Maximal Noncompactness of Singular Integral Operators on L^2 Spaces with Some Khvedelidze Weights

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To Professor Naum Krupnik on the occasion of his 90th birthday

Abstract. Let Γ be a contour in the complex plane consisting of a finite number of circular arcs joining the endpoints -1 and 1, possibly including the segment $[-1, 1]$. We consider the singular integral operator $A = aI + bS_\Gamma$ with constant coefficients $a, b \in \mathbb{C}$, where S_Γ is the Cauchy singular integral operator over Γ. We provide a detailed proof of the maximal noncompactness of the operator A on L^2 spaces with the Khvedelidze weights $\varrho(t) = |t - 1|^\beta |t + 1|^{-\beta}$ satisfying $-1 < \beta < 1$. This result was announced by Naum Krupnik in 2010, but its proof has never been published.

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1. Introduction and the main result

Let X be a Banach space, $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X, and $\mathcal{K}(X)$ be the two-sided ideal of all compact operators on X. The norm of an operator $A \in \mathcal{B}(X)$ is denoted by $\|A\|_{\mathcal{B}(X)}$. The essential norm of $A \in \mathcal{B}(X)$ is defined by

$$|A|_{\mathcal{B}(X)} := \inf \left\{ \|A + K\|_{\mathcal{B}(X)} \ : \ K \in \mathcal{K}(X) \right\}.$$

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It is clear that $|A|_{\mathcal{B}(X)} \leq \|A\|_{\mathcal{B}(X)}$. An operator $A \in \mathcal{B}(X)$ is said to be maximally noncompact on X if

$$|A|_{\mathcal{B}(X)} = \|A\|_{\mathcal{B}(X)}.$$

Maximal noncompactness of one-dimensional singular integral operators with the Cauchy kernel on weighted Lebesgue spaces was studied in a series of papers by Naum Krupnik and his coauthors. We refer to [4, Section 13.5], [8, Section 6], [9] and to the references therein. The first author and Shargorodsky extended further the results on the maximal noncompactness of the operators $aI + bS_\Gamma$ with $a, b \in \mathbb{C}$ and $\Gamma \in \{\mathbb{T}, \mathbb{R}\}$, where $\mathbb{T}$ is the unit circle, to the setting of rearrangement-invariant Banach function spaces with nontrivial Boyd indices [6, Theorem 1.1], [7, Corollary 5.4 and formula (5.5)].

The aim of this paper is to discuss one of Krupnik's results (see [9, Theorem 5.9]) on the maximal noncompactness of one-dimensional singular integral operators $aI + bS_\Gamma$ with $a, b \in \mathbb{C}$ on spaces $L^2(\Gamma, \varrho)$ with some Khvedelidze weights ϱ over some piecewise Lyapunov curves Γ.

Let us define precisely the setting. Let Γ be a contour in the complex plane consisting of a finite number of circular arcs joining the endpoints -1 and 1, possibly including the segment $[-1, 1]$ as in Figure 1.

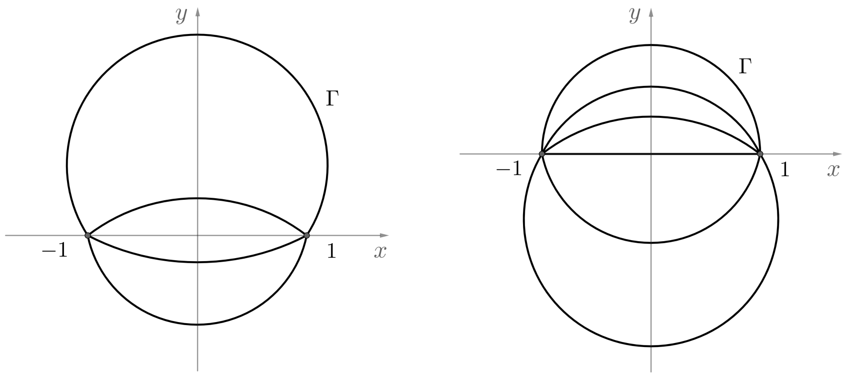

Figure 1. Examples of a possible contour Γ.

Consider the Khvedelidze weight of the form

$$w(t) := |t-1|^{\alpha_1}|t+1|^{\alpha_2}, \quad t \in \Gamma, \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

The space $L^2(\Gamma, w)$ consists of all measurable functions $f : \Gamma \to \mathbb{C}$ such that

$$\|f\|_{L^2(\Gamma,w)} := \left(\int_\Gamma |f(t)|^2 w(t)|dt|\right)^{1/2} < \infty.$$

It follows from Khvedelidze's theorem (see, e.g., [3, Ch. 1, Theorem 4.1, Corollary 4.1]) that the Cauchy singular integral operator S_Γ, defined by

$$(S_\Gamma f)(t) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma,$$

where $\Gamma(t,\varepsilon) := \{\tau \in \Gamma : |\tau - t| < \varepsilon\}$, is bounded on $L^2(\Gamma, w)$ if and only if

$$-1 < \alpha_1, \alpha_2 < 1.$$

The following result was announced in [9, Theorem 5.9].

Theorem 1.1 (Main result, Krupnik, 2010). *Let Γ be a contour in the complex plane consisting of a finite number of circular arcs joining the endpoints -1 and 1, possibly including the segment $[-1,1]$. Suppose the Khvedelidze weight ϱ is given by*

$$\varrho(t) := \left|\frac{t-1}{t+1}\right|^{\beta}, \quad t \in \Gamma, \quad -1 < \beta < 1.$$

Then the singular integral operator $A := aI + bS_\Gamma$ with constant coefficients $a, b \in \mathbb{C}$ is maximally noncompact on the space $L^2(\Gamma, \varrho)$, that is,

$$|A|_{\mathcal{B}(L^2(\Gamma,\varrho))} = \|A\|_{\mathcal{B}(L^2(\Gamma,\varrho))}.$$

Unfortunately, a detailed proof of this theorem has never been published. The aim of this paper is to fill in this gap in the literature, correcting also a misleading inaccuracy in the hint given in [9, p. 382].

Krupnik wrote on [9, p. 382] the following: "... *This theorem follows from Lemma* 1.2 (see below) *with*

$$(R_n f)(z) := \frac{2n^{\beta+1/2}}{n+1+z(n-1)} f\left(\frac{(n+1)z+n-1}{n+1+z(n-1)}\right). \tag{1.1}$$

A particular case of this statement was obtained in [1]...". Notice also that the proof of the particular case of Theorem 1.1 in the non-weighted case given in [1, Theorem 2] is not much more detailed than the above quotation. It only says that the sequence given by (1.1) with $\beta = 0$ satisfies the conditions of Lemma 1.2 below.

Lemma 1.2 ([8, Theorem 4.3])**.** *Let X be a Banach space and $A \in \mathcal{B}(X)$. Suppose that there exists a sequence $R_n \in \mathcal{B}(X)$ such that*

(a) $\|R_n f\|_X = \|f\|_X$ *for all $f \in X$ and all $n \in \mathbb{N}$;*
(b) $AR_n = R_n A$ *for all $n \in \mathbb{N}$;*
(c) *the sequence $\{R_n\}$ converges weakly to the zero operator.*

Then the operator A is maximally noncompact on the space X, that is,

$$|A|_{\mathcal{B}(X)} = \|A\|_{\mathcal{B}(X)}.$$

The factor $2n^{\beta+1/2}$ is, unfortunately, chosen incorrectly in (1.1) (see Corollary 2.3 below). In the next section we suggest the correction for the sequence $\{R_n\}$ given by (1.1) that satisfies all the conditions of Lemma 1.2. We feel that for the convenience of the reader it is important to provide the calculations that were missed in [1, 9].

2. Proof of the main result

To make further calculations easier, let us rewrite the elements of sequence (1.1), when $n \geq 2$, in the form

$$(R_n f)(z) = \frac{2n^{\beta+1/2}}{n-1} \cdot \frac{1}{z + \frac{n+1}{n-1}} f\left(\frac{\frac{n+1}{n-1}z + 1}{z + \frac{n+1}{n-1}}\right) \tag{2.1}$$

and then, defining

$$\lambda := \lambda(n) = \frac{n+1}{n-1},$$

introduce the operator

$$(Rf)(z) := \frac{\mu}{z+\lambda} f\left(\frac{\lambda z + 1}{z+\lambda}\right), \tag{2.2}$$

which structurally "generalizes" all operators R_n for $n \geq 2$.

Here, we put an unknown coefficient $\mu \in \mathbb{R}$ in place of the numerical factor from the rewritten expression for R_n. Exactly this factor is responsible for $\{R_n\}$ being a sequence of isometries. With the unknown μ in definition (2.2), we will further assume that the operator R is isometric and derive a condition on μ, necessary and sufficient for this assumption to hold. It will allow us to show that the choice of

$$\mu = \frac{2n^{\beta+1/2}}{n-1}$$

taken in (2.1) leads to the situation that the sequence $\{R_n\}$ is not a sequence of isometries, whence Lemma 1.2 cannot be applied with this choice of $\{R_n\}$. Then we will suggest a correction for the factor μ.

To realize this idea technically, we will need to make a fractional-linear change of variables in the integral over the contour Γ; and the following auxiliary statement concerning this change is a very useful result to have in advance.

2.1. Möbius transformation mapping each arc of Γ onto itself

Definition (2.2) of the operator R features the Möbius (or fractional-linear) transformation

$$t(z) = \frac{\lambda z + 1}{z + \lambda}, \quad \lambda > 1, \tag{2.3}$$

as the argument of f. According to the next lemma, $t = t(z)$ turns out to possess a property particularly important for our purposes: it maps the contour Γ onto itself.

Lemma 2.1. *Transformation* (2.3) *maps each circular arc with the endpoints* -1 *and* 1 *onto itself and the segment* $[-1, 1]$ *onto itself.*

Proof. (i) Consider first a circular arc γ joining the endpoints -1 and 1, assuming it to be part of a full circle γ_C. Note that $t(1) = 1$ and $t(-1) = -1$ for any $\lambda > 1$, so ± 1 are the fixed points of the transformation $t = t(z)$.

Next, it is a well-known property of the Möbious transformation that it maps circles and lines into circles and lines. In our case, since the point

$z = -\lambda$, which $t = t(z)$ maps into $z = \infty$, does not belong to γ_C (because $-\lambda < -1$), the image of γ_C is a circle passing through the points ± 1. Together with the fact that $t = t(z)$ is continuous, this leads us to the conclusion that the arc γ is being mapped onto either of the two arcs of the "image-circle" $t(\gamma_C)$, which are enclosed between the points -1 and 1.

Since a circular arc is uniquely determined by three points (two endpoints and one more arbitrary point lying on the arc), and we already know that $t = t(z)$ preserves the endpoints ± 1, it suffices to show that any other point $\tilde{z} \in \gamma$ different from ± 1 has its image $t(\tilde{z})$ belonging the same arc γ: then we will be able to conclude that $t(\gamma) = \gamma$.

Suppose first that

$$\gamma \setminus \{-1, 1\} \subset \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Let $\tilde{z} = i(c + \sqrt{c^2+1})$ be the point at which γ intersects the imaginary axis, with ic being the center of the full circle γ_C (see Fig. 2).

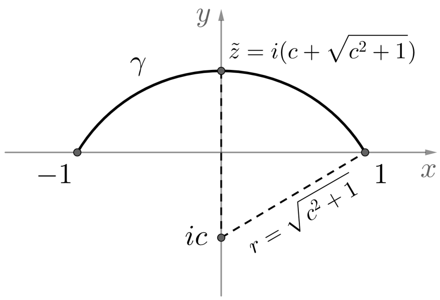

FIGURE 2. Choice of the point $\tilde{z}$.

The distance between the image $t(\tilde{z})$ and the center of γ_C is

$$\begin{aligned}
|t(\tilde{z}) - ic| &= \left| \frac{i\lambda(c + \sqrt{c^2+1}) + 1}{i(c + \sqrt{c^2+1}) + \lambda} - ic \right| \\
&= \left| \frac{i\lambda c + i\lambda\sqrt{c^2+1} + 1 + c^2 + c\sqrt{c^2+1} - i\lambda c}{i(c + \sqrt{c^2+1}) + \lambda} \right| \\
&= \frac{\sqrt{c^2+1}}{|i|} \cdot \frac{|c + \sqrt{c^2+1} + i\lambda|}{|c + \sqrt{c^2+1} - i\lambda|} \\
&= \sqrt{c^2+1},
\end{aligned}$$

hence $t(\tilde{z}) \in \gamma_C$. Moreover, for any $y \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
\operatorname{Im}(t(iy)) &= \operatorname{Im}\left(\frac{i\lambda y + 1}{iy + \lambda} \right) = \operatorname{Im}\left(\frac{(i\lambda y + 1)(\lambda - iy)}{\lambda^2 + y^2} \right) \\
&= \frac{y(\lambda^2 - 1)}{\lambda^2 + y^2} = \begin{cases} > 0 \text{ if } y > 0, \\ < 0 \text{ if } y < 0 \end{cases}
\end{aligned}$$

(because $\lambda > 1$), which guarantees, together with the previous assertion, that $t(\tilde{z}) \in \gamma$.

If

$$\gamma \setminus \{-1, 1\} \subset \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\},$$

then we choose $\tilde{z} = i(c - \sqrt{c^2+1})$ and show as before that $t(\tilde{z}) \in \gamma_C$ and $t(\tilde{z}) \in \mathbb{C}_-$, whence $t(\tilde{z}) \in \gamma$. This finishes the proof that the transformation (2.3) maps any circular arc with the endpoints -1 and 1 onto itself.

(ii) Now consider the case when the circular arc degenerates into the segment $[-1, 1]$ contained in the real line. The transformation $t = t(z)$, having all its coefficients real-valued, maps the real line to the real line, preserving the points ± 1. Therefore, there are two possibilities for the image $t([-1, 1])$: it either coincides with the segment $[-1, 1]$ or with the remaining part of the "generalized circle" $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, that is, with $\dot{\mathbb{R}} \setminus (-1, 1)$. Checking that $t(0) = 1/\lambda < 1$, we conclude that $t([-1, 1]) = [-1, 1]$. $\square$

2.2. Isometric operator R generated by the Möbius transformation

Now that the handy Lemma 2.1 is proved, we can proceed with the verification of Condition (a) of Lemma 1.2.

Lemma 2.2. *Let Γ and ϱ be as in Theorem 1.1. Suppose that $\lambda > 1$ and $\mu \in \mathbb{R}$. Then the operator R given by (2.2) is an isometry on $L^2(\Gamma, \varrho)$ if and only if*

$$\mu^2 = (\lambda - 1)^{1+\beta}(\lambda + 1)^{1-\beta}. \tag{2.4}$$

Proof. Making the change of variables

$$t = \frac{\lambda z + 1}{z + \lambda}$$

and taking into account that this transformation maps Γ onto itself in view of Lemma 2.1, we see that

$$z = \frac{\lambda t - 1}{\lambda - t}, \quad |dz| = \frac{\lambda^2 - 1}{|\lambda - t|^2}|dt| \tag{2.5}$$

and

$$\begin{aligned}
\|Rf\|^2_{L^2(\Gamma,\varrho)} &= \int_\Gamma \left| \frac{\mu}{z+\lambda} f\left(\frac{\lambda z + 1}{z + \lambda}\right)\right|^2 \cdot \left|\frac{z-1}{z+1}\right|^\beta |dz| \\
&= \int_\Gamma \frac{\mu^2}{\left|\frac{\lambda t - 1}{\lambda - t} + \lambda\right|^2} |f(t)|^2 \cdot \left| \frac{\frac{\lambda t - 1}{\lambda - t} - 1}{\frac{\lambda t - 1}{\lambda - t} + 1}\right|^\beta \cdot \frac{\lambda^2 - 1}{|\lambda - t|^2}|dt| \\
&= \int_\Gamma \frac{\mu^2(\lambda^2 - 1)}{|\lambda t - 1 + \lambda^2 - \lambda t|^2} |f(t)|^2 \cdot \left|\frac{\lambda t - 1 - \lambda + t}{\lambda t - 1 + \lambda - t}\right|^\beta |dt| \\
&= \frac{\mu^2}{\lambda^2 - 1} \int_\Gamma |f(t)|^2 \left|\frac{(\lambda+1)(t-1)}{(\lambda-1)(t+1)}\right|^\beta |dt| \\
&= \mu^2 \frac{(\lambda+1)^{\beta-1}}{(\lambda-1)^{\beta+1}} \cdot \int_\Gamma |f(t)|^2 \left|\frac{t-1}{t+1}\right|^\beta |dt|
\end{aligned}$$

$$= \mu^2 \frac{(\lambda+1)^{\beta-1}}{(\lambda-1)^{\beta+1}} \cdot \|f\|^2_{L^2(\Gamma,\varrho)}.$$

It shows that R is isometric if and only if (2.4) holds. □

2.3. Mistake in the choice of sequence (1.1)

This lemma immediately implies that the sequence $\{R_n\}$ given by (1.1) does not satisfy Condition (a) of Lemma 1.2.

Corollary 2.3. *Let* Γ *and* ϱ *be as in Theorem* 1.1*. If* $n \geq 2$*, then the operator* R_n *defined by* (1.1) *is not an isometry on the space* $L^2(\Gamma, \varrho)$*.*

Proof. It follows from (2.1) that R_n is of the form of R given by (2.2) with

$$\lambda = \frac{n+1}{n-1}, \quad \mu = \frac{2n^{\beta+1/2}}{n-1}.$$

It is clear that

$$(\lambda-1)^{1+\beta}(\lambda+1)^{1-\beta} = \left(\frac{n+1}{n-1} - 1\right)^{1+\beta} \left(\frac{n+1}{n-1} + 1\right)^{1-\beta} = \frac{2^2 n^{1-\beta}}{(n-1)^2} \tag{2.6}$$

and

$$\mu^2 = \frac{2^2 n^{2\beta+1}}{(n-1)^2},$$

which implies that (2.4) does not hold in general. Hence the result immediately follows from Lemma 2.2. □

2.4. Correcting the mistake in the choice of sequence (1.1)

Taking into account Lemma 2.2 and identity (2.6), we can choose μ to be

$$\mu = \sqrt{(\lambda-1)^{1+\beta}(\lambda+1)^{1-\beta}} = \frac{2n^{(1-\beta)/2}}{n-1},$$

making it positive for convenience.

This factor μ and the one suggested by Krupnik differ by their numerators $2n^{(1-\beta)/2}$ and $2n^{\beta+1/2}$, respectively. But while sequence (1.1) does not consists of all isometries (as we have seen in Corollary 2.3), each element of the sequence defined by (2.2) with the chosen value of μ, that is

$$(R_n f)(z) := \frac{2n^{(1-\beta)/2}}{n+1+z(n-1)} f\left(\frac{(n+1)z+n-1}{n+1+z(n-1)}\right) \tag{2.7}$$

for all $n \geq 2$, is an isometry as we proved in Lemma 2.2. When $n = 1$, formula (2.7) gives R_1 as coinciding with the identity operator I, which is also isometric.

Overall, (2.7) redefines the sequence $\{R_n\}_{n\in\mathbb{N}}$ as a sequence of isometries. We suggest this new sequence with the altered factor to be the announced correction for the initial version of $\{R_n\}$ given by (1.1). Referring further to R_n, we will understand them in the sense of definition (2.7); Conditions (b) and (c) of Lemma 1.2, which are to be checked ahead, will be verified for this renewed sequence.

2.5. The proof of $AR_n = R_nA$

We continue by proving that $\{R_n\}$ meets Condition (b) of Lemma 1.2.

For $n = 1$, the equality $AR_1 = R_1A$ obviously holds since $R_1 = I$. All the other operators R_n, starting from $n = 2$, are representable in the form of the operator R defined by (2.2) with the corresponding $\lambda = \lambda(n) > 1$. Therefore, to prove that $AR_n = R_nA$ when $n \geq 2$, it is sufficient to check the equality $AR = RA$ for an arbitrary $\lambda > 1$ and a positive constant μ.

The following lemma is analogous to [2, Lemma 3.1].

Lemma 2.4. *Let Γ and ϱ be as in Theorem 1.1. The Cauchy singular integral operator S_Γ commutes with the operator R defined by (2.2) for any $\lambda > 1$ and $\mu > 0$, i.e., $S_\Gamma Rf = RS_\Gamma f$ for all $f \in L^2(\Gamma, \varrho)$.*

Proof. Resorting again to the change of variables

$$t = \frac{\lambda\tau + 1}{\tau + \lambda},$$

which preserves the contour Γ and implies the identities

$$\tau = \frac{\lambda t - 1}{\lambda - t}, \quad d\tau = \frac{\lambda^2 - 1}{(\lambda - t)^2} dt,$$

we establish that for $f \in L^2(\Gamma, \varrho)$ and $z \in \Gamma$,

$$\begin{aligned}
(S_\Gamma Rf)(z) &= \frac{1}{\pi i} \int_\Gamma \frac{\mu}{\tau + \lambda} f\left(\frac{\lambda\tau + 1}{\tau + \lambda}\right) \frac{d\tau}{\tau - z} \\
&= \frac{1}{\pi i} \int_\Gamma \frac{\mu}{\frac{\lambda t - 1}{\lambda - t} + \lambda} f(t) \cdot \frac{\lambda^2 - 1}{(\lambda - t)^2} \cdot \frac{dt}{\frac{\lambda t - 1}{\lambda - t} - z} \\
&= \frac{1}{\pi i} \int_\Gamma \frac{\mu f(t) dt}{\lambda t - 1 - \lambda z + tz} = \frac{1}{\pi i} \int_\Gamma \frac{\mu f(t) dt}{t(z + \lambda) - (\lambda z + 1)} \\
&= \frac{\mu}{z + \lambda} \cdot \frac{1}{\pi i} \int_\Gamma \frac{f(t)}{t - \frac{\lambda z + 1}{z + \lambda}} dt = (RS_\Gamma f)(z),
\end{aligned}$$

where all the integrals should be understood in the sense of the Cauchy principal value. □

Now, employing this lemma, we obtain the equality

$$\begin{aligned}
AR = (aI + bS_\Gamma)R &= aIR + bS_\Gamma R = aRI + bRS_\Gamma \\
&= R(aI) + R(bS_\Gamma) = R(aI + bS_\Gamma) = RA
\end{aligned}$$

for an arbitrary singular integral operator $A = aI + bS_\Gamma$ with $a, b \in \mathbb{C}$. The verification of the fact that all R_n commute with A is thus complete.

2.6. Preparing the proof of the weak convergence of R_n to the zero operator

From this moment on, we are working toward confirmation of the last and most complicated Condition (c) of Lemma 1.2 for the sequence $\{R_n\}$.

We start this subsection, which comprises two crucial technical results, with the following simple lemma.

Lemma 2.5. *Let $\beta > -1$. Then*

$$\lim_{n\to\infty} \frac{n^{(1-\beta)/2}}{n-1} \int_{-1}^{1} \frac{dt}{\left|t - \frac{n+1}{n-1}\right|} = 0.$$

Proof. Using the fundamental theorem of calculus, we first compute the integral

$$\int_{-1}^{1} \frac{dt}{\left|t - \frac{n+1}{n-1}\right|} = \int_{-1}^{1} \frac{dt}{\frac{n+1}{n-1} - t} = -\ln\left|\frac{n+1}{n-1} - t\right|\Big|_{-1}^{1}$$
$$= -\ln\frac{2}{n-1} + \ln\frac{2n}{n-1} = \ln n$$

for all $n \geq 2$. Consequently, the limit given in the statement of this lemma is equal to the following two limits

$$\lim_{n\to\infty} \frac{n^{(1-\beta)/2}}{n-1} \int_{-1}^{1} \frac{dt}{\left|t - \frac{n+1}{n-1}\right|} = \lim_{n\to\infty} \frac{n^{(1-\beta)/2}\ln n}{n-1} = \lim_{n\to\infty} \frac{\ln n}{n^{(\beta+1)/2}},$$

with the second equality being true by the equivalence $(n-1) \sim n$ as $n \to \infty$. Finally, the equality

$$\lim_{n\to\infty} \frac{\ln n}{n^{(\beta+1)/2}} = 0 \tag{2.8}$$

follows from the equivalence of Cauchy's and Heine's definitions of the limit and the L'Hôpital rule. □

The next theorem is a version of the above lemma: it addresses the limit value of the analogous expression, this time containing the integral over a circle passing through the points ± 1 instead of simply the segment $[-1, 1]$.

Theorem 2.6. *Let $\beta > -1$ and γ_C be a circle in the complex plane passing through the points -1 and 1. Then*

$$\lim_{n\to\infty} \frac{n^{(1-\beta)/2}}{n-1} \int_{\gamma_C} \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|} = 0.$$

Proof. We investigate the limiting behavior of the expression

$$I_n := \frac{n^{(1-\beta)/2}}{n-1} \int_{\gamma_C} \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|},$$

assuming as before that $n \geq 2$.

(i) Let r denote the radius of γ_C. Suppose that the center ic of the circle γ_C belongs to the upper half-plane ($c > 0$). Looking at the modulus

$$\operatorname{dist}(t) := \left|t - \frac{n+1}{n-1}\right|$$

as the distance from an arbitrary point $t \in \gamma_C$ to the "floating" point $\frac{n+1}{n-1}$ on the real axis, we open up a way to build our investigation on illustrative geometric reasoning.

Let us draw the line segment joining ic to $\frac{n+1}{n-1}$ and extend it to the point of its other intersection with the circle γ_C (see Fig. 3). This extended segment notionally dissects γ_C into the "upper" and "lower" semicircles γ_u and γ_l, respectively. Next, choose an arbitrary point t on the circle.

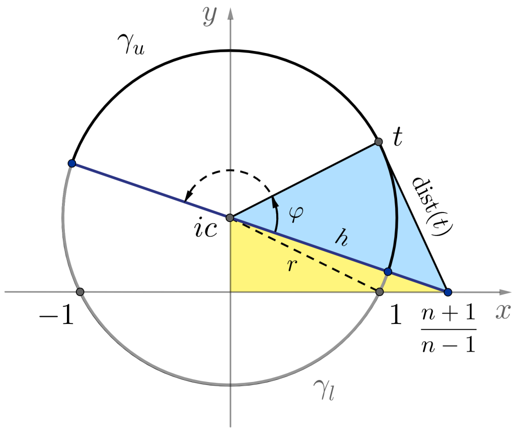

FIGURE 3. Dissection of γ_C and an example of a point t chosen on the "upper" semicircle γ_u.

Consider the yellow triangle depicted in Fig. 3. Its hypotenuse is

$$h = h(n) = \sqrt{c^2 + \left(\frac{n+1}{n-1}\right)^2} = \sqrt{r^2 - 1 + \left(\frac{n+1}{n-1}\right)^2}, \tag{2.9}$$

where we used that $c = \sqrt{r^2 - 1}$. Note that $h = h(n) > r$ for any n, and $h = h(n) \to r$ as $n \to \infty$.

Then, applying the law of cosines in the blue triangle above, for any $\varphi \in [0, \pi]$ we obtain

$$\begin{aligned} \operatorname{dist}(t) &= \sqrt{h^2 + r^2 - 2hr\cos\varphi} = \sqrt{h^2 + r^2 - 2hr\left(1 - 2\sin^2\frac{\varphi}{2}\right)} \\ &= \sqrt{(h-r)^2 + 4hr\sin^2\frac{\varphi}{2}} = (h-r)\sqrt{1 + \frac{4hr}{(h-r)^2}\sin^2\frac{\varphi}{2}}. \end{aligned}$$

Here we assume that $t \in \gamma_u$, and in fact, from now on it will be enough to restrict our attention to the "upper" semicircle only. Since γ_u and γ_l are congruent curves, and the function $\operatorname{dist}(t)$ takes the same values when t travels along either the "upper" semicircle or the "lower" one (perhaps only in reverse order if a common orientation is chosen on the whole circle; see Fig. 4), then

$$\int_{\gamma_u} \frac{|dt|}{\operatorname{dist}(t)} = \int_{\gamma_l} \frac{|dt|}{\operatorname{dist}(t)}.$$

Hence, the integral over γ_C contained in the expression for I_n is twice the

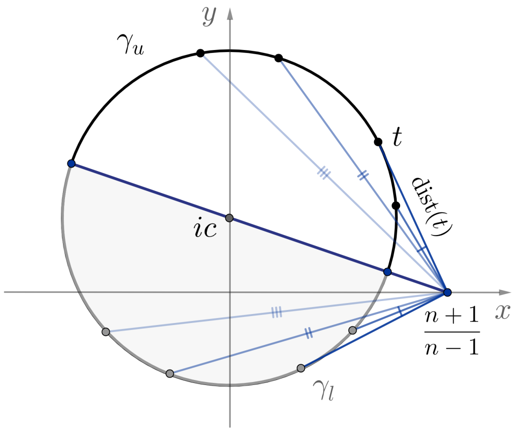

FIGURE 4. Equal distances to the points on γ_u and γ_l symmetric about the dissecting segment.

integral over the "upper" semicircle, and this gives the simpler formula

$$I_n = \frac{2n^{(1-\beta)/2}}{n-1} \int_{\gamma_u} \frac{|dt|}{\operatorname{dist}(t)}. \tag{2.10}$$

Let us now parametrize γ_u by the angle φ, since the integrand has been already expressed as a function of this angle. Denote the lower acute angle of the yellow triangle by α_n, as shown in Fig. 5.

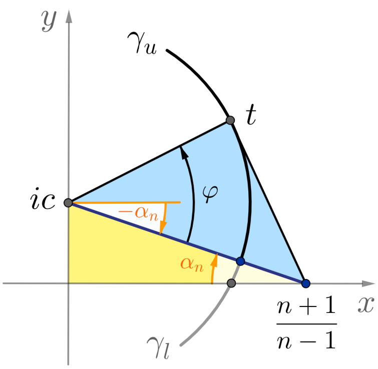

FIGURE 5. Choice of the angle α_n.

Then

$$t(\varphi) = ic + r(\cos(\varphi - \alpha_n) + i\sin(\varphi - \alpha_n)), \quad \text{where } \varphi \in [0, \pi],$$

is a parametrization of the semicircle γ_u with $|dt| = r d\varphi$. Its application to (2.10) yields

$$I_n = \frac{2n^{(1-\beta)/2}}{n-1}\int_0^\pi \frac{r\,d\varphi}{(h-r)\sqrt{1+\frac{4hr}{(h-r)^2}\sin^2\frac{\varphi}{2}}}.$$

Since

$$\sin\frac{\varphi}{2} \geq \frac{\varphi}{\pi} \quad \text{for all} \quad \varphi\in[0,\pi]$$

and $n-1 \geq n/2$ for all $n \geq 2$, we get

$$\begin{aligned}
0 < I_n &\leq \frac{4}{n^{(\beta+1)/2}}\frac{r}{h-r}\int_0^\pi \frac{d\varphi}{\sqrt{1+\frac{4hr}{\pi^2(h-r)^2}\varphi^2}} \\
&= \frac{4}{n^{(\beta+1)/2}}\cdot\frac{r}{h-r}\cdot\frac{\pi(h-r)}{2\sqrt{hr}}\int_0^{\frac{2\sqrt{hr}}{h-r}}\frac{dx}{\sqrt{1+x^2}} \\
&= \frac{2\pi}{n^{(\beta+1)/2}}\sqrt{\frac{r}{h}}\int_0^{\frac{2\sqrt{hr}}{h-r}}\frac{dx}{\sqrt{1+x^2}} =: \widetilde{I}_n. \qquad (2.11)
\end{aligned}$$

Applying the formula

$$\int\frac{dx}{\sqrt{1+x^2}} = \ln\left(x+\sqrt{1+x^2}\right)+C$$

(see, e.g., [5, formula 2.01.18, page 64]), we get

$$\begin{aligned}
\widetilde{I}_n &= \frac{2\pi}{n^{(\beta+1)/2}}\sqrt{\frac{r}{h}}\ln\left(\frac{2\sqrt{hr}}{h-r}+\sqrt{1+\frac{4hr}{(h-r)^2}}\right) \\
&= \frac{2\pi}{n^{(\beta+1)/2}}\sqrt{\frac{r}{h}}\ln\left(\frac{2\sqrt{hr}}{h-r}+\frac{h+r}{h-r}\right) \\
&= \frac{2\pi}{n^{(\beta+1)/2}}\sqrt{\frac{r}{h}}\ln\left(\frac{(2\sqrt{hr}+h+r)(h+r)}{h^2-r^2}\right).
\end{aligned}$$

It follows from (2.9) that $h = h(n) \sim r$ and

$$\frac{1}{h^2-r^2} = \frac{1}{\left(\frac{n+1}{n-1}\right)^2-1} = \frac{(n-1)^2}{4n} \sim \frac{n}{4}$$

as $n\to\infty$. Hence

$$\widetilde{I}_n \sim \frac{2\pi}{n^{(\beta+1)/2}}\ln(2r^2n) \quad \text{as} \quad n\to\infty. \qquad (2.12)$$

Since $\beta > -1$, combining (2.11), (2.12), and (2.8), we arrive at $\lim\limits_{n\to\infty} I_n = 0$, as was to be proved.

(ii) The case when $c \leq 0$, that is when the center of γ_C lies in the closed lower half-plane, is treated analogously. Expressions for h and $\operatorname{dist}(t)$, when dissecting the circle and choosing the angle φ as shown in Fig. 6, remain the same. We will only need to make a slight adjustment to the parametrization of the "upper" semicircle. When the angles α_n are chosen as in Fig. 6, the contour γ_u is parametrized by

$$t(\varphi) = ic + r(\cos(\varphi+\alpha_n) + i\sin(\varphi+\alpha_n)), \quad \varphi \in [0,\pi].$$

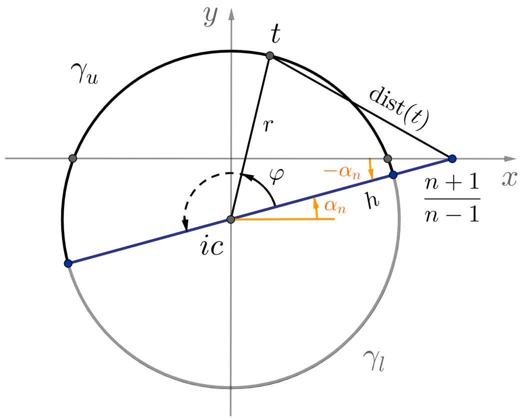

FIGURE 6. Natural adjustments for the case $c \le 0$.

This alteration, however, does not affect the differential $|dt| = rd\varphi$. Thus, when $c \le 0$,

$$I_n = \frac{2n^{(1-\beta)/2}}{n-1}\int_{\gamma_u}\frac{|dt|}{\operatorname{dist}(t)} = \frac{2n^{(1-\beta)/2}}{n-1}\int_0^{\pi}\frac{rd\varphi}{(h-r)\sqrt{1+\frac{4hr}{(h-r)^2}\sin^2\frac{\varphi}{2}}},$$

which coincides precisely with the quantity I_n for the case $c > 0$. Regarding the latter, we have already proved that $I_n \to 0$ as $n \to \infty$, and this finishes the proof for the case $c \le 0$ as well. □

2.7. Weak convergence of the sequence R_n to the zero operator

Having made all the necessary preparations, we are finally ready to prove that sequence (2.7) satisfies the last condition of Lemma 1.2, i.e., converges weakly to zero.

According to the definition of weak convergence, we need to show that for any function $f \in L^2(\Gamma,\varrho)$, the sequence $\{R_n f\}$ converges weakly to the zero function. This, in turn, can be reformulated as follows: for every pair $f \in L^2(\Gamma,\varrho)$ and $g \in (L^2(\Gamma,\varrho))^* = L^2(\Gamma,\varrho^{-1})$, there must hold

$$\langle R_n f, g\rangle = \int_\Gamma \frac{2n^{(1-\beta)/2}}{n+1+z(n-1)} f\left(\frac{(n+1)z+n-1}{n+1+z(n-1)}\right)\cdot\overline{g(z)}\cdot|dz| \to 0 \quad (2.13)$$

as $n \to 0$. Fortunately, it suffices to verify (2.13) for all f and g belonging only to dense subsets of $L^2(\Gamma,\varrho)$ and $L^2(\Gamma,\varrho^{-1})$, respectively, in order to have this result for the entire $L^2(\Gamma,\varrho)$ [10, Lemma 1.4.1(i)].

The theorem below, which is a particular case of [3, Ch. 1, Theorem 1.2], implies that one can choose the set of all continuous functions on Γ as such a dense subset for both $L^2(\Gamma, \varrho)$ and $L^2(\Gamma, \varrho^{-1})$ whenever $-1 < \beta < 1$.

Theorem 2.7. *Let Γ be a contour in the complex plane consisting of a finite number of circular arcs joining the endpoints -1 and 1, possibly including the segment $[-1, 1]$. Suppose that $w(t) = |t-1|^{\alpha_1}|t+1|^{\alpha_2}$ with $\alpha_1, \alpha_2 \in \mathbb{R}$. If $\alpha_1, \alpha_2 > -1$, then the set $C(\Gamma)$ of all continuous functions on Γ is dense in the space $L^2(\Gamma, w)$.*

Given all of this, it remains to prove the following concluding theorem.

Theorem 2.8. *Let $\beta > -1$ and the contour Γ be as in Theorem 1.1. Then for any $f, g \in C(\Gamma)$, convergence (2.13) is valid.*

Proof. Since the contour Γ is a compact set, arbitrary continuous functions f and g are bounded on Γ, that is, there exist nonnegative constants L and M such that $|f(z)| \le L$ and $|\overline{g(z)}| \le M$ for all $z \in \Gamma$.

Applying these estimates together with the usual for this work change of variables

$$t = \frac{\lambda z + 1}{z + \lambda}, \quad \text{where } \lambda = \frac{n+1}{n-1},$$

and its implications (2.5), we obtain for $n \ge 2$ that

$$\begin{aligned} |\langle R_n f, g\rangle| &\le M \int_\Gamma \frac{2n^{(1-\beta)/2}}{|n+1+z(n-1)|} \cdot \left| f\left(\frac{(n+1)z+n-1}{n+1+z(n-1)}\right)\right| \cdot |dz| \\ &= M \int_\Gamma \frac{2n^{(1-\beta)/2}}{(n-1)|z+\lambda|} \cdot \left| f\left(\frac{\lambda z+1}{z+\lambda}\right)\right| \cdot |dz| \\ &= 2M \cdot \frac{n^{(1-\beta)/2}}{n-1} \int_\Gamma \frac{|f(t)|}{|t-\lambda|} |dt| \le 2LM \cdot \frac{n^{(1-\beta)/2}}{n-1} \int_\Gamma \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|} \\ &= \sum_{i=1}^{k} 2LM \cdot \frac{n^{(1-\beta)/2}}{n-1} \int_{\gamma_i} \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|}, \end{aligned} \tag{2.14}$$

where $\Gamma = \gamma_1 \cup \ldots \cup \gamma_k$ with γ_i $(1 \le i \le k)$ being either a circular arc with the endpoints ± 1 or the segment $[-1, 1]$.

If $\gamma_i = [-1, 1]$, then the corresponding term of sum (2.14) containing the integral over $[-1, 1]$ goes to zero by Lemma 2.5. If, alternatively, γ_i is a circular arc, we can obviously guarantee that

$$2LM \cdot \frac{n^{(1-\beta)/2}}{n-1} \int_{\gamma_i} \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|} \le 2LM \cdot \frac{n^{(1-\beta)/2}}{n-1} \int_{\gamma_C} \frac{|dt|}{\left|t - \frac{n+1}{n-1}\right|},$$

where the last expression with the integral over the whole circle $\gamma_C \supset \gamma_i$ tends to zero according to Theorem 2.6. This ensures that the left-hand side with the integral over the arc goes to zero as well.

Thus, all terms of the finite sum (2.14) approach zero as $n \to \infty$, and hence the same is true for the entire sum. This immediately implies the convergence $|\langle R_n f, g\rangle| \to 0$, and therefore (2.13). □

References

[1] R. Avendaño and N. Krupnik. On the connection between the norm and the essential norm of the singular operator. *Doklady Rasshirennyh Zasedaniy Seminara Instituta Prikladnoy Matematiki im. I. N. Vekua*, 3:5–8, 1988 (in Russian).

[2] M. Bertola, A. Katsevich, and A. Tovbis. On the spectral properties of the Hilbert transform operator on multi-intervals. *J. Spectr. Theory*, 12(2):339–379, 2022.

[3] I. Gohberg and N. Krupnik. *One-dimensional linear singular integral equations. I Introduction*, volume 53 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1992.

[4] I. Gohberg and N. Krupnik. *One-dimensional linear singular integral equations. II General theory and applications*, volume 54 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1992.

[5] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015.

[6] O. Karlovych and E. Shargorodsky. On the essential norms of singular integral operators with constant coefficients and of the backward shift. *Proc. Amer. Math. Soc. Ser. B*, 9:60–70, 2022.

[7] O. Karlovych and E. Shargorodsky. Discrete Riesz transforms on rearrangement-invariant Banach sequence spaces and maximally noncompact operators. *Pure and Applied Functional Analysis*, 2023 (to appear).

[8] N. Krupnik. *Banach algebras with symbol and singular integral operators*, volume 26 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1987.

[9] N. Krupnik. Survey on the best constants in the theory of one-dimensional singular integral operators. In *Topics in operator theory. Volume 1. Operators, matrices and analytic functions*, volume 202 of *Operator Theory: Advances and Applications*, pages 365–393. Birkhäuser Verlag, Basel, 2010.

[10] S. Roch, P. A. Santos, and B. Silbermann. *Non-commutative Gelfand theories. A tool-kit for operator theorists and numerical analysts*. Universitext. Springer-Verlag London, Ltd., London, 2011.

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