A location invariant probability weighted moment EVI-estimator*

Frederico Caeiro, M. Ivette Gomes
and
Lígia Henriques-Rodrigues

Abstract

The *peaks over random threshold* (PORT) methodology and the *Pareto probability weighted moments* (PPWM) of largest observations are used to build a class of location-invariant estimators of the *extreme value index* (EVI), the primary parameter in *statistics of extremes.* The asymptotic behaviour of such a class of EVI-estimators, the so-called PORT PPWM EVI-estimators, is derived, and an alternative class of location-invariant EVI-estimators, the *generalized Pareto probability weighted moments* (GPPWM) EVI-estimators is considered as an alternative. These two classes of estimators, the PORT-PPWM and the GPPWM, jointly with the classical Hill EVI-estimator and a recent class of minimum variance reduced bias estimators are compared for finite samples, through a large-scale Monte-Carlo simulation study. An adaptive choice of the tuning parameters under play is put forward and applied to simulated and real data sets.

AMS 2000 subject classification. Primary 62G32, 62E20; Secondary 65C05.

Keywords. Adaptive semi-parametric estimation; asymptotic properties; bootstrap methodology; extreme value index; heavy tails; location/scale invariant estimation; Monte Carlo simulation; statistics of extremes.

*Research partially supported by National Funds through **FCT** — Fundação para a Ciência e a Tecnologia, projects PEst-OE/MAT/UI0006/2011, PEst-OE/MAT/UI0297/2011, EXTREMA project, PTDC/FEDER and grant SFRH/BPD/77319/2011.

1 Introduction, preliminaries and scope of the paper

The *extreme value index* (EVI) is the parameter $\gamma \in \mathbb{R}$ in the general *extreme value* distribution function (d.f.)

$$\mathrm{EV}_\gamma(x) := \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), & 1+\gamma x > 0, \quad \text{if} \quad \gamma \neq 0, \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \quad \text{if} \quad \gamma = 0. \end{cases} \tag{1}$$

The EV_γ d.f. appears as the limiting d.f., whenever such a non-degenerate limit exists, of the suitably linearly normalised maximum $X_{n:n}$, of any random sample, $\underline{X}_n := (X_1, \ldots, X_n)$, from either independent, identically distributed (i.i.d.) or even stationary weakly dependent random variables (r.v.'s) from an underlying model F. When such a non-degenerate limit exists, we say that F is in the *max-domain of attraction* of the d.f. in (1) and use the notation $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\gamma)$. Let us further denote by $(X_{1:n} \le \cdots \le X_{n:n})$ the sample of ascending order statistics (o.s.'s) associated to the available random sample $\underline{X}_n$.

1.1 Heavy tails and first-order conditions

We shall now consider heavy right-tails, i.e. a positive EVI, in (1). Then, as first proved by Gnedenko (1943), the right-tail function is of regular variation with an index of regular variation equal to $-1/\gamma$, i.e.

$$F \in \mathcal{D}_\mathcal{M}(EV_\gamma)_{\gamma>0} =: \mathcal{D}_\mathcal{M}^+ \equiv \mathcal{D}_{\mathcal{M}|1}^+ \quad \Longleftrightarrow \quad \overline{F} := 1 - F \in \mathrm{RV}_{-1/\gamma}, \tag{2}$$

where the notation RV_a stands for the class of *regularly varying* functions at infinity with an *index of regular variation* equal to $a \in \mathbb{R}$, i.e. positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^a$, for all $x > 0$.

With the notation

$$U(t) := F^{\leftarrow}(1 - 1/t),\ t \ge 1 \qquad \text{and} \qquad F^{\leftarrow}(y) := \inf\{x : F(x) \ge y\}, \tag{3}$$

for the generalized inverse function of the d.f. F, condition (2) is equivalent to saying that $U \in \mathrm{RV}_\gamma$ (de Haan, 1984), i.e. we often assume the validity of the so-called first-order condition,

$$F \in \mathcal{D}_\mathcal{M}^+ \quad \Longleftrightarrow \quad \overline{F} \in \mathrm{RV}_{-1/\gamma} \quad \Longleftrightarrow \quad U \in \mathrm{RV}_\gamma. \tag{4}$$

1.2 The estimators under study and a second-order condition

One of the first classes of semi-parametric estimators of a positive EVI was considered in Hill (1975). Hill's estimators are based on the log-excesses over an o.s. $X_{n-k:n}$, and have the functional form

$$\hat{\gamma}^{\mathrm{H}}_{k,n} \equiv \hat{\gamma}^{\mathrm{H}}_{k,n}(\underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k}\left\{\ln X_{n-i+1:n} - \ln X_{n-k:n}\right\}, \quad k = 1, 2, \ldots, n-1. \tag{5}$$

Consistency is achieved in the whole $\mathcal{D}^+_{\mathcal{M}}$ provided that $X_{n-k:n}$ is an *intermediate* o.s., i.e. we need to have

$$k = k_n \to \infty \quad \text{and} \quad k/n \to 0, \text{ as } n \to \infty. \tag{6}$$

We shall also consider the *Pareto probability weighted moments* (PPWM) EVI-estimators, recently introduced in Caeiro and Gomes (2011) and revisited in Caeiro *et al.* (2012). They are valid for heavy right-tails with $\gamma < 1$, compare favourably with the Hill estimator, in (5), for a wide variety of underlying models F, and are given by

$$\hat{\gamma}^{\mathrm{PPWM}}_{k,n} \equiv \hat{\gamma}^{\mathrm{PPWM}}_{k,n}(\underline{X}_n) := 1 - \frac{\hat{a}_1(k;\underline{X}_n)}{\hat{a}_0(k;\underline{X}_n) - \hat{a}_1(k;\underline{X}_n)}, \tag{7}$$

with

$$\hat{a}_r(k;\underline{X}_n) \equiv \hat{a}_r(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^r \frac{X_{n-i+1:n}}{X_{n-k:n}}, \qquad r = 0, 1.$$

Again, consistency is achieved under the first-order framework in (4) and intermediate k-values, i.e. whenever (6) holds.

In order to derive the asymptotic normality of the estimators either in (5) or in (7), it is convenient to slightly restrict the class $\mathcal{D}^+_{\mathcal{M}|1}$, assuming the validity of a second-order condition either on $\overline{F}$, in (2), or on U, in (3). We then guarantee the existence of a function $A(t)$, going to zero as $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{8}$$

where $\rho \le 0$ is a second-order parameter, which measures the rate of convergence in the first-order condition, in (4). For such a class of models, we use the notation $\mathcal{D}^+_{\mathcal{M}|2}$. If the limit in the left hand side of Eq. (8) exists, it is necessarily of the above mentioned type and $|A| \in \mathrm{RV}_\rho$ (Geluk and de Haan, 1987). If we assume the validity of the second-order framework in (8), the aforementioned EVI-estimators are asymptotically normal, provided that $\sqrt{k}A(n/k) \to \lambda_{_A}$, finite, as $n \to \infty$, with A given in (8). Indeed, if we denote $\hat{\gamma}^{\bullet}_{k,n}$, either the Hill estimator in (5) or the PPWM estimator in (7), we have, with $Z^{\bullet}_k$ asymptotically standard normal and for adequate $(b_{\bullet},\ \sigma_{\bullet}) \in (\mathbb{R},\ \mathbb{R}^+)$, the validity of the asymptotic distributional representation

$$\hat{\gamma}^{\bullet}_{k,n} \overset{d}{=} \gamma + \sigma_{\bullet} Z^{\bullet}_k/\sqrt{k} + b_{\bullet}\ A(n/k)(1 + o_p(1)), \quad \text{as} \quad n \to \infty. \tag{9}$$

In this paper, we shall often further assume that $\rho < 0$, in (8), and we shall use the following parameterization in $(\gamma, \beta, \rho) \in (\mathbb{R}^+, \mathbb{R} - \{0\}, \mathbb{R}^-)$,

$$A(t) =: \gamma\beta t^{\rho} \tag{10}$$

for the function A, in (8). This is equivalent to say that

$$U(t) = C\Big(1 + \gamma\beta t^{\rho}/\rho + o(t^{\rho})\Big), \quad \text{as } t \to \infty.$$

Further note that the classes of estimators, either in (5) or in (7), are scale-invariant but not location-invariant, as often desired, and this contrarily to the PORT-Hill estimators, introduced in Araújo Santos *et al.* (2006) and further studied in Gomes *et al.* (2008a), with PORT standing for *peaks over random thresholds*. The class of PORT-Hill estimators is based on a *sample of excesses* over a random threshold $X_{n_q:n}$, $n_q := \lfloor nq \rfloor + 1$, where $\lfloor x \rfloor$ denotes, as usual, the integer part of x, i.e. it is based on

$$\underline{X}^{(q)}_n := \big(X_{n:n} - X_{n_q:n}, \ldots, X_{n_q+1:n} - X_{n_q:n}\big). \tag{11}$$

We can have $0 < q < 1$, for d.f.'s with finite or infinite left endpoint $x_{_F} := \inf\{x : F(x) > 0\}$ (*the random threshold is then any empirical quantile*), and $0 \le q < 1$, for d.f.'s with finite left endpoint $x_{_F}$ (*the random threshold can also be the minimum*). Other results on PORT EVI-estimation can be found in Fraga Alves *et al.* (2009) and Gomes *et al.* (2011, 2012a, 2013). Such a methodology leads to location-invariant estimation, and the unshifted model F_0, underlying the r.v. X_0, plays thus a central and prominent role. In what follows, we use the notation $\chi_q = F_0^{\leftarrow}(q)$ for the q-quantile of the d.f. F_0. Then (see van der Vaart, 1998, p.308, among others), and for $X \frown F_0$,

$$X_{n_q:n} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \chi_q = F_0^{\leftarrow}(q) = U_0(1/(1-q)) \quad \text{for} \quad 0 \le q < 1 \quad \big(F_0^{\leftarrow}(0) = x_{_F}\big). \tag{12}$$

In this article, just as already initiated in Gomes *et al.* (2012a), we consider the application of the PORT methodology to the PPWM EVI-estimators, in (7), deriving the so-called

PORT-PPWM estimators. They have the same functional form of the PPWM estimators in (7), but with the original sample $\underline{X}_n$ replaced everywhere by the sample of excesses $\underline{X}_n^{(q)}$, in (11). Consequently, such estimators are given by the functional equation,

$$\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q} := \hat{\gamma}_{k,n}^{\mathrm{PPWM}}(\underline{X}_n^{(q)}), \tag{13}$$

with $\hat{\gamma}_{k,n}^{\mathrm{PPWM}}(\underline{X}_n)$ and $\underline{X}_n^{(q)}$ given in (7) and (11), respectively. These estimators are now invariant for both changes of location and scale, and depend on this *tuning parameter* q, which only influences the asymptotic bias of $\hat{\gamma}_{k,n}^{\mathrm{PPWM}}$, in (7), making this new class highly flexible, and able to compare favourably with the *generalized Pareto probability weighted moment* estimators (GPPWM), for a large variety of underlying models F in the max-domain of attraction of the EV d.f., in (1). These GPPWM EVI-estimators have been studied in de Haan and Ferreira (2006), are also scale and location invariant and are given by

$$\hat{\gamma}_{k,n}^{\mathrm{GPPWM}} = 1 - \frac{2\hat{a}_1^\star(k)}{\hat{a}_0^\star(k) - 2\hat{a}_1^\star(k)}, \tag{14}$$

with $k = 1, 2, \ldots, n-1$, $\gamma < 1$, and

$$\hat{a}_r^\star(k) \equiv \hat{a}_r^\star(k; \underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k} \left(\frac{i}{k}\right)^r (X_{n-i+1:n} - X_{n-k:n}), \quad r = 0, 1.$$

See also Caeiro and Gomes (2011), for an asymptotic comparison at optimal levels of the PPWM and GPPWM EVI-estimators, in (7) and (14), respectively.

1.3 Scope of the article

In Section 2, the asymptotic properties of the PORT-PPWM EVI-estimators, in (13), are derived. In Section 3, to obtain the behaviour of these estimators for finite samples, a large-scale Monte-Carlo simulation is performed and some overall conclusions are drawn. Section 4 is dedicated to a bootstrap data-driven choice of the *tuning parameters* under play. After a brief review of the role of the bootstrap methodology in the estimation of optimal sample fractions, we provide a double-bootstrap algorithm for the adaptive EVI-estimation through the PORT-PPWM estimators, also valid, with slight modifications, for the Hill and the GPPWM EVI-estimators. We further computationally validate this bootstrap data-driven estimation algorithm, and we apply it to simulated EV and Student's-t samples and to a real data set in the field of finance.

2 Asymptotic properties of the EVI-estimators

We first state well-known results in the field of statistics of extremes, related to the asymptotic behaviour of the Hill and the PPWM EVI-estimators. The notation $\mathcal{N}(\mu, \sigma^2)$ is used for a normal r.v. with mean μ and variance σ^2.

Theorem 1 (de Haan and Peng, 1998). *Under the second-order framework in* (8)*, i.e. whenever working in* $\mathcal{D}^+_{\mathcal{M}|2}$*, and for intermediate* k*, i.e. if* (6) *holds, the asymptotic distributional representation of* $\hat{\gamma}^{\rm H}_{k,n}$*, in* (5)*, is given in* (9)*, with*

$$\sigma^2_{\rm H} = \gamma^2, \qquad b_{\rm H} = \frac{1}{1-\rho}$$

and $Z^{\rm H}_k = \sqrt{k}\left(\sum_{i=1}^k E_i/k - 1\right)$*, with* $\{E_i\}$ *i.i.d. standard exponential r.v.'s. Consequently, if we choose* k *such that* $\sqrt{k}\,A(n/k) \to \lambda_A$*, finite and not necessarily null, then, as* $n \to \infty$*,*

$$\sqrt{k}(\hat{\gamma}^{\rm H}_{k,n} - \gamma) \stackrel{d}{\longrightarrow} \mathcal{N}\left(\lambda_A b_{\rm H}, \sigma^2_{\rm H}\right).$$

Theorem 2 (Caeiro and Gomes, 2011). *In* $\mathcal{D}^+_{\mathcal{M}|2}$*, i.e. under the second order framework in* (8)*, with* $0 < \gamma < 1/2$*, and for intermediate* k *such that* $\sqrt{k}\,A(n/k) \to \lambda_A$*, finite, we can guarantee the asymptotic normality of* $\hat{\gamma}^{\rm PPWM}_{k,n}$ *and* $\hat{\gamma}^{\rm GPPWM}_{k,n}$*, in* (7) *and* (14)*, respectively. Indeed, with* $\bullet$ *denoting either* PPWM *or* GPPWM*, the distributional representation in* (9) *holds, with*

$$\sigma^2_{\rm PPWM} := \frac{\gamma^2(1-\gamma)(2-\gamma)^2}{(1-2\gamma)(3-2\gamma)}, \quad b_{\rm PPWM} := \frac{(1-\gamma)(2-\gamma)}{(1-\gamma-\rho)(2-\gamma-\rho)},$$

and

$$\sigma^2_{\rm GPPWM} := \frac{(1-\gamma+2\gamma^2)(1-\gamma)(2-\gamma)^2}{(1-2\gamma)(3-2\gamma)}, \quad b_{\rm GPPWM} := \frac{(\gamma+\rho)\ b_{PPWM}}{\gamma}.$$

Consequently,

$$\sqrt{k}\left(\hat{\gamma}^{\bullet}_{k,n} - \gamma\right) \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} \mathcal{N}\left(\lambda_A b_{\bullet}, \sigma^2_{\bullet}\right).$$

Remark 1. *Note again that* $\sigma^2_{\rm H} < \sigma^2_{\rm PPWM}$ *for all* $\gamma > 0$*. The other way round,* $b_{\rm H} \geq b_{\rm PPWM}$ *for all* $\gamma < 1/2$*, with* $b_{\rm H} = b_{\rm PPWM}$ *only if* $\rho = 0$*. As can be seen in* Caeiro and Gomes (2011)*,* $\hat{\gamma}^{\rm PPWM}_{k,n}$ *can asymptotically outperform* $\hat{\gamma}^{\rm H}_{k,n}$ *at optimal levels in the sense of minimal root mean square error* (RMSE)*, in a wide and relevant region of the* (γ, ρ)*-plane.*

2.1 Asymptotic behaviour of PORT-PPWM EVI-estimators

Note first that if there is a possible shift s in the model, i.e. if the d.f. $F(x) \equiv F_s(x) = F(x;s)$ depends on (x,s) through the difference $x-s$, the parameter ρ, as well as the A-function, in (8), depend on such a shift s, i.e. $\rho = \rho_s$, $A = A_s$, and

$$(A_s(t), \rho_s) := \begin{cases} \big(\gamma s/U_0(t), -\gamma\big), & \text{if } \ \gamma + \rho_0 < 0 \ \wedge \ s \neq 0, \\ \big(A_0(t) + \gamma s/U_0(t), \rho_0\big), & \text{if } \ \gamma + \rho_0 = 0 \ \wedge \ s \neq 0, \\ \big(A_0(t), \rho_0\big), & \text{otherwise.} \end{cases}$$

To study the asymptotic properties of the PORT-PPWM EVI-estimators, it is convenient to study first the behaviour of the statistics,

$$Q_r(k;q) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^r \frac{X_{n-i+1:n} - X_{n_q:n}}{X_{n-k:n} - X_{n_q:n}}, \qquad r \geq 0, \tag{15}$$

for $X = X_0 \frown F_0$.

It is also worth noting that, as already detected in Fraga Alves *et al.* (2009) for invariant versions of the mixed moment EVI-estimator, due to the fact that $X_{\lfloor nq \rfloor+1:n} - U_0(1/(1-q))$ $=O_p(1/\sqrt{n})$, the EVI-estimator $\hat{\gamma}_{k,n}^{\mathrm{PPWM|q}}$, in (13), has exactly the same asymptotic behaviour of the estimator defined as $\hat{\gamma}_{k,n}^{\mathrm{PPWM}}$, in (7), but with $X_{n-i+1:n}$ replaced everywhere by $X_{n-i+1:n} - U_0(1/(1-q))$, $1 \leq i \leq n$. The same comment applies to $Q_r(k;q)$, in (15), where $X_{n_q:n}$ can be replaced by $\chi_q = U_0(1/(1-q))$, already defined in (12). The asymptotic behaviour of the statistics $Q_r(k;q)$, in (15), comes then straightforwardly from the behaviour of the non-shifted statistics, studied in Caeiro and Gomes (2011), as stated in the following proposition.

Proposition 1. *Under the second order framework in* (8)*, and for intermediate k, i.e. whenever* (6) *holds, we can guarantee the asymptotic normality of $Q_r(k;q)$, in* (15)*. Indeed, we can write, for $r > \gamma - 1/2$,*

$$Q_r(k;q) \stackrel{d}{=} \frac{1}{1+r-\gamma} + \frac{\sigma_r}{\sqrt{k}}W_k^{(r)} + \frac{A_0(n/k)(1+o_p(1))}{(1+r-\gamma)(1+r-\gamma-\rho_0)} + \frac{\gamma\chi_q(1+o_p(1))}{(1+r)(1+r-\gamma)U_0(n/k)}, \tag{16}$$

where $W_k^{(r)}$ is an asymptotically standard normal r.v., and

$$\sigma_r^2 := \frac{\gamma^2}{(1+r-\gamma)^2(1+2r-2\gamma)}. \tag{17}$$

Proof. Since $U_0(X_{i:n}) \stackrel{d}{=} Y_{i:n}$, where Y is a standard Pareto r.v., with d.f. $F_Y(y) = 1 - 1/y$, $y > 1$, $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$, under the second order framework in (8), and thinking on the fact that we are now working with $s = 0$ due to the location invariance property of the statistics in (15), we can write

$$\frac{X_{n-i+1:n}}{X_{n-k:n}} \stackrel{d}{=} \frac{U_0(\frac{Y_{n-i+1:n}}{Y_{n-k:n}} Y_{n-k:n})}{U_0(Y_{n-k:n})} \stackrel{d}{=} (Y_{k-i+1:k})^\gamma \left(1 + \tfrac{Y_{k-i+1:k}^\rho - 1}{\rho} A_0(Y_{n-k:n})(1 + o_p(1))\right).$$

Next, with the notation $\chi_q = U_0(1/(1-q))$, already introduced in (12),

$$\begin{aligned}
\frac{X_{n-i+1:n} - \chi_q}{X_{n-k:n} - \chi_q} &= \frac{X_{n-i+1:n}}{X_{n-k:n}} \left(\frac{1 - \chi_q/X_{n-i+1:n}}{1 - \chi_q/X_{n-k:n}}\right) \\
&= \frac{X_{n-i+1:n}}{X_{n-k:n}} \left(1 - \frac{\chi_q}{X_{n-i+1:n}}\right) \left(1 + \frac{\chi_q}{X_{n-k:n}}(1 + o_p(1))\right) \\
&= \frac{X_{n-i+1:n}}{X_{n-k:n}} \left(1 + \frac{\chi_q}{X_{n-k:n}} \left(1 - \frac{X_{n-k:n}}{X_{n-i+1:n}}\right)(1 + o_p(1))\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
Q_r(k;q) &\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} \left(\frac{i}{k}\right)^r (Y_{k-i+1:k})^\gamma \left(1 + \frac{\chi_q}{U_0(n/k)}\left(1 - Y_{k-i+1:k}^{-\gamma}\right)(1 + o_p(1))\right) \\
&\quad + \frac{1}{k}\sum_{i=1}^{k} \left(\frac{i}{k}\right)^r (Y_{k-i+1:k})^\gamma \frac{(Y_{k-i+1:k})^\rho - 1}{\rho} A_0(n/k)(1 + o_p(1)) \\
&\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} \left(1 - \frac{i}{k}\right)^r Y_{i:k}^\gamma + \frac{\chi_q}{U_0(n/k)} \frac{1}{k}\sum_{i=1}^{k} \left(1 - \frac{i}{k}\right)^r (Y_{i:k}^\gamma - 1)(1 + o_p(1)) \\
&\qquad + \frac{1}{k}\sum_{i=1}^{k} \left(1 - \frac{i}{k}\right)^r Y_{i:k}^\gamma \frac{Y_{i:k}^\rho - 1}{\rho} A_0(n/k)(1 + o_p(1)).
\end{aligned}$$

Since

$$\frac{1}{k}\sum_{i=1}^{k} \left(1 - \frac{i}{k}\right)^r Y_{i:k}^\gamma \; (Y_{i:k}^\rho - 1) \xrightarrow{\mathbb{P}} \frac{\rho}{(1 + r - \gamma)(1 + r - \gamma - \rho)},$$

equation (16) follows. Moreover, σ_r^2 is given in (17), a result already proved in Caeiro and Gomes (2011). ■

We next state the main theoretical result in this paper, related with the shift invariant version of the EVI-estimators in (7), i.e. the estimators in (13). The asymptotic variance of $\hat{\gamma}_{k,n}^{\mathrm{PPWM|q}}$ is kept at the same level of the PPWM-estimator $\hat{\gamma}_{k,n}^{\mathrm{PPWM}}$, but the dominant component of bias changes only in a few cases, as already detected in Araújo Santos *et al.* (2006) for PORT-Hill and PORT-moment EVI-estimators, and in Fraga Alves *et al.* (2009), for similar location-invariant versions of the mixed-moment EVI-estimators.

Theorem 3. *Under the second order framework in* (8), *with* $0 < \gamma < 1/2$, *and for intermediate* k, *i.e. if* (6) *holds, the asymptotic bias of the* PORT-PPWM EVI-*estimators, in* (13), *is going to be ruled by*

$$B(t) = \begin{cases} \gamma\chi_q/U_0(t), & \text{if } \gamma+\rho_0 < 0 \wedge \chi_q \neq 0, \\ A_0(t) + \gamma\chi_q/U_0(t), & \text{if } \gamma+\rho_0 = 0 \wedge \chi_q \neq 0, \\ A_0(t), & \text{otherwise,} \end{cases}$$

with χ_q *defined in* (12). *If we assume that* $\sqrt{k}\ A_0(n/k) \to \lambda_{_A}$ *and/or* $\sqrt{k}/U_0(n/k) \to \lambda_{_U}$, *finite, as* $n \to \infty$

$$\sqrt{k}\left(\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q} - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(b_{\mathrm{PPWM}|q}, \sigma^2_{\mathrm{PPWM}}\right),$$

where

$$b_{\mathrm{PPWM}|q} = \begin{cases} \frac{\gamma(1-\gamma)(2-\gamma)\chi_q}{2}\lambda_{_U}, & \text{if } \gamma+\rho_0 < 0 \wedge \chi_q \neq 0, \\ \frac{(1-\gamma)(2-\gamma)}{2}\ \lambda_{_A} + \frac{\gamma(1-\gamma)(2-\gamma)\chi_q}{2}\lambda_{_U}, & \text{if } \gamma+\rho_0 = 0, \\ \frac{(1-\gamma)(2-\gamma)}{(1-\gamma-\rho_0)(2-\gamma-\rho_0)}\ \lambda_{_A}, & \text{otherwise.} \end{cases}$$

Proof. We can write,

$$\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q} = 1 - \left(\frac{Q_0(k;q)}{Q_1(k;q)} - 1\right)^{-1}$$

with $Q_r(k;q)$ $r = 0, 1$, defined in (15). Using (16) with $r = 1$ and Taylor's expansion $(1+x)^{-1} = 1 - x + o(x)$, as $x \to 0$, we get

$$Q_1^{-1}(k;q) \overset{d}{=} (2-\gamma)\left\{1 - \frac{(2-\gamma)\sigma_1}{\sqrt{k}}W_k^{(1)} - \frac{A_0(n/k)}{2-\gamma-\rho_0}(1+o_p(1)) - \frac{\gamma\chi_q}{2U_0(n/k)}(1+o_p(1))\right\}.$$

Using again the previous Taylor expansion, and with $\overline{\sigma}_r := (1+r-\gamma)\sigma_r$, $r = 0, 1$,

$$\begin{aligned}\left(\frac{Q_0(k;q)}{Q_1(k;q)} - 1\right)^{-1} \overset{d}{=}\ & (1-\gamma)\Big\{1 - \frac{(2-\gamma)}{\sqrt{k}}\Big(\overline{\sigma}_0 W_k^{(0)} - \overline{\sigma}_1 W_k^{(1)}\Big) \\ & - \frac{(2-\gamma)A_0(n/k)}{(1-\gamma-\rho_0)(2-\gamma-\rho_0)}(1+o_p(1)) - \frac{\gamma(2-\gamma)\chi_q}{2U_0(n/k)}(1+o_p(1))\Big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q} \overset{d}{=}\ & \gamma - \frac{(1-\gamma)(2-\gamma)}{\sqrt{k}}\Big(\overline{\sigma}_0 W_k^{(0)} - \overline{\sigma}_1 W_k^{(1)}\Big) \\ & + \frac{(1-\gamma)(2-\gamma)A_0(n/k)}{(1-\gamma-\rho_0)(2-\gamma-\rho_0)}(1+o_p(1)) + \frac{\gamma(1-\gamma)(2-\gamma)\chi_q}{2U_0(n/k)}(1+o_p(1))\end{aligned}$$

and the result in the theorem follows. ∎

3 Simulated behaviour of the EVI estimators.

In this section, we have enlarged the multi-sample Monte-Carlo simulation in Gomes *et al.* (2012a), proceeding with the implementation of a multi-sample simulation experiment of size 5000×20, to obtain the distributional behaviour of the EVI estimators $\hat{\gamma}_{k,n}^{\rm H}$, $\hat{\gamma}_{k,n}^{\rm PPWM}$, $\hat{\gamma}_{k,n}^{{\rm PPWM}|q}$ and $\hat{\gamma}_{k,n}^{\rm GPPWM}$ in (5), (7), (13) and (14), respectively, for the following underlying parents in the scope of Theorem 3,

1. Student's-t_ν with $\nu = 3, 4$ ($\gamma = 1/\nu$, $\rho = -2/\nu$),

2. Fréchet(γ) parents X, with d.f. $F(x) = \exp(-x^{-1/\gamma})$, $x > 0$, $\gamma > 0$, with $\gamma = 0.25$ ($\rho = -1$). We have further considered the shifted Fréchet models $Y = X - 0.5$ and $Y = X - 1$, with $\rho = -\gamma = -0.25$,

3. Burr(γ, ρ) parents X, with d.f. $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x > 0$, now for $(\gamma, \rho) \in \{0.25\} \times \{-0.25, -0.5, -1, -2\}$. Similarly to what we have done for the Fréchet parents, we have now considered the shifted Burr models $Y = X - 0.5$ and $Y = X - 1$, also with $\rho = -\gamma = -0.25$,

4. EV(γ) parents, with d.f. in (1), with $\gamma = 0.25$ ($\rho = -\gamma = -0.25$).

We have further considered the following two parents with $\gamma = 1/2$, both out of the scope of Theorem 3:

5. Student's-t_ν with $\nu = 2$ ($\gamma = 0.5$, $\rho = -1$) degrees of freedom;

6. EV(γ) models with $\gamma = 0.5$ ($\rho = -\gamma = -0.5$).

For comparison, we also picture the same characteristics for a minimum-variance reduced-bias (MVRB) estimator, denoted CH, with CH standing for *corrected-Hill.* The MVRB estimator considered is the one introduced in Caeiro *et al.* (2005), given by

$$\widehat{\gamma}_{k,n}^{\rm CH}(\hat{\beta}, \hat{\rho}) := \hat{\gamma}_{k,n}^{\rm H}\big(1 - \hat{\beta}(n/k)^{\hat{\rho}}/(1 - \hat{\rho})\big), \tag{18}$$

with $(\hat{\beta}, \hat{\rho})$ adequate estimators of the vector of 'scale' and 'shape' second-order parameters (β, ρ) in (10). We have again used the class of β-estimators in Gomes and Martins (2002) and the simplest class of ρ-estimators in Fraga Alves *et al.* (2003). In the simulations, given a sample, $\underline{\mathbf{X}}_n$, and since all simulated models under study are such that $|\rho| \leq 1$, the case where alternatives to the Hill EVI-estimator are welcome due to the high bias of Hill's estimators for moderate up to large values of k, we shall essentially work with the *tuning* parameter $\tau = 0$ in the simplest class of ρ-estimators in Fraga Alves *et al.* (2003), given by

$$\hat{\rho}_\tau(k) \equiv \hat{\rho}_\tau(k; \underline{\mathbf{X}}_n) := \min\left(0, \ \frac{3(W_n^{(\tau)}(k; \underline{\mathbf{X}}_n) - 1)}{W_n^{(\tau)}(k; \underline{\mathbf{X}}_n) - 3}\right), \tag{19}$$

and dependent on the statistics

$$W_n^{(\tau)}(k; \underline{\mathbf{X}}_n) := \begin{cases} \dfrac{\left(M_n^{(1)}(k;\underline{\mathbf{X}}_n)\right)^\tau - \left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)^{\tau/2} - \left(M_n^{(3)}(k;\underline{\mathbf{X}}_n)/6\right)^{\tau/3}}, & \text{if} \quad \tau \neq 0, \\[2ex] \dfrac{\ln\left(M_n^{(1)}(k;\underline{\mathbf{X}}_n)\right) - \frac{1}{2}\ln\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)}{\frac{1}{2}\ln\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right) - \frac{1}{3}\ln\left(M_n^{(3)}(k;\underline{\mathbf{X}}_n)/6\right)}, & \text{if} \quad \tau = 0 \end{cases}$$

where

$$M_n^{(j)}(k; \underline{\mathbf{X}}_n) := \frac{1}{k}\sum_{i=1}^{k} \left\{\ln X_{n-i+1:n} - \ln X_{n-k:n}\right\}^j, \quad j = 1, 2, 3.$$

As already suggested in previous papers, we have here decided for the computation of $\hat{\rho}_\tau(k)$ at $k = k_1$, given by

$$k_1 = \lfloor n^{1-\epsilon} \rfloor, \quad \epsilon = 0.001, \tag{20}$$

the threshold used in Caeiro *et al.* (2005) and Gomes and Pestana (2007). Interesting alternative classes of ρ-estimators have recently been introduced in Goegebeur *et al.* (2008, 2010), Ciuperca and Mercadier (2010) and Caeiro and Gomes (2012a,b).

For the estimation of the scale second-order parameter β, in (10), and again on the basis of a sample $\underline{\mathbf{X}}_n$, we shall here consider

$$\hat{\beta}_{\hat{\rho}}(k) \equiv \hat{\beta}_{\hat{\rho}}(k; \underline{\mathbf{X}}_n) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{d_{\hat{\rho}}(k)\ D_0(k) - D_{\hat{\rho}}(k)}{d_{\hat{\rho}}(k)\ D_{\hat{\rho}}(k) - D_{2\hat{\rho}}(k)}, \tag{21}$$

dependent on the estimator $\hat{\rho} = \hat{\rho}_0(k_1; \underline{\mathbf{X}}_n)$, suggested before and where, for any $\alpha \leq 0$,

$$d_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} \quad \text{and} \quad D_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} U_i, \quad U_i := i\left(\ln\frac{X_{n-i+1:n}}{X_{n-i:n}}\right),$$

with $U_i,\ 1 \leq i \leq k$, the *scaled log-spacings* associated with $\underline{\mathbf{X}}_n$. Details on the distributional behaviour of the estimator in (21) can be found in Gomes and Martins (2002) and more recently in Gomes *et al.* (2008b) and Caeiro *et al.* (2009). Alternative estimators of β can be found in Caeiro and Gomes (2006), Gomes *et al.* (2010) and Caeiro and Gomes (2012c).

For other possible reliable estimation methods of (β, ρ), see the algorithms in Gomes and Pestana (2007) and in Gomes *et al.* (2008b). Recent overviews of statistics of univariate extremes can be found in Beirlant *et al.* (2012) and Scarrot and McDonald (2012).

3.1 Simulated mean values and RMSEs of the EVI-estimators

To illustrate the finite sample behaviour of the EVI estimators as functions of k, the number of top o.s.'s used, we next present the simulated mean value (E) and RMSE patterns of $\hat{\gamma}_{k,n}^{\mathrm{H}}$, $\hat{\gamma}_{k,n}^{\mathrm{PPWM}}$, $\hat{\gamma}_{k,n}^{\mathrm{GPPWM}}$, $\widehat{\gamma}_{k,n}^{\mathrm{CH}}$ and $\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q}$, for a few values of q, usually only the one leading to minimal RMSE among the values considered from $q = 0$ until $q = 0.4$, with step 0.05. The illustration is done for a sample size $n = 1000$ and for some of the aforementioned models. For simplicity, we shall denote the EVI estimators in (5), (7) and (18), respectively by H(0), P(0) and CH(0), whenever dealing with an unshifted model ($s = 0$) and by H(s), P(s) and CH(s), whenever dealing with a s-shifted parent. The PORT-PPWM EVI-estimators in (13) are location invariant, i.e. independent on any shift s imposed to the data, but depend on the tuning parameter q and will be generally denoted by P$|q$. The location-invariant GPPWM EVI-estimators in (14) will be denoted GP.

For models with an infinite left endpoint, like the Student-t_ν, and particularly when ν is small, we should pay special attention to the choice of q. Indeed, when q approaches 0, and surely due to a closeness to inconsistency, the minimum RMSE is attained at k close to $n-1$. Figure 1 is related to a Student-t_3 underlying parent. It is clear the non-consistency of P$|0$, the PORT-PPWM estimator associated to a shift induced by the minimum of the sample. For $q = 0.05$, we are led to a minimum RMSE reasonably close to $k = n - 1$, and with a pattern reasonably above the RMSE of the PPWM EVI-estimator for moderate k-values, including the one that leads to the minimum RMSE of the PPWM EVI-estimator. Even for $q = 0.1$, we are led to the same type of RMSE pattern, as a function of k. We would thus advise the choice of P$|0.2$. Then, at the optimal level for the PPWM EVI-estimation, the RMSE of PPWM$|0.2$ is below the RMSE of PPWM. A similar comment applies to all other simulated Student-t_ν random samples. The value of q depends on ν, as can be seen in Figure 2, related to a Student t_4 parent. Here the best performance is achieved by P$|0.1$ if we take into account the RMSE criterion, but the highest relative efficiency (REFF) is obtained throuhgh P$|0.05$. Figure 3 is related to a Student-t_2 underlying parent, a model out of the scope of Theorem 3. Then, even for $q = 0.1$, we are led to a minimum RMSE quite close to $k = n - 1$. Following the same kind of reasoning, we would thus advise the choice of P|0.4, despite of the fact that then the PORT-PPWM does not beat the CH EVI-estimator.

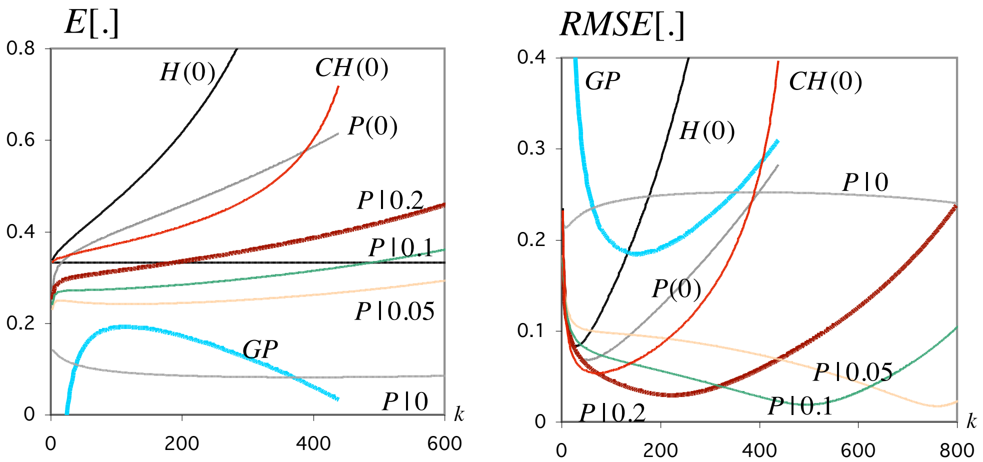

Figure 1: E and RMSE patterns of the EVI-estimators under study for a Student-t_ν underlying parent with $\nu = 3$ ($\gamma = 1/\nu = 0.333, \rho = -2/\nu = -0.666$).

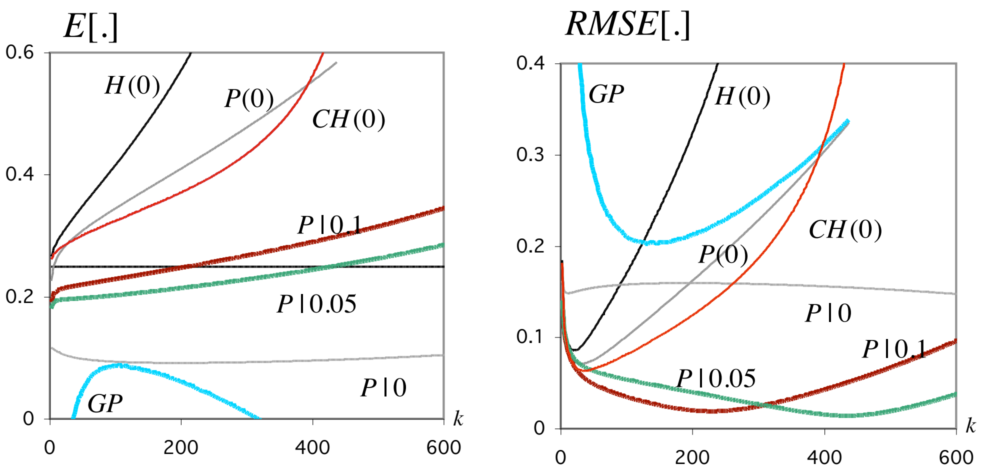

Figure 2: E and RMSE patterns of the EVI-estimators under study for a Student-t_ν underlying parent with $\nu = 4$ ($\gamma = 1/\nu = 0.25, \rho = -2/\nu = -0.5$).

We next present Figures 4 and 5, with the sample paths of the different EVI-estimators, when the underlying parent is an EV(γ), with $\gamma = 0.25$ and $\gamma = 0.5$, respectively. Despite

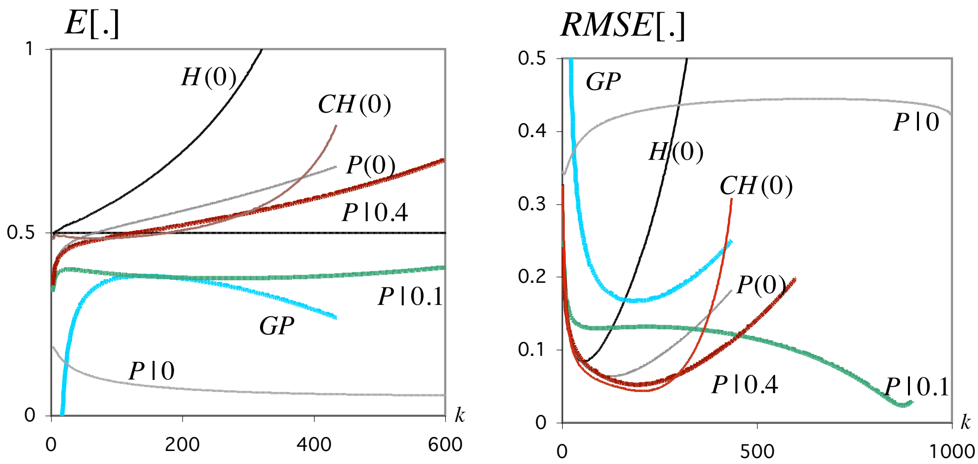

Figure 3: E and RMSE patterns of the EVI-estimators under study for a Student-t_ν underlying parent with $\nu = 2$ $(\gamma = 1/\nu = 0.5, \rho = -2/\nu = -1)$.

of the fact that this second model is out of scope of Theorem 3, the best performance has in both cases been achieved by P|0.

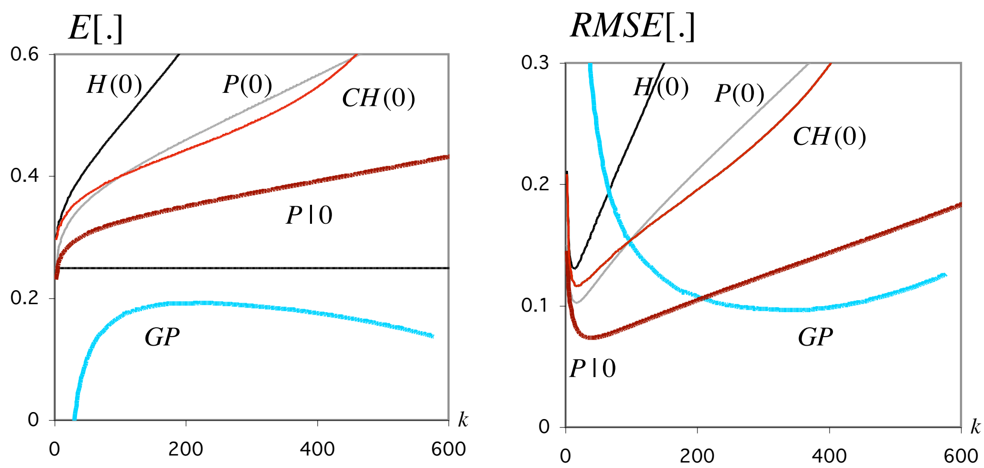

Figure 4: E and RMSE patterns of the EVI-estimators under study for an EV(γ) underlying parent with $\gamma = 0.25$ $(\rho = -\gamma = -0.25)$.

In Figure 6, we present sample paths associated to unshifted and shifted $(s = -1)$ Fréchet parents with $\gamma = 0.25$. Note that when we are dealing with the unshifted parent, and due to the fact that the support is $[0, \infty)$, we cannot improve the performance of the PPWM EVI-estimator when we use the PORT-PPWM methodology. But when we have a shifted

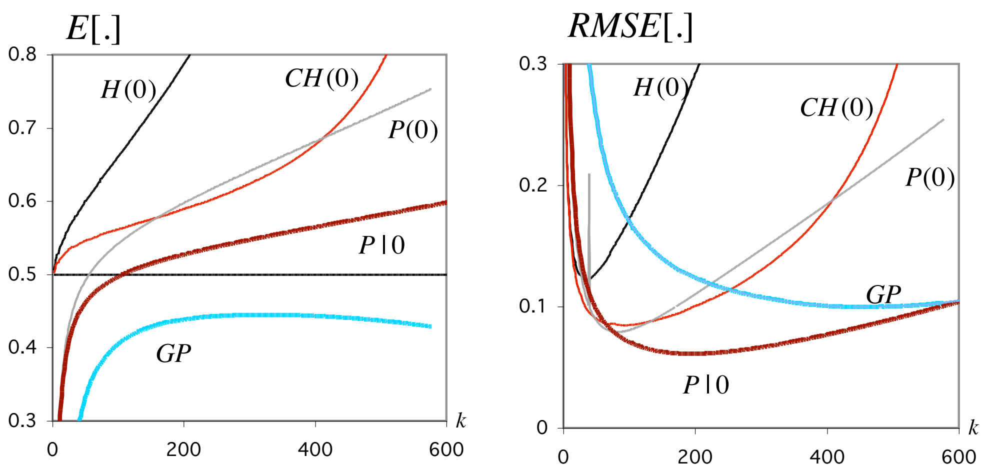

Figure 5: E and RMSE patterns of the EVI-estimators under study for an EV(γ) underlying parent with $\gamma = 0.5$ ($\rho = -\gamma = -0.5$).

Fréchet parent the location-invariant PPWM(0) EVI-estimator outperforms not only the Hill but also the CH EVI-estimators.

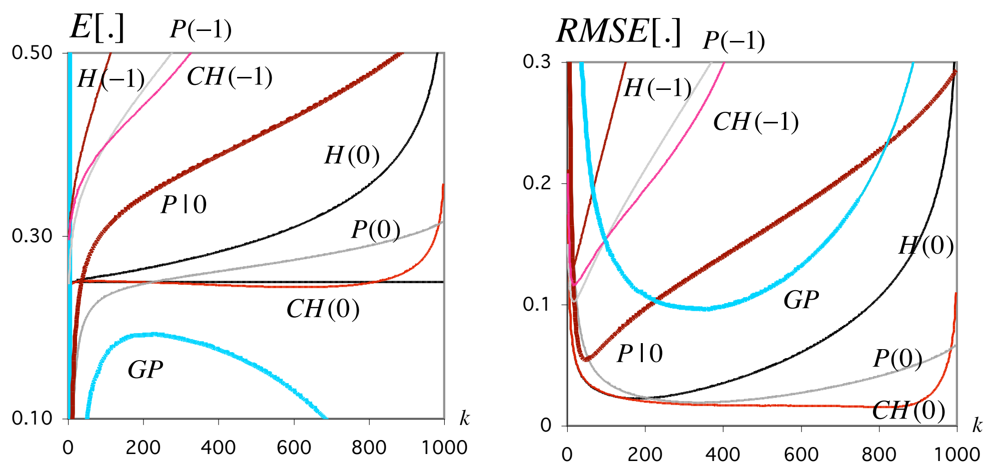

Figure 6: E and RMSE patterns of the EVI-estimators under study for unshifted ($s = 0$) and shifted ($s = 1$) Fréchet underlying parent with $\gamma = 0.25$

Among the Burr(γ, ρ), we present Figure 7 where $\rho = -\gamma = -0.25$, to show that for these models the PPWM estimators are never able to beat the GPPWM, even when we have shifted Burr data. However, this does not happen when we have $\gamma \neq -\rho$, as can be seen in Figure 8.

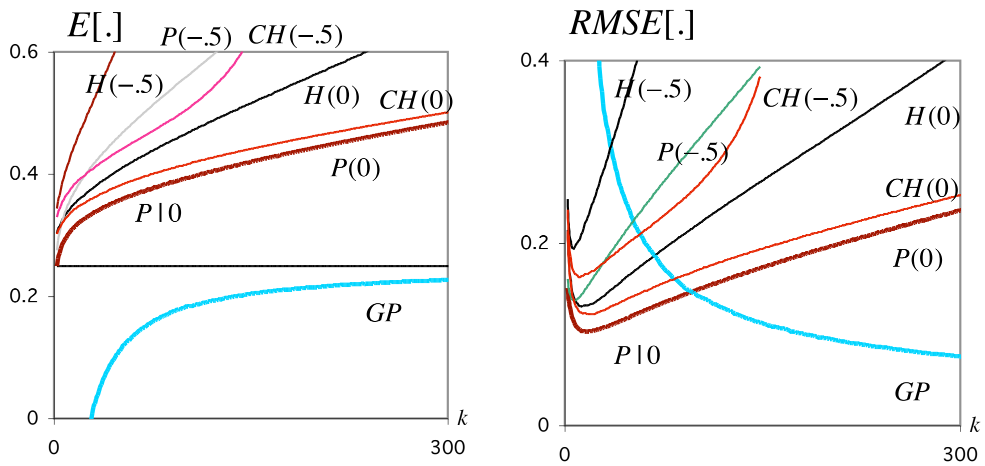

Figure 7: E and RMSE patterns of the EVI-estimators under study for a Burr(γ, ρ) underlying parent with $(\gamma, \rho) = (0.25, -0.25)$.

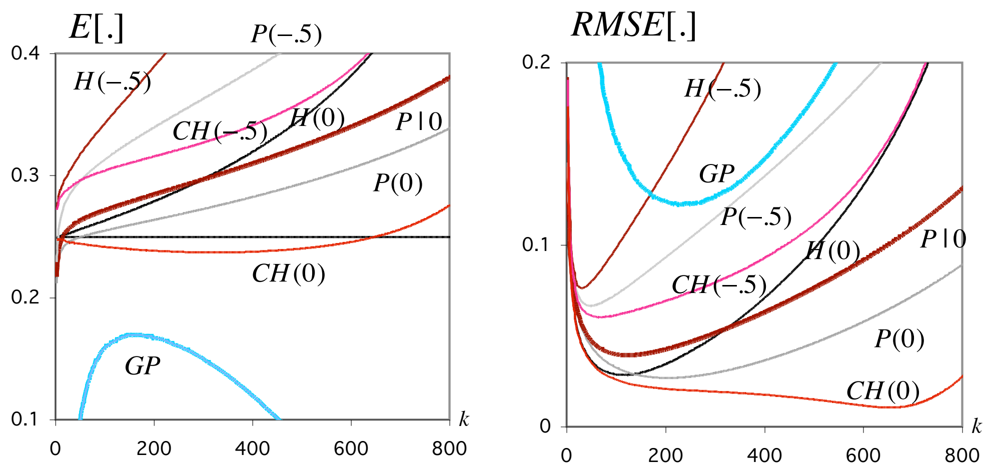

Figure 8: E and RMSE patterns of the EVI-estimators under study for a Burr(γ, ρ) underlying parent with $(\gamma, \rho) = (0.25, -1)$.

3.1.1 Finite sample behaviour of the EVI-estimators at simulated optimal levels

In Table 1, we present for the EV parents the simulated mean values of the above mentioned EVI-estimators, generally denoted $\mathrm{E}_{k,n}$, at their simulated optimal levels $k_{0|\mathrm{E}} := \arg\min_k \mathrm{RMSE}(\mathrm{E}_{k,n})$. In Table 2, we present, or each model and up to the second last row, the simulated relative efficiencies (REFF) of $\mathrm{E}_{k,n}$, comparatively with the Hill estimator, whenever computed at their simulated optimal levels, i.e., the simulated values of

$$\mathrm{REFF}_{\mathrm{E}_0|H_0} := \frac{\mathrm{RMSE}\big(\mathrm{H}_{k_{0|\mathrm{H}},n}\big)}{\mathrm{RMSE}\big(\mathrm{E}_{k_{0|\mathrm{E}},n}\big)} =: \frac{\mathrm{RMSE}\big(\mathrm{H}_0\big)}{\mathrm{RMSE}\big(\mathrm{E}_0\big)}.$$

In the last row, we present the simulated value of $\text{RMSE}(\text{H}_0)$, so that we can easily recover the RMSE of the other estimators. In all tables the "best" values are written in **bold**.

Table 1: Simulated mean values of the semi-parametric EVI-estimators under consideration, at their simulated optimal levels for underlying EV parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
			EV(γ) **parent,** $\gamma = 0.25$			
H	0.427 ± 0.0012	0.392 ± 0.0026	0.365 ± 0.0019	0.348 ± 0.0012	0.335 ± 0.0013	0.321 ± 0.0010
PPWM	0.344 ± 0.0008	0.331 ± 0.0013	0.323 ± 0.0009	0.318 ± 0.0008	0.313 ± 0.0007	0.305 ± 0.0006
GPPWM	0.018 ± 0.0027	0.102 ± 0.0015	0.160 ± 0.0007	0.186 ± 0.0008	0.203 ± 0.0009	$\underline{\mathbf{0.219}} \pm 0.0004$
PPWM|0	$\underline{\mathbf{0.322}} \pm 0.0013$	$\underline{\mathbf{0.315}} \pm 0.0008$	$\underline{\mathbf{0.308}} \pm 0.0005$	$\underline{\mathbf{0.302}} \pm 0.0006$	$\underline{\mathbf{0.297}} \pm 0.0005$	0.289 ± 0.0003
PPWM|0.1	0.329 ± 0.0016	0.323 ± 0.0011	0.316 ± 0.0008	0.311 ± 0.0007	0.306 ± 0.0005	0.299 ± 0.0005
CH	0.382 ± 0.0027	0.372 ± 0.0021	0.353 ± 0.0014	0.342 ± 0.0017	0.330 ± 0.0008	0.317 ± 0.0008
			EV(γ) **parent,** $\gamma = 0.5$			
H	0.654 ± 0.0032	0.624 ± 0.0033	0.596 ± 0.0011	0.579 ± 0.0011	0.565 ± 0.0010	0.551 ± 0.0010
PPWM	0.553 ± 0.0009	0.549 ± 0.0009	0.545 ± 0.0005	0.542 ± 0.0006	0.539 ± 0.0005	0.535 ± 0.0004
GPPWM	0.282 ± 0.0035	0.371 ± 0.0010	0.419 ± 0.0007	0.441 ± 0.0006	0.456 ± 0.0005	0.469 ± 0.0003
PPWM|0	$\underline{\mathbf{0.544}} \pm 0.0008$	$\underline{\mathbf{0.540}} \pm 0.0006$	$\underline{\mathbf{0.536}} \pm 0.0004$	$\underline{\mathbf{0.533}} \pm 0.0005$	$\underline{\mathbf{0.530}} \pm 0.0004$	$\underline{\mathbf{0.525}} \pm 0.0003$
PPWM|0.1	0.548 ± 0.0008	0.545 ± 0.0005	0.541 ± 0.0005	0.538 ± 0.0005	0.535 ± 0.0005	0.531 ± 0.0003
CH	0.637 ± 0.0037	0.619 ± 0.0032	0.595 ± 0.0021	0.579 ± 0.0020	0.565 ± 0.0011	0.551 ± 0.0010

Table 2: Simulated values of the $\text{REFF}_{\bullet|\text{H}}$ (from first to sixth row of each entry) and $\text{RMSE}(\text{H}_0)$ for underlying EV parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
			EV(γ) **parent,** $\gamma = 0.25$			
PPWM	1.550 ± 0.0069	1.435 ± 0.0063	1.321 ± 0.0051	1.262 ± 0.0050	1.212 ± 0.0050	1.172 ± 0.0031
GPPWM	0.779 ± 0.0110	0.988 ± 0.0070	1.207 ± 0.0075	1.382 ± 0.0081	1.565 ± 0.0094	$\underline{\mathbf{1.833}} \pm 0.0124$
PPWM|0	$\underline{\mathbf{2.020}} \pm 0.0082$	$\underline{\mathbf{1.909}} \pm 0.0077$	$\underline{\mathbf{1.817}} \pm 0.0062$	$\underline{\mathbf{1.774}} \pm 0.0066$	$\underline{\mathbf{1.741}} \pm 0.0057$	1.721 ± 0.0060
PPWM|0.1	1.778 ± 0.0073	1.645 ± 0.0062	1.521 ± 0.0049	1.455 ± 0.0061	1.402 ± 0.0048	1.358 ± 0.0034
CH	1.328 ± 0.0108	1.237 ± 0.0056	1.171 ± 0.0042	1.130 ± 0.0031	1.101 ± 0.0021	1.072 ± 0.0020
RMSE(H_0)	0.246 ± 0.3905	0.200 ± 0.3126	0.157 ± 0.2504	0.133 ± 0.2150	0.113 ± 0.1865	0.092 ± 0.1557
			EV(γ) **parent,** $\gamma = 0.5$			
PPWM	1.985 ± 0.0081	1.757 ± 0.0078	1.526 ± 0.0079	1.395 ± 0.0078	1.289 ± 0.0078	1.188 ± 0.0079
GPPWM	0.836 ± 0.0124	1.024 ± 0.0081	1.154 ± 0.0067	1.219 ± 0.0062	1.278 ± 0.0080	1.351 ± 0.0087
PPWM|0	$\underline{\mathbf{2.416}} \pm 0.0110$	$\underline{\mathbf{2.193}} \pm 0.0108$	$\underline{\mathbf{1.976}} \pm 0.0111$	$\underline{\mathbf{1.862}} \pm 0.0123$	$\underline{\mathbf{1.772}} \pm 0.0128$	$\underline{\mathbf{1.700}} \pm 0.0102$
PPWM|0.1	2.208 ± 0.0091	1.965 ± 0.0091	1.722 ± 0.0092	1.586 ± 0.0100	1.475 ± 0.0095	1.373 ± 0.0080
CH	1.492 ± 0.0258	1.501 ± 0.0097	1.476 ± 0.0059	1.452 ± 0.0059	1.417 ± 0.0057	1.359 ± 0.0052
RMSE(H_0)	0.256 ± 0.3846	0.202 ± 0.3086	0.151 ± 0.2508	0.122 ± 0.2197	0.100 ± 0.1939	0.077 ± 0.1656

Table 3 and Table 4 are similar to Table 1 and Table 2, respectively, but for Student underlying parents. And a similar comment applies to all other pairs of tables. We have always restricted the choice of the optimal k to values smaller than or equal to the number of positive elements in the original sample. Table 5 and Table 6 are associated with unshifted and shifted Fréchet models, and finally, Table 7 and Table 8 are associated with unshifted and shifted Burr models. Note that the GPPWM and the PORT-PPWM EVI-estimators are invariant for changes in location. Consequently their mean values at optimal levels are the same as for the unshifted model. However, due to high increase in the RMSE at optimal levels of the Hill EVI-estimator, their efficiencies increase proportionally to such a new RMSE(H_0).

3.2 Some overall comments

We think sensible to provide the following comments, which in a certain sense justify some of the parents chosen in this section.

- For all models with a left endpoint greater than or equal to zero, the PORT-PPWM EVI-estimators cannot improve the performance of $\hat{\gamma}^{\mathrm{PPWM}}$, in (7), as had already happened with the PORT-Hill estimators when compared with the Hill estimator H (see Figures 6, 7 and 8. However, if we induce a shift in the model, things change drastically and the PORT-PPWM provide an interesting estimation procedure, as can be seen in the aforementioned figures.
- The PORT-PPWM estimators can even outperform the MVRB-estimator under consideration (see Figure 2 and Figure 5, associated with a Sudent-t_4 and an EV(0.5) underlying parent, respectively).
- For models with a left endpoint equal to infinity, like the Student model, the value $q = 0$ should be discarded due to inconsistency (see the patterns of PPWM|0 in Figure 2, and Gomes *et al.*, 2008a, for further details on the subject).
- We can often find a value of q that provides the best estimator of γ, regarding for instance minimum RMSE, through the use of the new class of estimators $\hat{\gamma}_{k,n}^{\mathrm{PPWM}|\mathrm{q}}$, in (13), like the value $q = 0.1$, in Figure 2, and the value $q = 0$, in Figure 5.
- An adaptive choice of k and q is thus an important topic, to be dealt with in Section 4, where we consider again the use of a double bootstrap methodology, similar to the one used in Gomes and Oliveira (2001), among others, for classical EVI-estimation, in

Table 3: Simulated mean values of the semi-parametric EVI-estimators under consideration, at their simulated optimal levels for underlying Student-t_ν parents.

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
Student-t_4 parent ($\gamma = 0.25$)						
H	0.361 ± 0.0009	0.339 ± 0.0026	0.317 ± 0.0016	0.306 ± 0.0013	0.296 ± 0.0009	0.286 ± 0.0007
PPWM	0.310 ± 0.0018	0.305 ± 0.0012	0.298 ± 0.0005	0.292 ± 0.0005	0.287 ± 0.0004	0.280 ± 0.0003
GPPWM	-0.240 ± 0.0022	-0.090 ± 0.0016	0.028 ± 0.0011	0.083 ± 0.0009	0.122 ± 0.0013	0.157 ± 0.0008
PPWM|0	0.234 ± 0.0004	0.242 ± 0.0004	0.196 ± 0.0004	0.167 ± 0.0003	0.142 ± 0.0003	0.114 ± 0.0002
PPWM|0.05	0.235 ± 0.0013	0.242 ± 0.0009	$\underline{\mathbf{0.249}} \pm 0.0005$	$\underline{\mathbf{0.251}} \pm 0.0001$	$\underline{\mathbf{0.250}} \pm 0.0001$	$\underline{\mathbf{0.250}} \pm 0.0001$
PPWM|0.1	$\underline{\mathbf{0.265}} \pm 0.0005$	$\underline{\mathbf{0.257}} \pm 0.0002$	0.256 ± 0.0001	0.254 ± 0.0001	0.252 ± 0.0001	0.251 ± 0.0000
PPWM|0.25	0.288 ± 0.0011	0.282 ± 0.0006	0.274 ± 0.0004	0.269 ± 0.0002	0.265 ± 0.0002	0.260 ± 0.0001
CH	0.311 ± 0.0023	0.310 ± 0.0009	0.301 ± 0.0013	0.294 ± 0.0008	0.288 ± 0.0006	0.281 ± 0.0004
Student-t_3 parent ($\gamma = 0.3(3)$)						
H	0.439 ± 0.0030	0.417 ± 0.0019	0.396 ± 0.0019	0.385 ± 0.0011	0.375 ± 0.0009	0.365 ± 0.0007
PPWM	0.385 ± 0.0009	0.381 ± 0.0008	0.375 ± 0.0007	0.371 ± 0.0006	0.366 ± 0.0004	0.360 ± 0.0004
GPPWM	-0.123 ± 0.0018	0.023 ± 0.0014	0.135 ± 0.0012	0.187 ± 0.0010	0.224 ± 0.0013	0.257 ± 0.0007
PPWM|0	0.151 ± 0.0004	0.142 ± 0.0007	0.139 ± 0.0009	0.137 ± 0.0007	0.135 ± 0.0007	0.135 ± 0.0007
PPWM|0.05	0.255 ± 0.0004	0.261 ± 0.0006	0.265 ± 0.0004	0.268 ± 0.0003	0.269 ± 0.0002	0.271 ± 0.0001
PPWM|0.1	0.299 ± 0.0008	0.310 ± 0.0009	0.319 ± 0.0007	0.323 ± 0.0004	0.326 ± 0.0003	0.329 ± 0.0002
PPWM|0.15	$\underline{\mathbf{0.332}} \pm 0.0010$	$\underline{\mathbf{0.344}} \pm 0.0004$	$\underline{\mathbf{0.339}} \pm 0.0001$	$\underline{\mathbf{0.337}} \pm 0.0001$	$\underline{\mathbf{0.335}} \pm 0.0001$	$\underline{\mathbf{0.334}} \pm 0.0001$
PPWM|0.2	0.355 ± 0.0006	0.350 ± 0.0003	0.345 ± 0.0003	0.341 ± 0.0001	0.338 ± 0.0001	0.336 ± 0.0001
CH	0.362 ± 0.0035	0.377 ± 0.0013	0.370 ± 0.0008	0.364 ± 0.0008	0.359 ± 0.0006	0.354 ± 0.0003
Student-t_2 parent ($\gamma = 0.5$)						
H	0.602 ± 0.0039	0.577 ± 0.0027	0.556 ± 0.0011	0.544 ± 0.0008	0.536 ± 0.0010	0.526 ± 0.0005
PPWM	0.541 ± 0.0006	0.536 ± 0.0006	0.531 ± 0.0004	0.526 ± 0.0003	0.521 ± 0.0002	0.514 ± 0.0002
GPPWM	0.093 ± 0.0028	0.231 ± 0.0011	0.332 ± 0.0013	0.379 ± 0.0010	0.409 ± 0.0006	0.435 ± 0.0006
PPWM|0	0.194 ± 0.0009	0.189 ± 0.0009	0.186 ± 0.0011	0.184 ± 0.0009	0.180 ± 0.0007	0.172 ± 0.0006
PPWM|0.25	0.465 ± 0.0021	$\underline{\mathbf{0.485}} \pm 0.0011$	$\underline{\mathbf{0.502}} \pm 0.0007$	$\underline{\mathbf{0.506}} \pm 0.0002$	$\underline{\mathbf{0.503}} \pm 0.0001$	$\underline{\mathbf{0.502}} \pm 0.0001$
PPWM|0.3	$\underline{\mathbf{0.491}} \pm 0.0025$	0.515 ± 0.0013	0.514 ± 0.0002	0.510 ± 0.0002	0.506 ± 0.0001	0.503 ± 0.0001
PPWM|0.35	0.516 ± 0.0029	0.524 ± 0.0004	0.518 ± 0.0002	0.514 ± 0.0002	0.510 ± 0.0001	0.505 ± 0.0001
PPWM|0.4	0.533 ± 0.0012	0.528 ± 0.0005	0.523 ± 0.0003	0.518 ± 0.0002	0.514 ± 0.0001	0.508 ± 0.0002
CH	0.464 ± 0.0123	0.506 ± 0.0020	0.512 ± 0.0001	0.507 ± 0.0006	0.504 ± 0.0006	0.502 ± 0.0003

Gomes *et al.* (2012b), for reduced-bias estimation and in Brilhante *et al.* (2013), for a mean-of-order-p EVI-estimator. Such methods, despite of computationally intensive, can indeed provide reliable data-driven choices of the tuning parameters under play.

Table 4: Simulated values of the $\text{REFF}_{\bullet|\text{H}}$ (from first to seventh row of each entry) and $\text{RMSE}(\text{H}_0)$ for underlying Student-t_ν parents.

	$n=100$	$n=200$	$n=500$	$n=1000$	$n=2000$	$n=5000$
Student-t_4 parent ($\gamma=0.25$)						
PPWM	1.470 ± 0.0076	1.362 ± 0.0059	1.257 ± 0.0061	1.201 ± 0.0045	1.159 ± 0.0045	1.120 ± 0.0035
GPPWM	0.318 ± 0.0026	0.364 ± 0.0028	0.402 ± 0.0027	0.424 ± 0.0016	0.443 ± 0.0028	0.461 ± 0.0027
PPWM|0	2.712 ± 0.0214	2.224 ± 0.0191	1.414 ± 0.0085	0.906 ± 0.0044	0.608 ± 0.0037	0.385 ± 0.0013
PPWM|0.05	$\underline{\mathbf{3.594}} \pm 0.0505$	$\underline{\mathbf{4.169}} \pm 0.0441$	$\underline{\mathbf{5.106}} \pm 0.0382$	$\underline{\mathbf{5.863}} \pm 0.0294$	$\underline{\mathbf{6.746}} \pm 0.0426$	$\underline{\mathbf{8.205}} \pm 0.0483$
PPWM|0.1	3.293 ± 0.0183	3.476 ± 0.0222	3.890 ± 0.0281	4.343 ± 0.0252	4.924 ± 0.0287	5.928 ± 0.0366
PPWM|0.25	2.150 ± 0.0095	2.092 ± 0.0092	2.080 ± 0.0130	2.120 ± 0.0105	2.193 ± 0.0096	2.350 ± 0.0100
CH	1.435 ± 0.0468	1.398 ± 0.0084	1.362 ± 0.0053	1.322 ± 0.0056	1.283 ± 0.0057	1.236 ± 0.0048
RMSE(H_0)	0.183 ± 0.5264	0.143 ± 0.4352	0.106 ± 0.3562	0.085 ± 0.3117	0.070 ± 0.2753	0.054 ± 0.2366
Student-t_3 parent ($\gamma=0.3(3)$)						
PPWM	1.561 ± 0.0067	1.412 ± 0.0080	1.271 ± 0.0087	1.200 ± 0.0063	1.149 ± 0.0063	1.104 ± 0.0048
GPPWM	0.345 ± 0.0024	0.391 ± 0.0028	0.431 ± 0.0028	0.453 ± 0.0030	0.470 ± 0.0030	0.486 ± 0.0032
PPWM|0	0.983 ± 0.0053	0.705 ± 0.0044	0.501 ± 0.0030	0.393 ± 0.0023	0.314 ± 0.0019	0.233 ± 0.0013
PPWM|0.05	1.898 ± 0.0132	1.720 ± 0.0137	1.438 ± 0.0113	1.227 ± 0.0075	1.029 ± 0.0060	0.803 ± 0.041
PPWM|0.1	2.585 ± 0.0290	2.871 ± 0.0411	$\underline{\mathbf{3.305}} \pm 0.0466$	$\underline{\mathbf{3.769}} \pm 0.0355$	$\underline{\mathbf{4.285}} \pm 0.0425$	$\underline{\mathbf{5.207}} \pm 0.0608$
PPWM|0.15	$\underline{\mathbf{2.831}} \pm 0.0270$	$\underline{\mathbf{3.011}} \pm 0.0239$	3.235 ± 0.0024	3.526 ± 0.0216	3.916 ± 0.0260	4.576 ± 0.0226
PPWM|0.2	2.589 ± 0.0151	2.560 ± 0.0199	2.643 ± 0.0194	2.804 ± 0.0175	3.043 ± 0.0188	3.489 ± 0.0131
CH	1.433 ± 0.0416	1.511 ± 0.0088	1.551 ± 0.0084	1.571 ± 0.0066	1.569 ± 0.0070	1.537 ± 0.0067
RMSE(H_0)	0.189 ± 0.0513	0.145 ± 0.0422	0.106 ± 0.0344	0.084 ± 0.0299	0.067 ± 0.0263	0.050 ± 0.0223
Student-t_2 parent ($\gamma=0.5$)						
PPWM	1.789 ± 0.0094	1.570 ± 0.0088	1.383 ± 0.0086	1.299 ± 0.0058	1.269 ± 0.0058	1.271 ± 0.0056
GPPWM	0.404 ± 0.0047	0.454 ± 0.0025	0.488 ± 0.0027	0.503 ± 0.0038	0.517 ± 0.0031	0.519 ± 0.0030
PPWM|0	0.605 ± 0.0037	0.451 ± 0.0024	0.317 ± 0.0016	0.246 ± 0.0015	0.192 ± 0.0008	0.138 ± 0.0006
PPWM|0.25	1.971 ± 0.0352	2.088 ± 0.0197	$\underline{\mathbf{2.280}} \pm 0.0161$	$\underline{\mathbf{2.371}} \pm 0.0097$	$\underline{\mathbf{2.533}} \pm 0.0135$	$\underline{\mathbf{2.814}} \pm 0.0141$
PPWM|0.3	2.106 ± 0.0326	$\underline{\mathbf{2.112}} \pm 0.0131$	2.034 ± 0.0135	2.048 ± 0.0087	2.152 ± 0.0113	2.354 ± 0.0114
PPWM|0.35	$\underline{\mathbf{2.109}} \pm 0.0204$	1.941 ± 0.0107	1.813 ± 0.0118	1.786 ± 0.0076	1.839 ± 0.0093	1.968 ± 0.0093
PPWM|0.4	2.000 ± 0.0105	1.792 ± 0.0100	1.636 ± 0.0103	1.581 ± 0.0066	1.595 ± 0.0075	1.666 ± 0.0073
CH	0.980 ± 0.1394	1.418 ± 0.0172	1.706 ± 0.0152	1.944 ± 0.0162	2.227 ± 0.0179	2.641 ± 0.0218
RMSE(H_0)	0.203 ± 0.4920	0.153 ± 0.4029	0.108 ± 0.3264	0.084 ± 0.2827	0.065 ± 0.2467	0.047 ± 0.2091

4 Adaptive estimation of the EVI

With $\mathbb{E}$ denoting the mean value operator, a possible substitute for the MSE of any classical EVI-estimator $\hat{\gamma}^{\bullet}_{k,n}$ is, cf. equation (9),

$$\text{AMSE}(\hat{\gamma}^{\bullet}_{k,n}) := \mathbb{E}\big(\sigma_\bullet\ \overline{Z}_k/\sqrt{k} + b_\bullet\ A(n/k)\big)^2 = \sigma^2_\bullet/k + b^2_\bullet\ \gamma^2\ \beta^2\ (n/k)^{2\rho},$$

Table 5: Simulated mean values of the semi-parametric EVI-estimators under consideration, at their simulated optimal levels for unshifted and shifted Fréchet underlying parents

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
Fréchet parent ($\gamma = 0.25, s = 0$)						
H	0.272 ± 0.0007	0.271 ± 0.0007	0.262 ± 0.0004	0.260 ± 0.0002	0.257 ± 0.0001	0.257 ± 0.0001
PPWM	0.273 ± 0.0004	0.269 ± 0.0002	0.262 ± 0.0001	0.260 ± 0.0001	0.257 ± 0.0001	$\mathbf{0.256} \pm 0.0001$
GPPWM	0.022 ± 0.0020	0.101 ± 0.0016	0.186 ± 0.0008	0.203 ± 0.0009	0.219 ± 0.0004	0.219 ± 0.0004
PPWM|0	0.322 ± 0.0013	0.315 ± 0.0008	0.302 ± 0.0006	0.297 ± 0.0005	0.290 ± 0.0003	0.269 ± 0.0002
PPWM|0.1	0.329 ± 0.0016	0.323 ± 0.0011	0.311 ± 0.0007	0.39 ± 0.0005	0.2986 ± 0.0005	0.270 ± 0.0004
PPWM|0.25	0.332 ± 0.0010	0.327 ± 0.0009	0.315 ± 0.0008	0.310 ± 0.0006	0.302 ± 0.0006	0.270 ± 0.0004
CH	$\mathbf{0.245} \pm 0.0007$	$\mathbf{0.247} \pm 0.0099$	$\mathbf{0.250} \pm 0.0002$	$\mathbf{0.250} \pm 0.0001$	$\mathbf{0.250} \pm 0.0001$	0.270 ± 0.0004
Fréchet parent ($\gamma = 0.25, s = -0.5$)						
H	0.427 ± 0.0012	0.392 ± 0.0026	0.365 ± 0.0019	0.348 ± 0.0012	0.335 ± 0.0013	0.321 ± 0.0010
PPWM	0.344 ± 0.0008	0.331 ± 0.0013	0.323 ± 0.0009	0.318 ± 0.0008	0.313 ± 0.0007	0.305 ± 0.0006
CH	0.382 ± 0.0027	0.372 ± 0.0021	0.353 ± 0.0014	0.342 ± 0.0017	0.330 ± 0.0008	0.317 ± 0.0008

Table 6: Simulated values of the $\mathrm{REFF}_{\bullet|\mathrm{H}}$ (from first to seventh row of each entry) and $\mathrm{RMSE(H_0)}$ for unshifted and shifted Fréchet underlying parents

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
Fréchet parent ($\gamma = 0.25, s = 0$)						
PPWM	1.109 ± 0.0052	1.075 ± 0.0064	1.035 ± 0.0035	1.023 ± 0.0034	1.018 ± 0.0034	1.093 ± 0.0039
GPPWM	0.178 ± 0.0014	0.201 ± 0.0017	0.238 ± 0.0011	0.248 ± 0.0019	0.260 ± 0.0018	0.260 ± 0.0018
PPWM|0	0.434 ± 0.0026	0.388 ± 0.0032	0.305 ± 0.0017	0.275 ± 0.0014	0.244 ± 0.0011	0.309 ± 0.0015
PPWM|0.1	0.382 ± 0.0024	0.335 ± 0.0026	0.250 ± 0.0015	0.222 ± 0.0012	0.192 ± 0.0010	0.256 ± 0.0013
PPWM|0.25	0.352 ± 0.0023	0.308 ± 0.0024	0.229 ± 0.0014	0.203 ± 0.0011	0.176 ± 0.0010	0.238 ± 0.0013
CH	$\mathbf{1.257} \pm 0.0072$	$\mathbf{1.238} \pm 0.1605$	$\mathbf{1.460} \pm 0.0078$	$\mathbf{1.574} \pm 0.0123$	$\mathbf{1.795} \pm 0.0097$	$\mathbf{1.955} \pm 0.0001$
RMSE(H$_0$)	0.053 ± 0.3784	0.041 ± 0.3125	0.023 ± 0.2150	0.018 ± 0.1865	0.013 ± 0.1557	0.013 ± 0.1456
Fréchet parent ($\gamma = 0.25, s = -0.5$)						
PPWM	1.550 ± 0.0069	1.435 ± 0.0063	1.321 ± 0.0051	1.262 ± 0.0050	1.212 ± 0.0050	1.172 ± 0.0031
GPPWM	0.829 ± 0.0066	0.988 ± 0.0085	1.637 ± 0.0078	1.839 ± 0.0138	1.836 ± 0.0157	1.833 ± 0.0129
PPWM|0	$\mathbf{2.020} \pm 0.0119$	$\mathbf{1.909} \pm 0.0155$	$\mathbf{2.100} \pm 0.0117$	$\mathbf{2.046} \pm 0.0108$	$\mathbf{2.107} \pm 0.0097$	$\mathbf{2.185} \pm 0.0104$
PPWM|0.1	1.778 ± 0.0111	1.645 ± 0.0130	1.723 ± 0.0103	1.648 ± 0.0088	1.662 ± 0.0090	1.806 ± 0.0095
PPWM|0.25	1.638 ± 0.0107	1.514 ± 0.0120	1.579 ± 0.0096	1.508 ± 0.0082	1.520 ± 0.0090	1.679 ± 0.0095
CH	1.328 ± 0.0108	1.237 ± 0.0057	1.171 ± 0.0042	1.130 ± 0.0031	1.101 ± 0.0021	1.072 ± 0.0020
RMSE(H_0)	0.246 ± 0.3905	0.200 ± 0.3126	0.157 ± 0.2504	0.133 ± 0.2150	0.113 ± 0.1865	0.092 ± 0.1557

depending on n and k, and with AMSE standing for *asymptotic mean square error*. We get (Dekkers and de Haan, 1993)

$$\begin{aligned} k_{0|\hat\gamma^\bullet}(n) &:= \arg\min_k \mathrm{AMSE}\big(\hat\gamma^\bullet_{k,n}\big) \\ &= \Big((-2\rho)\; b_\bullet^2\; \gamma^2\beta^2\; n^{2\rho}\big/\sigma_\bullet^2\Big)^{-1/(1-2\rho)} = k_0^{\hat\gamma^\bullet}(n)(1+o(1)). \qquad (22) \end{aligned}$$

Table 7: Simulated mean values of the semi-parametric EVI-estimators under consideration, at their simulated optimal levels for unshifted and shifted Burr underlying parents

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
			BURR($\gamma = 0.25, \rho = -0.25, s = 0$)			
H	0.419 ± 0.0024	0.390 ± 0.0028	0.365 ± 0.0018	0.348 ± 0.0016	0.335 ± 0.0012	0.320 ± 0.0011
PPWM	0.339 ± 0.0010	0.330 ± 0.0011	0.323 ± 0.0013	0.318 ± 0.0009	0.313 ± 0.0008	0.305 ± 0.0005
GPPWM	$\underline{\mathbf{0.182}} \pm 0.0006$	$\underline{\mathbf{0.217}} \pm 0.0006$	$\underline{\mathbf{0.237}} \pm 0.0005$	$\underline{\mathbf{0.243}} \pm 0.0002$	$\underline{\mathbf{0.247}} \pm 0.0001$	$\underline{\mathbf{0.249}} \pm 0.0002$
PPWM|0	0.340 ± 0.0010	0.331 ± 0.0011	0.324 ± 0.0013	0.318 ± 0.0009	0.313 ± 0.0008	0.305 ± 0.0005
PPWM|0.1	0.344 ± 0.0010	0.333 ± 0.0014	0.324 ± 0.0013	0.319 ± 0.0010	0.313 ± 0.0008	0.306 ± 0.0005
PPWM|0.25	0.353 ± 0.0010	0.333 ± 0.0009	0.326 ± 0.0013	0.320 ± 0.0010	0.315 ± 0.0007	0.307 ± 0.0006
CH	0.406 ± 0.0030	0.382 ± 0.0017	0.360 ± 0.0017	0.345 ± 0.0018	0.333 ± 0.0013	0.319 ± 0.0009
			BURR($\gamma = 0.25, \rho = -0.25, s = -0.5$)			
H	0.595 ± 0.0071	0.515 ± 0.0012	0.420 ± 0.0023	0.390 ± 0.0029	0.370 ± 0.0029	0.347 ± 0.0015
PPWM	0.397 ± 0.0073	0.392 ± 0.0010	0.341 ± 0.0012	0.331 ± 0.0009	0.326 ± 0.0009	0.318 ± 0.0007
CH	0.392 ± 0.0076	0.387 ± 0.0061	0.386 ± 0.0022	0.374 ± 0.0023	0.360 ± 0.0020	0.344 ± 0.0015
			BURR($\gamma = 0.25, \rho = -0.5, s = 0$)			
H	0.324 ± 0.0022	0.311 ± 0.0012	0.297 ± 0.0010	0.289 ± 0.0006	0.283 ± 0.0007	0.275 ± 0.0006
PPWM	$\mathbf{0.300} \pm 0.0006$	$\mathbf{0.291} \pm 0.0005$	$\mathbf{0.287} \pm 0.0006$	$\mathbf{0.282} \pm 0.0004$	$\mathbf{0.278} \pm 0.0003$	0.273 ± 0.0002
GPPWM	-0.073 ± 0.0016	0.025 ± 0.0015	0.101 ± 0.0010	0.138 ± 0.0009	0.164 ± 0.0007	0.187 ± 0.0008
PPWM|0	0.303 ± 0.0009	0.297 ± 0.0004	0.289 ± 0.0006	0.283 ± 0.0004	0.279 ± 0.0003	0.273 ± 0.0002
PPWM|0.1	0.311 ± 0.0010	0.307 ± 0.0008	0.299 ± 0.0008	0.294 ± 0.0005	0.289 ± 0.0004	0.283 ± 0.0003
PPWM|0.25	0.320 ± 0.0011	0.314 ± 0.0008	0.306 ± 0.0009	0.301 ± 0.0007	0.295 ± 0.0005	0.289 ± 0.0003
CH	0.307 ± 0.0012	0.297 ± 0.0010	0.288 ± 0.0007	0.283 ± 0.0004	0.278 ± 0.0005	$\mathbf{0.272} \pm 0.0004$
			BURR($\gamma = 0.25, \rho = -0.5, s = -0.5$)			
H	0.386 ± 0.0025	0.360 ± 0.0024	0.338 ± 0.0014	0.324 ± 0.0014	0.314 ± 0.0013	0.301 ± 0.0008
PPWM	0.326 ± 0.0013	0.318 ± 0.0011	0.310 ± 0.0009	0.305 ± 0.0007	0.300 ± 0.0006	0.293 ± 0.0004
CH	0.349 ± 0.0023	0.340 ± 0.0015	0.327 ± 0.0014	0.317 ± 0.0010	0.308 ± 0.0010	0.298 ± 0.0007

For the Hill estimator, we have, in (9), $\sigma_{\rm H} = \gamma$ and $b_{\rm H} = 1/(1-\rho)$. Consequently, with $(\hat\beta, \hat\rho)$ any consistent estimator of the vector (β, ρ) of second-order parameters, (22) justifies asymptotically the estimator

$$\hat k_0^{\rm H} := \left\lfloor \left((1-\hat\rho)^2 n^{-2\hat\rho}/(-2\hat\rho\hat\beta^2)\right)^{1/(1-2\hat\rho)} \right\rfloor. \qquad (23)$$

Moreover, provided that $\sqrt{k}\ (n/k)^\rho \to \lambda$, finite, and with $b_{k,n,\rho} = 1 + \beta(n/k)^\rho/(1-\rho)$, $\sqrt{k}\{\hat\gamma^{\rm H}_{k,n}/\gamma - b_{k,n,\rho}\}$ is approximately $\mathcal{N}(0,1)$. We may then get approximate $100(1-\alpha)\%$ confidence intervals (CI's) for γ,

$$\left(\frac{\hat\gamma^{\rm H}_{k,n}}{b_{k,n,\rho} + \xi_{1-\alpha/2}/\sqrt{k}},\ \frac{\hat\gamma^{\rm H}_{k,n}}{b_{k,n,\rho} - \xi_{1-\alpha/2}/\sqrt{k}}\right), \qquad (24)$$

Table 8: Simulated values of the $\text{REFF}_{\bullet|\text{H}}$ (from first to seventh row of each entry) and $\text{RMSE}(\text{H}_0)$ for unshifted and shifted Burr underlying parents

	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
BURR($\gamma = 0.25, \rho = -0.25, s = 0$)						
PPWM	1.517 ± 0.0061	1.413 ± 0.0057	1.314 ± 0.0061	1.256 ± 0.0045	1.212 ± 0.0045	1.166 ± 0.0045
GPPWM	**<u>1.642</u>** ± 0.0118	**<u>2.055</u>** ± 0.0136	**<u>2.676</u>** ± 0.0155	**<u>3.244</u>** ± 0.0174	**<u>3.945</u>** ± 0.0249	**<u>5.099</u>** ± 0.0343
PPWM|0	1.514 ± 0.0061	1.411 ± 0.0057	1.314 ± 0.0061	1.256 ± 0.0050	1.212 ± 0.0045	1.166 ± 0.0045
PPWM|0.1	1.483 ± 0.0060	1.385 ± 0.0057	1.290 ± 0.0062	1.233 ± 0.0050	1.190 ± 0.0045	1.144 ± 0.0045
PPWM|0.25	1.428 ± 0.0058	1.341 ± 0.0059	1.249 ± 0.0064	1.194 ± 0.0050	1.152 ± 0.0044	1.107 ± 0.0044
CH	1.148 ± 0.0049	1.118 ± 0.0026	1.088 ± 0.0025	1.069 ± 0.0023	1.057 ± 0.0018	1.042 ± 0.0012
RMSE(H_0)	0.237 ± 0.2639	0.196 ± 0.2142	0.155 ± 0.1672	0.131 ± 0.1397	0.112 ± 0.1172	0.092 ± 0.0931
BURR($\gamma = 0.25, \rho = -0.25, s = -0.5$)						
PPWM	1.837 ± 0.1695	1.706 ± 0.0054	1.521 ± 0.0067	1.417 ± 0.0060	1.334 ± 0.0060	1.255 ± 0.0044
GPPWM	**<u>2.851</u>** ± 0.0205	**<u>3.512</u>** ± 0.0233	**<u>4.124</u>** ± 0.0239	**<u>4.860</u>** ± 0.0260	**<u>5.768</u>** ± 0.0364	**<u>7.310</u>** ± 0.0049
PPWM|0	2.629 ± 0.0106	2.411 ± 0.0097	2.025 ± 0.0095	1.881 ± 0.0075	1.772 ± 0.0065	1.671 ± 0.0060
PPWM|0.1	2.576 ± 0.0104	2.367 ± 0.0097	1.988 ± 0.0095	1.847 ± 0.0075	1.739 ± 0.0065	1.640 ± 0.0062
PPWM|0.25	2.480 ± 0.0101	2.292 ± 0.0101	1.926 ± 0.0098	1.789 ± 0.0075	1.684 ± 0.0064	1.588 ± 0.0064
CH	1.100 ± 0.1911	1.107 ± 0.1808	1.264 ± 0.0092	1.195 ± 0.0078	1.146 ± 0.0036	1.100 ± 0.0030
RMSE(H_0)	0.412 ± 0.5130	0.334 ± 0.5092	0.239 ± 0.3007	0.197 ± 0.2315	0.164 ± 0.1864	0.132 ± 0.1442
BURR($\gamma = 0.25, \rho = -0.5, s = 0$)						
PPWM	1.299 ± 0.0050	1.230 ± 0.0051	1.168 ± 0.0053	1.134 ± 0.0039	1.106 ± 0.0039	1.085 ± 0.0037
GPPWM	0.312 ± 0.0020	0.350 ± 0.0030	0.391 ± 0.0027	0.415 ± 0.0023	0.434 ± 0.0026	0.456 ± 0.0034
PPWM|0	1.223 ± 0.0052	1.173 ± 0.0051	1.130 ± 0.0052	1.105 ± 0.0040	1.084 ± 0.0038	1.070 ± 0.0036
PPWM|0.1	1.048 ± 0.0054	0.978 ± 0.0054	0.909 ± 0.0046	0.866 ± 0.0034	0.828 ± 0.0031	0.789 ± 0.0029
PPWM|0.25	0.927 ± 0.0052	0.850 ± 0.0056	0.788 ± 0.0043	0.744 ± 0.0034	0.705 ± 0.0028	0.665 ± 0.0028
CH	**<u>1.422</u>** ± 0.0066	**<u>1.383</u>** ± 0.0065	**<u>1.337</u>** ± 0.0052	**<u>1.300</u>** ± 0.0057	**<u>1.262</u>** ± 0.0045	**<u>1.230</u>** ± 0.0048
RMSE(H_0)	0.119 ± 0.4294	0.095 ± 0.3625	0.072 ± 0.2983	0.059 ± 0.2617	0.048 ± 0.2316	$0.038 \pm 0.1.994$
BURR($\gamma = 0.25, \rho = -0.5, s = -0.5$)						
PPWM	1.486 ± 0.0056	1.379 ± 0.0056	1.277 ± 0.0054	1.223 ± 0.0045	1.179 ± 0.0045	1.138 ± 0.0042
GPPWM	0.537 ± 0.0035	0.606 ± 0.0052	0.689 ± 0.0047	0.744 ± 0.0041	0.796 ± 0.0048	0.859 ± 0.0064
PPWM|0	**<u>2.105</u>** ± 0.0090	**<u>2.034</u>** ± 0.0089	**<u>1.988</u>** ± 0.0091	**<u>1.982</u>** ± 0.0072	**<u>1.986</u>** ± 0.0069	**<u>2.017</u>** ± 0.0068
PPWM|0.1	1.803 ± 0.0094	1.695 ± 0.0094	1.599 ± 0.0080	1.553 ± 0.0062	1.516 ± 0.0057	1.487 ± 0.0054
PPWM|0.25	1.597 ± 0.0090	1.488 ± 0.0097	1.387 ± 0.0075	1.335 ± 0.0061	1.292 ± 0.0051	1.254 ± 0.0053
CH	1.354 ± 0.0109	1.290 ± 0.0055	1.229 ± 0.0033	1.186 ± 0.0045	1.150 ± 0.0038	1.113 ± 0.0030
RMSE(H_0)	0.206 ± 0.4335	0.165 ± 0.3625	0.127 ± 0.2983	0.106 ± 0.2617	0.089 ± 0.2316	0.071 ± 0.1994

where ξ_p is the p-quantile of a $\mathcal{N}(0,1)$ d.f. If $\lambda = 0$, we need to replace in (24) the bias summand $\beta(n/k)^\rho/(1-\rho)$ by 0, i.e. we should consider $b_{k,n,\rho} = 1$, in (24).

The same does not happen with the PPWM and PORT-PPWM EVI-estimators, with an asymptotic variance ($\sigma_{\text{PPWM}|q}$) and a dominant component of bias ($b_{\text{PPWM}|q}$) dependent on γ.

In this situation, it is sensible to use the bootstrap methodology for the adaptive PPWM and PORT-PPWM EVI-estimation. Just as in Gomes and Oliveira (2001), for the estimation of γ through the Hill estimator, in Gomes *et al.* (2009, 2012b), for adaptive reduced-bias estimation and in Brilhante *et al.* (2012) for a MOP EVI-estimation, let us consider the auxiliary statistic,

$$T^\bullet_{k,n} := \hat\gamma^\bullet_{\lfloor k/2\rfloor,n} - \hat\gamma^\bullet_{k,n}, \quad k = 2, \ldots, n-1. \tag{25}$$

On the basis of results similar to the ones in Gomes *et al.* (2000) and Gomes and Oliveira (2001), we can get, for the auxiliary statistic $T^\bullet_{k,n}$, in (25), the asymptotic distributional representation,

$$T^\bullet_{k,n} \overset{d}{=} \sigma_\bullet\ Q^\bullet_k/\sqrt{k} + b_\bullet\ (2^\rho - 1)\ A(n/k) + o_p(A(n/k)),$$

with $Q^\bullet_k$ asymptotically standard normal, and $(b_\bullet, \sigma_\bullet)$ given in (9). The *AMSE* of $T^\bullet_{k,n}$ is thus minimal at a level $k_{0|T^\bullet}(n)$ such that $\sqrt{k}\ A(n/k) \to \lambda'_{_A} \neq 0$, i.e. a level of the type of the one in (22), with $b_\bullet$ replaced by $b_\bullet(2^\rho - 1)$, and we consequently have

$$k_{0|\hat\gamma^\bullet}(n) = k_{0|T^\bullet}(n)\,(1-2^\rho)^{\frac{1}{1-2\rho}}\,(1+o(1)).$$

Then, given the sample $\underline{X}_n = (X_1, \ldots, X_n)$ from an unknown model F, consider for any $n_1 = O(n^{1-\epsilon})$, with $0 < \epsilon < 1$, the bootstrap sample $\underline{X}^*_{n_1} = (X^*_1, \ldots, X^*_{n_1})$, from the model $F^*_n(x) = \frac{1}{n}\sum_{i=1}^n I_{\{X_i \le x\}}$, the empirical d.f. associated with the original sample $\underline{X}_n$. Next, associate to that bootstrap sample the corresponding bootstrap auxiliary statistic, denoted $T^*_{k_1,n_1}$, $1 < k_1 < n_1$. Then, with the obvious notation $k^*_{0|T}(n_1) = \arg\min_{k_1} \mathrm{AMSE}\big(T^*_{k_1,n_1}\big)$, $k^*_{0|T}(n_1)/k_{0|T}(n) = (n_1/n)^{-\frac{2\rho}{1-2\rho}}\,(1+o(1))$. Consequently, for another sample size $n_2 = \lfloor n_1^2/n\rfloor$, we have

$$\frac{\big(k^*_{0|T}(n_1)\big)^2}{k^*_{0|T}(n_2)} = k_{0|T}(n)(1+o(1)),\ \text{as } n\to\infty.$$

We are now able to estimate $k_0^{\widehat\gamma}(n)$, on the basis of any estimate $\hat\rho$ of ρ. With $\hat k^*_{0T}$ denoting the sample counterpart of k^*_{0T}, and taking into account (22), we can build the k_0-estimate,

$$\hat k^\bullet_{0*} \equiv \hat k^\bullet_{0*}(n; n_1) := \min\left(n-1,\ \left\lfloor \frac{(1-2^{\hat\rho})^{\frac{1}{1-2\hat\rho}}\big(\hat k^*_{0|T}(n_1)\big)^2}{\hat k^*_{0|T}(\lfloor n_1^2/n\rfloor + 1)} \right\rfloor + 1\right), \tag{26}$$

and the γ-estimate

$$\hat\gamma^\bullet_* \equiv \hat\gamma^\bullet_*(n; n_1) := \hat\gamma_{\hat k^\bullet_{0*}(n;n_1),n}. \tag{27}$$

4.1 An algorithm for the adaptive EVI-estimation

Now, and with $\hat{\gamma}_{k,n}^{\rm PPWM}$ defined in (7), the algorithm is the following:

1. Given a sample $(x_1, x_2, \ldots, x_n)$, compute, for tuning parameters $\tau = 0$ and $\tau = 1$, the observed values of $\hat{\rho}_\tau(k)$ in (19).

2. Consider $\{\hat{\rho}_\tau(k)\}_{k\in\mathcal{K}}$, with $\mathcal{K} = (\lfloor n^{0.995}\rfloor,\ \lfloor n^{0.999}\rfloor)$, compute their median, denoted η_τ, and compute $I_\tau := \sum_{k\in\mathcal{K}} \left(\hat{\rho}_\tau(k) - \eta_\tau\right)^2$, $\tau = 0, 1$. Next choose the *tuning parameter* $\tau^* = 0$ if $I_0 \le I_1$; otherwise, choose $\tau^* = 1$ and compute $\hat{\rho} \equiv \hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$, with k_1 given in (20).

3. Compute $\hat{\gamma}_{k,n}^{\rm PPWM}$, $k = 1, 2, \cdots, n-1$.

4. Next, consider the set $\{n_{1j}\}$ of the m values in $\mathcal{N} = (\lfloor n^{\theta_1}\rfloor,\ \lfloor n^{\theta_2}\rfloor)$ with $n^{\theta_i} = o(n)$, $i = 1, 2$, $\theta_1 < \theta_2$, say $\theta_1 = 0.95$ and $\theta_2 = 0.999$. For j from 1 until m do:

   4.1 Compute $n_{2j} = \lfloor n_{1j}^2/n\rfloor + 1$.

   4.2 For l from 1 until B, generate independently B bootstrap samples $(x_1^*, \ldots, x_{n_{2j}}^*)$ and $(x_1^*, \ldots, x_{n_{2j}}^*, x_{n_{2j}+1}^*, \ldots, x_{n_{1j}}^*)$, of sizes n_{2j} and n_{1j}, respectively, from the empirical d.f. $F_n^*(x) = \frac{1}{n}\sum_{i=1}^n I_{\{X_i \le x\}}$ associated with the observed sample $(x_1, \ldots, x_n)$.

   4.3 Denoting $T_{k,n}^*$ the bootstrap counterpart of $T_{k,n}^{\rm PPWM}$, defined in (25), obtain, for $1 \le l \le B$, $t_{k,n_{1j},l}^*$, $1 < k < n_{1j}$, and $t_{k,n_{2j},l}^*$, $1 < k < n_{2j}$, the observed values of the statistics $T_{k,n_{ij}}^*$, $i = 1, 2$. For $k = 2, \ldots, n_{ij} - 1$, compute

$$\mathrm{MSE}^*(n_{ij}, k) = \frac{1}{B}\sum_{l=1}^{B} \left(t_{k,n_{ij},l}^*\right)^2, \tag{28}$$

   and obtain $\hat{k}_{0|T}^*(n_{ij}) := \arg\min_{1<k<n_{ij}} \mathrm{MSE}^*(n_{ij}, k)$, $i = 1, 2$.

5. Obtain $j^* := \arg\min_{1<j<m} \mathrm{MSE}^*(n_{1j}, \hat{k}_{0|T}^*(n_{1j}))$

6. Compute the threshold estimate $\hat{k}_{0*} \equiv \hat{k}_{0*}^{\rm PPWM}$, in (26) using $\hat{k}_{0|T}^*(n_{ij^*})$, $i = 1, 2$.

7. Obtain $PPWM^* \equiv \hat{\gamma}_*^{\rm PPWM} \equiv \hat{\gamma}_*^{\rm PPWM}(n; n_{1j^*}) := \hat{\gamma}_{\hat{k}_{0*},n}$, already provided in (27).

A similar procedure can be used for the bootstrap data-driven estimation for the Hill estimator, in (5) and for the GPPWM estimator, in (14). Note also that bootstrap confidence intervals are easily associated with the estimates presented, through the replication of steps from 4. up to 7. of this algorithm r_1 times.

Remark 2. *Notice that the bootstrap mean square error of the auxiliary statistic,* T^*_{k,n_1} *defined in* (25)*, tends do decrease as* n_1 *increases. That tendency is usually affected by several peaks due to the effect of the random re-sampling. The adaptive choice of* j *in* STEP *5. may not be the best one, but at least it helps us to avoid values of* n_1 *with a large* $\text{MSE}^*(n_{1j}, k)$*. In practical applications we advise a sensitivity analysis of the performance of the methodology to the choice of the sample size* n_1*.*

4.2 Application to simulated samples

We now consider the performance of the Algorithm in Section 4.1 to the analysis of three simulated samples, of size $n = 1000$, from the models $EV(0.2)$, $EV(0.4)$ and Student's-t_3, generated from different seeds. We have selected the EVI-estimators $\widehat{\gamma}^{\text{H}}_{k,n}$, $\widehat{\gamma}^{\text{PPWM}}_{k,n}$ and $\hat{\gamma}^{\text{PPWM}|q}_{k,n}$, $q = 0$, 0.1 and 0.25, in (5), (7) and (13), respectively. Since Student's-t_3 has an infinite left endpoint, the EVI estimator $\hat{\gamma}^{\text{PPWM}|0}_{k,n}$ was excluded for the corresponding samples. Figure 10 presents the EVI-estimates, as a function of k, associated to $\widehat{\gamma}^{\text{H}}_{k,n}$, $\widehat{\gamma}^{\text{PPWM}}_{k,n}$ and $\hat{\gamma}^{\text{PPWM}|q}_{k,n}$. Notice that the PPWM and PPWM|q have a much smooth sample path and we have always a high positive bias, due to the fact that we are considering models with a second order parameter ρ, in (8), close to 0 ($|\rho| < 1$).

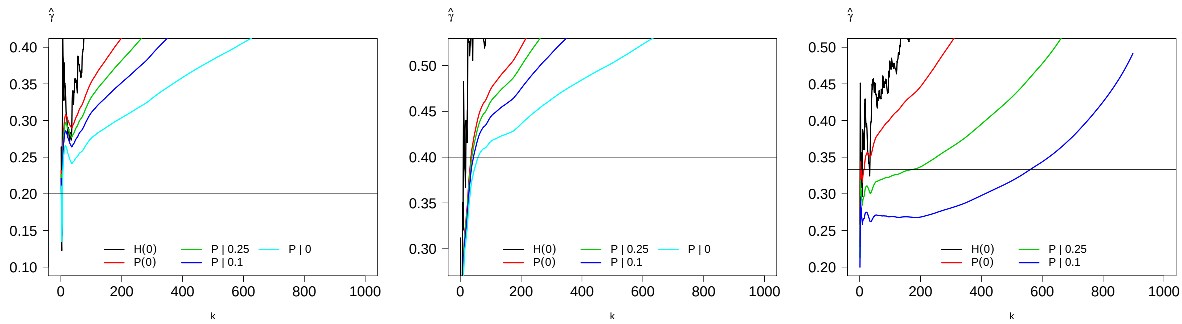

Figure 9: Adaptive bootstrap estimates of the EVI for one sample of size $n = 1000$ of the models $EV(0.2)$ (left), $EV(0.4)$ (center) and Student's-t_3 (right).

The above mentioned Algorithm was applied with θ_1=0.95, θ_2=1 and $B = 400$ bootstrap samples. In Table 9 we present the values of $\text{RMSE}^* := \sqrt{\text{MSE}^*(n_{1j^*}, \hat{k}^*_{0|T}(n_{1j^*}))}$, with j^* given in STEP 5, $\hat{k}_{0*}$ in STEP 6 and $\hat{\gamma}_*$ in STEP 7. The best adaptive EVI-estimates, in the sense of minimal squared bias, are presented in **bold**. This data analysis leads us to the conclusion that the lowest bias in the EVI-estimation is obtained with the PPWM|q estimators. For EV parents, the parameter $q = 0$ appears to be the best choice. Moreover, relying on the observed results for the EVI-estimates for Student's-t_3 sample, it is sensible to discard small values of q, probably due to the fact that the PPWM|0 is inconsistent. In Figures 10–12 we further provide the same simulated quantities, but as function of the sub-sample size n_1. The bootstrap Hill estimates are indeed much more unstable than the same estimates for PPWM and PPWM|q. For the PPWM and PPWM|q estimators, the bootstrap methodology is very resistant to the choice of n_1.

Table 9: Adaptive bootstrap estimates of the RMSE of the auxiliary statistic $T_{k,n}$, the optimal level, $\hat{k}_{0*}$, and the EVI, $\hat{\gamma}_*$, for the three samples under study.

	H	PPWM	PPWM|0	PPWM|0.1	PPWM|0.25
			EV(γ) **sample,** $\gamma = 0.2$		
$RMSE^*$	1.074	0.375	0.313	0.348	0.361
$\hat{k}_{0*}$	16	21	27	23	22
$\hat{\gamma}_*$	0.332	0.304	**0.250**	0.277	0.290
			EV(γ) **sample,** $\gamma = 0.4$		
$RMSE^*$	0.999	0.825	0.435	0.612	0.732
$\hat{k}_{0*}$	61	69	90	77	74
$\hat{\gamma}_*$	0.554	0.454	**0.413**	0.433	0.446
			Student-t_3 **sample** ($\gamma = 0.3(3)$)		
$RMSE^*$	0.978	0.664	——	0.221	0.366
$\hat{k}_{0*}$	51	22	——	98	85
$\hat{\gamma}_*$	0.451	0.354	——	0.269	**0.322**

4.3 Small scale simulation study

In this section, we have implemented a small Monte Carlo simulation, to obtain the distributional behaviour of the adaptive EVI-estimation through the EVI-estimators $\widehat{\gamma}_{k,n}^{\mathrm{H}}$, $\widehat{\gamma}_{k,n}^{\mathrm{PPWM}}$ and $\hat{\gamma}_{k,n}^{\mathrm{PPWM}|q}$ in (5), (7) and (13), respectively. Due to the time intensive nature of the adaptive algorithm, we have restricted the study to 100 runs, samples of size $n = 100$, 200, 500 and 1000 and to the following underlying parents:

1. Student's t_ν with $\nu = 2$, 3 and 4 degrees of freedom ($\gamma = 0.25$, $\rho = -0.5$);
2. EV d.f., in (1), with $\gamma = 0.2$, 0.4 and 0.6.

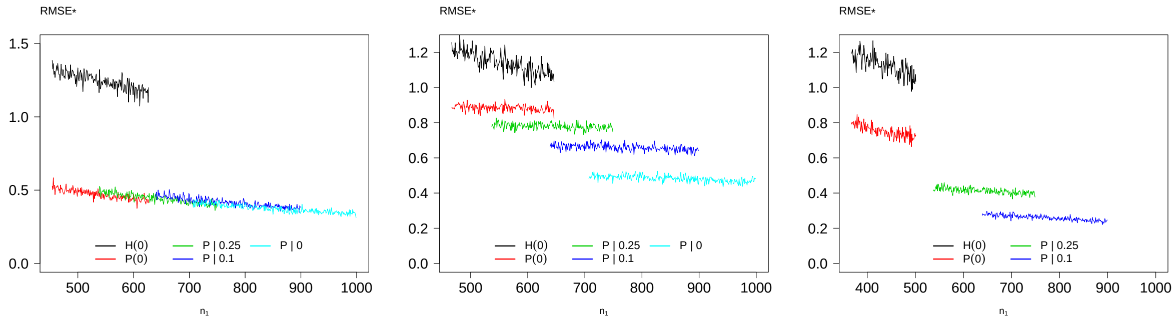

Figure 10: Adaptive bootstrap estimates of the RMSE of the auxiliary statistic $T_{k,n}$ as function of n_1, for the three samples of the models $EV(0.2)$ (left), $EV(0.4)$ (center) and Student's-t_3 (right).

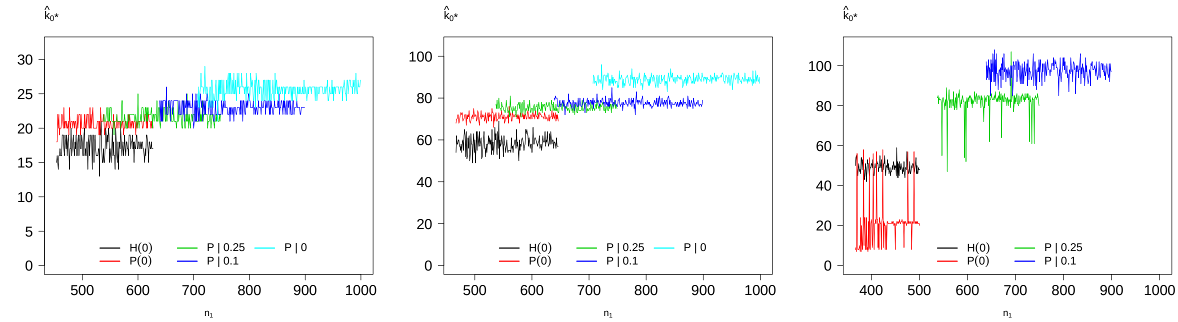

Figure 11: Adaptive bootstrap estimates of the optimal level, $\hat{k}_{0*}$, as function of n_1, for the three samples of the models $EV(0.2)$ (left), $EV(0.4)$ (center) and Student's-t_3 (right).

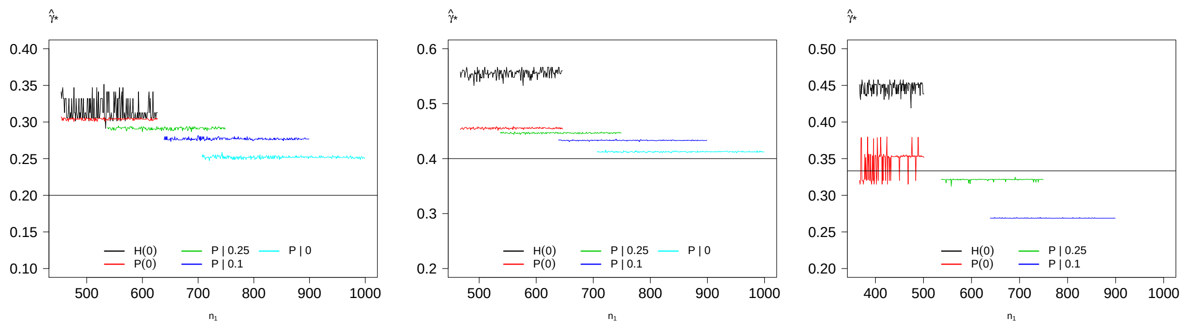

Figure 12: Adaptive bootstrap estimates of the EVI, $\hat{\gamma}_*$, as function of n_1, for the three samples of the models $EV(0.2)$ (left), $EV(0.4)$ (center) and Student's-t_3 (right).

Notice that both Student-t_2 and $EV(0.6)$ d.f.'s have $\gamma >= 0.5$ and are out of the scope of Theorem 3. The simulated mean values and RMSE are presented in Table 10. The smallest squared bias and RMSE are written in **bold**. As expected, the adaptive bootstrap algorithm performs worse with the Hill estimator. The smallest RMSE is almost always provided by the PPWM$|q$ estimators. Regarding squared bias, the smallest values are achieved by PPWM$|q$ estimators, for EV and Student's-t_2 parents and by PPWM estimators for Student's-t_3 and t_4 parents.

4.4 A case study

We shall finally consider an illustration of the performance of the algorithm is Section 4.1, with the $n = 2049$ daily negative log returns, $-100 \ln(x_i/x_{i-1})$, from the Euro–Swiss Franc exchange rate, from January 4, 1999 till December 29, 2006. In Figure 13, we picture, at the left, the values of the negative log returns. At the right, we present, a normal Q-Q plot of the data. The graphical analysis lead us to a immediate conclusion that the underlying model has heavier tails than the normal distribution. For this type of data, Student-t_ν with $\nu > 0$ or Skew$-t$, are possible candidates for the underlying model of the data. The probable infinite left endpoint led us to make use of the PORT-PPWM EVI-estimator with only $q = 0.25$.

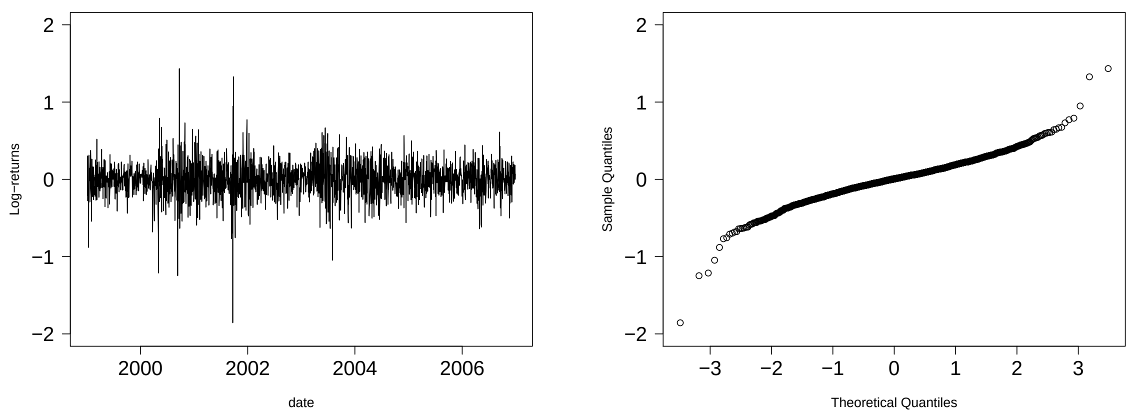

Figure 13: Daily negative log-returns (left) and corresponding Normal QQ Plot (right)

In Figure 14 (left) we present the EVI-estimates, as function of k, provided by H, PPWM and PPWM|0.25. The application of the algorithm led us to $\hat{\rho} = -0.675$ and to the estimates provided in Table 11. The EVI bootstrap estimates $\hat{\gamma}_*^H$ and $\hat{\gamma}_*^{\text{PPWM}}$ are very close and both slightly larger that $\hat{\gamma}_*^{\text{PPWM}|0.25}$. In Figure 14, we picture at the right, as a function of the sub-sample size n_1, ranging from $\lfloor n_+^{0.95} \rfloor$ until n_+, where n_+ denotes the number of positive values

Table 10: Simulated mean values / RMSE of the bootstrap adaptive EVI-estimators for underlying EV and Student t_ν parents.

n	H	PPWM	PPWM|0	PPWM|0.1	PPWM|0.25
			EV(γ) **parent,** $\gamma = 0.2$		
100	0.417 / 0.293	0.358 / 0.223	**0.302** / **0.153**	0.313 / 0.173	0.308 / 0.168
200	0.343 / 0.219	0.301 / 0.170	**0.279** / **0.124**	0.318 / 0.163	0.287 / 0.159
500	0.341 / 0.191	0.284 / 0.145	**0.262** / **0.110**	0.281 / 0.125	0.294 / 0.138
1000	0.350 / 0.188	0.283 / 0.134	0.255 / **0.086**	**0.250** / 0.108	0.296 / 0.125
			EV(γ) **parent,** $\gamma = 0.4$		
100	0.571 / 0.307	0.472 / 0.209	**0.431** / **0.150**	0.436 / 0.172	0.417 / 0.157
200	0.504 / 0.220	0.429 / 0.164	**0.406** / **0.131**	0.451 / 0.153	0.402 / 0.164
500	0.510 / 0.193	0.411 / 0.154	0.415 / **0.121**	**0.414** / 0.133	0.421 / 0.134
1000	0.516 / 0.176	0.424 / 0.137	**0.411** / **0.095**	0.380 / 0.137	0.447 / 0.093
			EV_γ **parent,** $\gamma = 0.6$		
100	0.745 / 0.340	**0.593** / 0.204	0.553 / **0.184**	0.565 / 0.193	0.532 / 0.189
200	0.681 / 0.244	0.548 / 0.185	0.558 / **0.148**	**0.580** / 0.164	0.527 / 0.197
500	0.667 / 0.188	0.548 / 0.187	**0.570** / 0.145	0.564 / **0.137**	0.555 / 0.157
1000	0.705 / 0.166	0.568 / 0.164	0.574 / 0.117	0.535 / 0.177	**0.597** / **0.086**
			Student t_4 **parent** ($\gamma = 0.25$)		
100	0.359 / 0.192	0.320 / 0.147	—— / ——	0.210 / **0.100**	**0.256** / 0.108
200	0.338 / 0.210	0.278 / 0.128	—— / ——	0.216 / **0.085**	**0.242** / 0.102
500	0.344 / 0.142	0.276 / 0.103	—— / ——	0.214 / **0.065**	**0.255** / 0.068
1000	0.329 / 0.106	0.280 / 0.089	—— / ——	0.223 / **0.056**	**0.248** / 0.073
			Student t_3 **parent** ($\gamma = 0.3(3)$)		
100	0.397 / 0.212	**0.359** / 0.169	—— / ——	0.264 / **0.120**	0.275 / 0.137
200	0.374 / 0.169	**0.323** / 0.142	—— / ——	0.265 / **0.099**	0.295 / 0.115
500	0.400 / 0.144	**0.332** / 0.133	—— / ——	0.263 / 0.091	0.321 / **0.059**
1000	0.401 / 0.100	**0.346** / 0.094	—— / ——	0.274 / 0.077	0.307 / **0.070**
			Student t_2 **parent** ($\gamma = 0.5$)		
100	0.597 / 0.240	**0.509** / **0.185**	—— / ——	0.364 / 0.186	0.395 / 0.204
200	0.566 / 0.191	**0.475** / 0.154	—— / ——	0.351 / 0.175	0.410 / **0.153**
500	0.549 / **0.126**	**0.493** / 0.141	—— / ——	0.365 / 0.150	0.421 / 0.133
1000	0.539 / 0.112	**0.472** / 0.130	—— / ——	0.370 / 0.138	0.446 / **0.091**

of the sample. The EVI-estimates $\hat{\gamma}_*^{\rm H}$ and $\hat{\gamma}_*^{\rm PPWM}$ have some volatility, while $\hat{\gamma}_*^{\rm PPWM|0.25}$ are almost independent of the choice of the sub-sample size n_1.

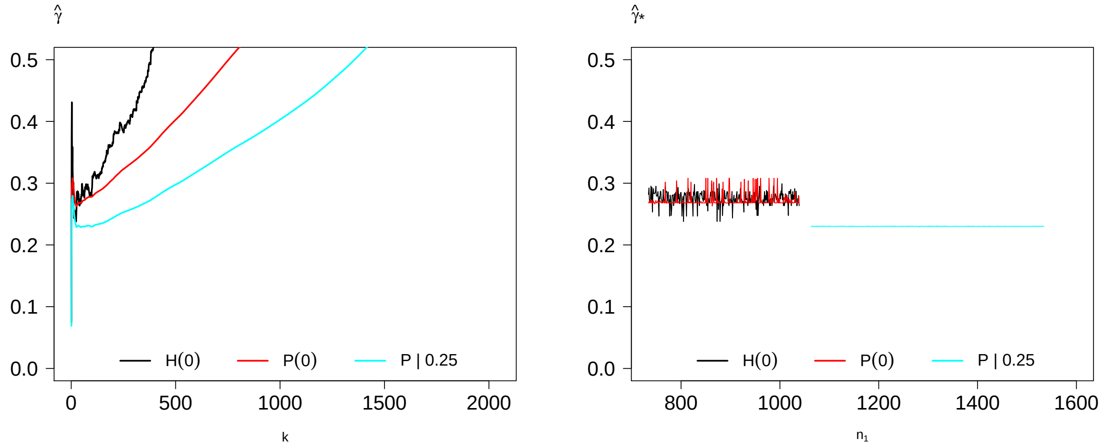

Figure 14: EVI-estimates for the daily negative log returns data, as a function of k (left) and adaptive EVI-estimates as function of the sub-sample size n_1.

Table 11: Adaptive bootstrap estimates of the optimal level, $\hat{k}_{0*}$, and the EVI, $\hat{\gamma}_*$, for the negative log returns data.

	H	PPWM	PPWM |0.25
k_0^*	40	48	67
$\hat{\gamma}^*$	0.264	0.268	0.230

References

[1] Araújo Santos, P., Fraga Alves, M.I. and Gomes, M.I. (2006). Peaks over random threshold methodology for tail index and quantile estimation. *Revstat* **4**:3, 227–247.

[2] Arnold, B.C., Balakrishnan, N. and Nagaraja H.N. (1992; 2008). *A First Course in Order Statistics*. John Wiley and Sons; SIAM.

[3] Beirlant, J., Caeiro, F. and Gomes, M.I. (2012). An overview and open research topics in the field of statistics of univariate extremes. *Revstat* **10**:1, 1–31.

[4] Brilhante, M.F., Gomes, M.I. and Pestana, D. (2013). A simple generalization of the Hill estimator. *Computational Statist. & Data Analysis* **57**:1, 518–535.

[5] Caeiro, F. and Gomes, M.I. (2006). A new class of estimators of a "scale" second order parameter. *Extremes* **9**, 193–211, 2007.

[6] Caeiro F. and Gomes M.I. (2011). Semi-Parametric Tail Inference through Probability-Weighted Moments. *J. Statist. Plann. and Inference* **141**:2, 9378-950.

[7] Caeiro, C. and Gomes, M.I. (2012a). *A Semi-parametric Estimator of a Shape Second Order Parameter.* Notas e Comunicações CEAUL 07/2012.

[8] Caeiro, F. and Gomes, M.I. (2012b). *Bias Reduction in the Estimation of a Shape Second-Order parameter of a Heavy Right Tail Model*, Preprint, CMA 22-2012, submitted.

[9] Caeiro, F. and Gomes, M.I. (2012c). A reduced bias estimator of a 'scale' second order parameter. In T. E. Simos, G. Psihoyios, Ch. Tsitouras and Z. Anastassi (eds.), AIP Conf. Proc. 1479 (Numerical Analysis and Applied Mathematics ICNAAM 2012), 1114–1117.

[10] Caeiro, F., Gomes, M.I. and Pestana, D.D. (2005). Direct reduction of bias of the classical Hill estimator. *Revstat* **3**:2, 111–136.

[11] Caeiro F., Gomes M.I. and Henriques-Rodrigues, L. (2009). Reduced-bias tail index estimators under a third order framework. *Commun. in Statist.–Theory and Methods* **38**:7, 1019–1040.

[12] Caeiro, F., Gomes, M.I. and Vandewalle, B. (2012). Semi-parametric probability-weighted moments estimation revisited. *Methodology and Computing in Applied Probability*, DOI: 10.1007/s11009-012-9295-6

[13] Ciuperca, G. and Mercadier, C. (2010). Semi-parametric estimation for heavy tailed distributions. *Extremes* **13**:1, 55–87.

[14] Figueiredo, F., Gomes, M.I., Henriques-Rodrigues, L. and Miranda, C. (2012). A computational study of a quasi-PORT methodology for VaR based on second-order reduced-bias estimation. *J. Statist. Comput. and Simul.* **82**:4, 587–602.

[15] Fraga Alves M.I., Gomes M.I. and de Haan L. (2003). A new class of semi-parametric estimators of the second order parameter. *Portugaliae Mathematica* **60**:2, 194–213.

[16] Fraga Alves, M.I., Gomes, M.I., de Haan, L. and Neves, C. (2009). Mixed moment estimators and location invariant alternatives. *Extremes* **12**, 149–185.

[17] Geluk J. and de Haan L. (1987). *Regular Variation, Extensions and Tauberian Theorems.* Tech. Report CWI Tract 40, Centre for Mathematics and Computer Science, Amsterdam, Netherlands.

[18] Gnedenko, B.V. (1943) Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44**, 423–453.

[19] Goegebeur, Y., Beirlant, J. and de Wet, T. (2008). Linking Pareto-tail kernel goodness-of-fit statistics with tail index at optimal threshold and second order estimation. *Revstat* **6**:1, 51–69.

[20] Goegebeur, Y., Beirlant, J. and de Wet, T. (2010). Kernel estimators for the second order parameter in extreme value statistics. *J. Statist. Plann. Inference* **140**:9, 2632–2654.

[21] Gomes M.I. and Martins M.J. (2002). "Asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *Extremes* **5**:1, 5–31.

[22] Gomes M.I. and Oliveira O. (2001). The bootstrap methodology in Statistics of Extremes: choice of the optimal sample fraction. *Extremes* **4**:4, 331-358.

[23] Gomes, M.I. and Pestana, D. (2007). A simple second order reduced-bias' tail index estimator. *J. Statist. Comp. and Simul.* **77**:6, 487–504.

[24] Gomes, M.I., Fraga Alves, M.I. and Araújo Santos, P. (2008a). PORT Hill and Moment Estimators for Heavy-Tailed Models. *Comm.Statist. – Simul. and Comput.* **37**, 1281–1306.

[25] Gomes, M.I., de Haan, L. and Henriques-Rodrigues, L. (2008b). Tail Index estimation for heavy-tailed models: accommodation of bias in weighted log-excesses. *J. Royal Statist. Soc.* **B70**:1, 31–52.

[26] Gomes, M.I., Henriques-Rodrigues, L., Pereira, H. and Pestana, D. (2010). Tail index and second order parameters' semi-parametric estimation based on the log-excesses. *J. Statist. Comput. and Simul.* **80**:6, 653–666.

[27] Gomes, M.I., Henriques-Rodrigues, L. and Miranda, C. (2011). Reduced-bias location-invariant extreme value index estimation: a simulation study. *Comm.Statist. – Simul. and Comput.* **40**:3, 424–447.

[28] Gomes, M.I., Caeiro, F. and Henriques-Rodrigues, L. (2012a). PORT-PPWM extreme value index estimation. In Colubi, A., Fokianos, K., Kontoghiorghes, E.J. and Gonzáez-Rodríuez, G. (eds.). *Proceedings of COMPSTAT* 2012, The International Statistical Institute/International Association for Statistical Computing, 259–270.

[29] Gomes, M.I., Figueiredo, F. and Neves, M.M. (2012b). Adaptive estimation of heavy right tails: the bootstrap methodology in action. *Extremes* **15**, 463–489.

[30] Gomes, M.I., Henriques-Rodrigues, L., Fraga Alves, M.I. and Manjunath, B.G. (2013). Adaptive PORT-MVRB estimation: an empirical comparison of two heuristic algorithms . *J. Statist. Comput. and Simul.* **83**:6, 1129–1144.

[31] de Haan, L. (1984). Slow variation and characterization of domains of attraction. In Tiago de Oliveira, ed., *Statistical Extremes and Applications*, D. Reidel, Dordrecht, 31–48.

[32] de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: an Introduction*. Springer Science+Business Media, LLC, New York.

[33] de Haan, L. and Peng, L. (1998). Comparison of tail index estimators. *Statistica Neerlandica*, **52**, 60–70.

[34] Hill, B.M. (1975). A Simple General Approach to Inference about the Tail of a Distribution. *Ann. Statist.* **3**(5), 1163–1174.

[35] Scarrot, C. and McDonald, A. (2012). A review of extreme value threshold estimation and uncertainty quantification. *Revstat* **10**:2, 33–60.

[36] van der Vaart, A.W. (1998). *Asymptotic Statistics*. Cambridge Univ. Press.