mathbb{R}_+))} &= \sup_{\substack{f\in X(\mathbb{R}_+)\\ \|f\|_{X(\mathbb{R}_+)}\leq 1}} \|(W(a)+K)f\|_{X(\mathbb{R}_+)} \\ &= \sup_{\substack{f\in X(\mathbb{R}_+)\\ \|f\|_{X(\mathbb{R}_+)}\leq 1}} \|\ell_+(W(a)+K)f\|_{X(\mathbb{R})} \\ &= \sup_{\substack{f\in X(\mathbb{R})\\ \|r_+f\|_{X(\mathbb{R}_+)}\leq 1}} \|\ell_+(W(a)+K)r_+f\|_{X(\mathbb{R})}\end{aligned}$$

$$\geq \sup_{\substack{f\in X(\mathbb{R})\\ \|f\|_{X(\mathbb{R})}\leq 1}} \|\ell_+(W(a)+K)r_+f\|_{X(\mathbb{R})}$$
$$= \|\ell_+(W(a)+K)r_+\|_{\mathcal{B}(X(\mathbb{R}))},$$

because $\|r_+\|_{\mathcal{B}(X(\mathbb{R}),X(\mathbb{R}_+))} \leq 1$. Taking into account that $\ell_+ r_+ = M(\chi_{\mathbb{R}_+})$, the above inequality becomes

$$\|W(a)+K\|_{\mathcal{B}(X(\mathbb{R}_+))} \geq \left\|M(\chi_{\mathbb{R}_+})W^0(a)M(\chi_{\mathbb{R}_+}) + \ell_+Kr_+\right\|_{\mathcal{B}(X(\mathbb{R}))}. \tag{8}$$

It is clear that since $K \in \mathcal{K}(X(\mathbb{R}_+))$, we have $\ell_+Kr_+ \in \mathcal{K}(X(\mathbb{R}))$. With this in mind, consider the sequence of translation operators $\mathcal{T} := (T_{h_n})_{n\in\mathbb{N}}$ defined by

$$(T_{h_n}f)(x) := f(x+h_n), \qquad n \in \mathbb{N},$$

where $(h_n)_{n\in\mathbb{N}}$ is any sequence of real numbers satisfying $h_n \to -\infty$ as $n \to \infty$. Lemma 4 yields that

$$\operatorname*{s-lim}_{n\to\infty} T_{-h_n} M(\chi_{\mathbb{R}_+}) T_{h_n} = I.$$

By Corollary 1, we have

$$\operatorname*{s-lim}_{n\to\infty} T_{-h_n} \ell_+ K r_+ T_{h_n} = 0.$$

On account of the above remarks, Proposition 1(ii), and the fact $W^0(a)$ is translation-invariant, we conclude that

$$\begin{aligned} &\operatorname*{s-lim}_{n\to\infty} T_{-h_n}\left(M(\chi_{\mathbb{R}_+})W^0(a)M(\chi_{\mathbb{R}_+}) + \ell_+Kr_+\right)T_{h_n} \\ &\quad = \operatorname*{s-lim}_{n\to\infty}\left[\left(T_{-h_n}M(\chi_{\mathbb{R}_+})T_{h_n}\right)W^0(a)\left(T_{-h_n}M(\chi_{\mathbb{R}_+})T_{h_n}\right) + T_{-h_n}\ell_+Kr_+T_{h_n}\right] \\ &\quad = W^0(a). \end{aligned}$$

Consequently, Proposition 1(i) yields that

$$\begin{aligned} &\left\|M(\chi_{\mathbb{R}_+})W^0(a)M(\chi_{\mathbb{R}_+}) + \ell_+Kr_+\right\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\quad \geq \left\|\left(M(\chi_{\mathbb{R}_+})W^0(a)M(\chi_{\mathbb{R}_+}) + \ell_+Kr_+\right)_{\mathcal{T}}\right\|_{\mathcal{B}(X(\mathbb{R}))} \\ &\quad = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}. \end{aligned} \tag{9}$$

Combining (8) and (9), we arrive at

$$\|W(a)+K\|_{\mathcal{B}(X(\mathbb{R}_+))} \geq \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

for any $K \in \mathcal{K}(X(\mathbb{R}_+))$, and thus

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}} \geq \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))} = \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R})),\mathrm{e}} \tag{10}$$

(see Theorem 3). Finally, inequalities (7) and (10) imply equality (6). □

Theorem 1 follows immediately from Theorems 3–5.

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