Mathematics

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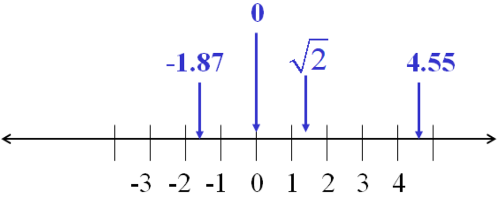
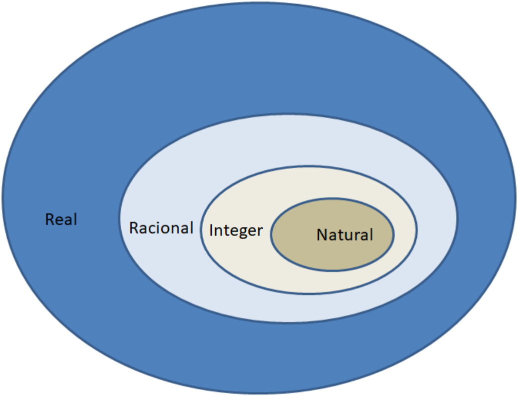
Real Numbers

Definition

Real Numbers $\mathbb{R}$ In mathematics, a real number is a value of a continuous quantity that can be represent along a line. The adjective real in this context was introduced in the 17th century by René Descartes, who distinguished between real and imaginary roots of polynomials.

The real numbers include all the rational numbers, such as the integer -5 and the fraction $\frac{3}{4}$, and all the irrational numbers, such as $\sqrt{2} = 1.41421356...$ (the square root of 2). Included within the irrationals are the transcendental numbers, such as π $(3.14159265...)$. Real numbers can be used to measure quantities such as time, mass, energy, velocity, and many more.


0
-1.87
$\sqrt{2}$
4.55
-3 -2 -1 0 1 2 3 4


Real
Racional
Integer
Natural

The real numbers form a metric space: the distance between x and y is defined as the absolute value $|x-y|$.

Definition of the absolute value or modulus:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example

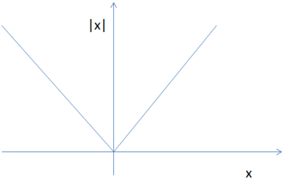
- $|-3| = 3$
- $|8| = 8$
- $|x-1| = 4 \Leftrightarrow x-1 = 4 \vee x-1 = -4 \Leftrightarrow x = 5 \vee x = -3$.

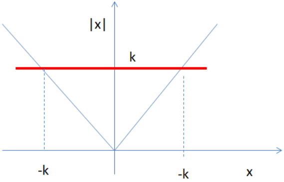
Properties of the absolute value or modulus:

- $|a| \geq 0$ Non-negativity
- $|a| = 0 \iff a = 0$ Positive-definiteness
- $|ab| = |a|\,|b|$. Multiplicativity
- $|a + b| \leq |a| + |b|$ Triangle inequality

$$|a| \geq b \iff a \leq -b \ \vee a \geq b$$

$$|a| \leq b \iff a \geq -b \ \wedge a \leq b \iff -b \leq a \leq b$$





Example

Solve

$$|x-4| = \sqrt{27}$$
$$|x-2| \geq 12/5$$
$$|x-3| \leq 7$$
$$|2+x| \geq 8$$
$$|1+x| \leq -2$$
$$|x-2| \leq |x-6|$$
$$\frac{|x-6|}{|x-1|} \geq 1$$

Operations with real numbers

Adding - $a + b$

1. Comutativity $a + b = b + a, \ \forall a, b \in \mathbb{R}$
   Example: $$4 + 11 = 11 + 4$$ $$12 + 13x = 13x + 12$$
2. Neutral element $a + 0 = 0 + a, \ \forall a \in \mathbb{R}$
   Example: $$3 + 0 = 0 + 3 = 3$$ $$x + 0 = 0 + x = x$$
3. Simmetric $a + (-a) = -a + a = 0, \ \forall a \in \mathbb{R}$
   Example: $$-3 + 3 = 0$$ $$x + (-x) = 0$$

4. Associativity $(a+b)+c=a+(b+c)$, $\forall a,b,c\in\mathbb{R}$

   Example:
   $$(2+3)+5=2+(3+5)$$
   $$(2+x)+3x=2+(x+3x)$$

Multipying - $a.b$

1. Comutativity $a.b = b.a, \ \forall a, b \in \mathbb{R}$
   Example:
   $2.3 = 3.2 = 0$
   $x.12 = 12x$

2. Neutral element $a.1 = 1.a, \ \forall a \in \mathbb{R}$
   Example:
   $3.1 = 1.3 = 3$
   $x.1 = 1.x = x$

3. Inverse $a.(1/a) = 1, \ \forall a \in \mathbb{R} \setminus \{0\}$
   Example:
   $3 * 1/3 = 1$
   $x/x = 1$

4. Associativity $(a.b).c = a.(b.c)$, $\forall a, b, c \in \mathbb{R}$
   Example:
   $$(2.3).5 = 2.(3.5)$$
   $$(2.x).3 = 2.(x.3) = 6x$$

5. Absorbing element (or, annihilating element $a.0 = 0.a = 0$, $\forall a, b, c \in \mathbb{R}$
   Example:
   $$2.3.0.x = 0$$
   $$x.y.4.0.z = 0$$

Distribuitivity properties

$$a(b+c) = ab + ac \,, \forall a, b, c \in \mathbb{R}$$

a) $4(x+5)$;

b) $7(\sqrt{5}+10)$;

c) $(x+2)\,33$;

d) $(y+3)(x+2)$;

e) $(2x+1)(2+y)$;

f) $(4+5z)(1+x)$.

a) $$\begin{aligned} 4(x+5) &= 4\cdot x + 4\cdot 5 \\ &= 4x+20 \end{aligned}$$

b) $$\begin{aligned} 7(\sqrt{5}+10) &= 7\sqrt{5}+7\cdot 10 \\ &= 7\sqrt{5}+70 \end{aligned}$$

c) $$\begin{aligned} (x+2)\,33 &= 33\,(x+2) \\ &= 33x+33\cdot 2 \\ &= 33x+66 \end{aligned}$$

d) $$\begin{aligned} (y+3)(x+2) &= (y+3)x+(y+3)2 \\ &= (yx+3x)+(2y+6) \\ &= yx+3x+2y+6 \end{aligned}$$

e) $$\begin{aligned} (2x+1)(2+y) &= (2x+1)2+(2x+1)y \\ &= (4x+2)+(2xy+y) \\ &= 4x+2+2xy+y \end{aligned}$$

f) $$\begin{aligned}(4+5z)(1+x) &= (4+5z)+(4+5z)x \\ &= (4+5z)+(4x+5zx) \\ &= 4+5z+4x+5zx\end{aligned}$$

Working with negative numbers

1. $(-1)a = -a$
2. $-(-a) = a$
3. $(-a)b = a(-b) = -ab$
4. $(-a)(-b) = ab$
5. $-(a+b) = -a-b$
6. $-(a-b) = -a+b$

Example

a) $(-1)13 = -13$;

b) $-(-45) = 45$;

c) $(-33)2 = -66 = 33(-2) = -(33 \cdot 2)$;

d) $(-20)(-15) = 300$;

e) $-(2x+1) = -2x - 1$;

f) $-(4z - 12x) = -4z - (-12x) = -4z + 12x = 12x - 4z$.

Working with fractions

1. Multiplying

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}.$$

2. Dividing

$$\frac{a/c}{b/d} = \frac{a}{c} \cdot \frac{d}{b} = \frac{ad}{bc}.$$

3. Adding

$$\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}.$$

4. Subtracting

$$\frac{a}{c} - \frac{b}{d} = \frac{ad - bc}{cd}.$$

5. Simplification

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

6. Equality

If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

a) $\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$;

b) $\frac{23}{31} \cdot \frac{10}{3} = \frac{23 \cdot 10}{31 \cdot 3} = \frac{230}{93}$;

c) $\frac{8}{3} : \frac{5}{11} = \frac{8}{3} \cdot \frac{11}{5} = \frac{8 \cdot 11}{3 \cdot 5} = \frac{88}{15}$;

d) $\frac{21}{24} = \frac{3 \cdot 7}{3 \cdot 8} = \frac{7}{8}$;

e) $\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5 + 3 \cdot 4}{15} = \frac{22}{15}$;

f) $\frac{9}{7} - \frac{1}{5} = \frac{9 \cdot 5 - 7 \cdot 1}{35} = \frac{38}{35}$.

Exercises

a) $\dfrac{2y}{y+1} \cdot \dfrac{4+4y}{y+2}$;

b) $\dfrac{2x+1}{x-1} \cdot \dfrac{2x-2}{3x}$;

c) $\dfrac{7+3y}{8y+4} \cdot \dfrac{2+4y}{35y+17}$;

d) $\dfrac{5z}{3z-1} \cdot \dfrac{9z-3}{z+31}$;

e) $\dfrac{2}{5y+11} \cdot \dfrac{10y+22}{45y+2}$;

f) $\dfrac{x}{x+3} \cdot \dfrac{2x+6}{x+4}$.

Solution:

a) $$\frac{2y}{y+1}\cdot\frac{4+4y}{y+2}=\frac{2y(4+4y)}{(y+1)(y+2)}=\frac{2y\cdot 4(1+y)}{(y+1)(y+2)}=\frac{8y}{y+2}.$$

b) $$\frac{2x+1}{x-1}\cdot\frac{2x-2}{3x}=\frac{(2x+1)(2x-2)}{3x(x-1)}=\frac{2(2x+1)(x-1)}{3x(x-1)}=\frac{2(2x+1)}{3x}.$$

c) $$\frac{7+3y}{8y+4}\cdot\frac{2+4y}{35y+17}=\frac{(7+3y)(2+4y)}{(8y+4)(35y+17)}=\frac{2(7+3y)(1+2y)}{4(2y+1)(35y+17)}=\frac{7+3y}{2(35y+17)}.$$

d) $\dfrac{5z}{3z-1}\cdot\dfrac{9z-3}{z+31}=\dfrac{5z(9z-3)}{(3z-1)(z+31)}=$
$\dfrac{15z(3z-1)}{(3z-1)(z+31)}=\dfrac{15z}{z+31}.$

e) $\dfrac{2}{5y+11}\cdot\dfrac{10y+22}{45y+2}=\dfrac{2(10y+22)}{(5y+11)(45y+2)}$
$=\dfrac{4(5y+11)}{(5y+11)(45y+2)}=\dfrac{4}{45y+2}.$

f) $\dfrac{x}{x+3}\cdot\dfrac{2x+6}{x+4}=\dfrac{x(2x+6)}{(x+3)(x+4)}=\dfrac{2x(x+3)}{(x+3)(x+4)}=$
$\dfrac{2x}{x+4}.$

Exercise

a) $\dfrac{x}{2} : \dfrac{x+2}{4}$;

b) $\dfrac{2y}{y+1} : \dfrac{4+4y}{y+2}$;

c) $\dfrac{2x+1}{x-1} : \dfrac{2x-2}{3x}$;

d) $\dfrac{7+3y}{8y+4} : \dfrac{2+4y}{35y+17}$;

e) $\dfrac{5z}{3y-1} : \dfrac{9z-3}{z+31}$;

f) $\dfrac{x}{x+3} : \dfrac{2x+6}{x+4}$.

Solution:

a) $\frac{x}{2} : \frac{x+2}{4} = \frac{x}{2} \cdot \frac{4}{x+2} = \frac{4x}{2(x+2)} = \frac{2x}{x+2}.$

$$\frac{\frac{x}{2}}{\frac{x+2}{4}} = \frac{x}{2} \cdot \frac{4}{x+2} = \frac{2x}{x+2}.$$

b) $\frac{2y}{y+1} : \frac{4+4y}{y+2} = \frac{2y}{y+1} \cdot \frac{y+2}{4+4y} = \frac{y(y+2)}{2(y+1)(1+y)}.$

c) $\frac{2x+1}{x-1} : \frac{2x-2}{3x} = \frac{2x+1}{x-1} \cdot \frac{3x}{2x-2} = \frac{3x(2x+1)}{(x-1)(2x-2)}.$

d) $\frac{7+3y}{8y+4} : \frac{2+4y}{35y+17} = \frac{7+3y}{8y+4} \cdot \frac{35y+17}{2+4y} = \frac{(7+3y)(35y}{8(2y+1)(1}$

e) $\dfrac{5z}{3y-1} : \dfrac{9z-3}{z+31} = \dfrac{5z}{3y-1} \cdot \dfrac{z+31}{9z-3} = \dfrac{5z(z+31)}{(3y-1)(9z-3)}.$

f) $\dfrac{x}{x+3} : \dfrac{2x+6}{x+4} = \dfrac{x}{x+3} \cdot \dfrac{x+4}{2x+6} = \dfrac{x(x+4)}{(x+3)(2x+6)}.$

Least common denominator

The lowest common denominator or least common denominator (abbreviated LCD) is the lowest common multiple of the denominators of a set of fractions. It simplifies adding, subtracting, and comparing fractions.

$$\frac{a}{c} + \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{d} + \frac{b}{d} \cdot \frac{c}{c} = \frac{ad + bc}{cd}.$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

$$\frac{a}{c} - \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{d} - \frac{b}{d} \cdot \frac{c}{c} = \frac{ad - bc}{cd}.$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$

"Prime Factorization" is finding which prime numbers multiply together to make the original number.
Example

9 , 6, 48, 12

Solve

$\frac{1}{9} + \frac{7}{6}$ and $\frac{7}{12} + \frac{5}{48}$

Exercise

a) $\dfrac{2}{35}+\dfrac{1}{70}$;

b) $\dfrac{23}{30}-\dfrac{13}{35}$;

c) $\dfrac{2}{5x+10}+\dfrac{2x}{x+2}$.

c) $\dfrac{x}{x-1}+\dfrac{3}{x^2-1}$.

Powers

Indices (or powers, or exponents) are very useful in mathematics. Indices are a convenient way of writing multiplications that have many repeated terms.
Example of an Index

For the example 5^3 we say that:

5 is the base and
3 is the index (or power, or exponent).

5^3 means "multiply 5 by itself 3 times".

Given $a \in \mathbb{R}$ and $n \in \mathbb{N}$ we define:

$$a^n = \underbrace{a \times a \times a \dots a \times a}_{n},$$

there is a^n is the product of n equal factors of value a.

Operating with powers

Natural exponets

Let a and b be real numbers and m e n natural numbers

1. $a^n \times a^m = a^{n+m}$.
2. $\dfrac{a^n}{a^m} = a^{n-m}$, if $a \neq 0$ and $n > m$.
3. $(a^n)^m = a^{n \times m}$.
4. $(a \times b)^n = a^n \times b^n$.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, if $b \neq 0$.

Exercises

a) $2^7 \cdot 2^3$;

b) $\frac{x^{10}}{x^3}$;

c) $(43^2)^5$;

d) $\left(\frac{8}{9}\right)^5$;

e) $(7x)^9$;

f) $(y^3)^4$.

Solution:

a) $2^7 \cdot 2^3 = 2^{7+3} = 2^{10}$

b) $\dfrac{x^{10}}{x^3} = x^{10-3} = x^7$

c) $(43^2)^5 = 43^{10}$

d) $\left(\dfrac{8}{9}\right)^5 = \dfrac{8^5}{9^5}$

e) $(7x)^9 = 7^9 \cdot x^9$

f) $(y^3)^4 = y^{12}$

Exercise

a) $\dfrac{(5\cdot 3)^{15}+3^{14}}{3^{12}}$;

b) $4b\,(3b)^5$;

c) $\dfrac{x^4y^6z^2}{xy^3z}$;

d) $\dfrac{49a^6b^7}{7ab^3}$;

e) $\dfrac{-32a^4b^5}{8a^2b}$;

f) $\left(\dfrac{45a^2b^7}{9ab^4}\right)^3$.

Solution:

a) $\dfrac{(5\cdot 3)^{15}+3^{14}}{3^{12}} = \dfrac{5^{15}\cdot 3^{15}+3^{14}}{3^{12}} = \dfrac{3^{14}(3\cdot 5^{15}+1)}{3^{12}} = 9(3\cdot 5^{15}+1)$.

b) $4b\,(3b)^5 = 4b\,3^5\,b^5 = 4\cdot 243\,b\,b^5 = 972\,b^6$.

c)

$$\frac{x^4y^6z^2}{xy^3z} = \frac{x^4}{x}\cdot\frac{y^6}{y^3}\cdot\frac{z^2}{z}.$$

$$\frac{x^4}{x}\cdot\frac{y^6}{y^3}\cdot\frac{z^2}{z} = x^3y^3z.$$

d)

$$\frac{49a^6b^7}{7ab^3} = \frac{49}{7}\cdot\frac{a^6}{a}\cdot\frac{b^7}{b^3} = 7a^5b^4.$$

e)
$$\frac{-32a^4b^5}{8a^2b} = \frac{-32}{8} \cdot \frac{a^4}{a^2} \cdot \frac{b^5}{b} = -4a^2b^4.$$

f) $\left(\frac{45a^2b^7}{9ab^4}\right)^3 = \left(\frac{45}{9} \cdot \frac{a^2}{a} \cdot \frac{b^7}{b^4}\right)^3 = \left(5ab^3\right)^3 = 5^3a^3(b^3)^3$
$125a^3b^9$.

Negative exponents

Let $a \in \mathbb{R} \setminus \{0\}$, m and n be natural numbers

1. $a^0 = 1$.
2. $a^{-n} = \dfrac{1}{a^n}$.

1. $a^{-n} \times a^{-m} = a^{-n+(-m)}$.

2. $\dfrac{a^{-n}}{a^{-m}} = a^{-n-(-m)} = a^{-n+m}$, if $a \neq 0$.

3. $(a^{-n})^{-m} = a^{n \times m}$.

4. $(a \times b)^{-n} = a^{-n} \times b^{-n}$.

5. $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{a^{-n}}{b^{-n}}$, if $a, b \neq 0$.

Exercise:

a) $7^{-2} \cdot 7^{-3}$;

b) $\dfrac{x^{-6}}{x^{-9}}$;

c) $(37^{-2})^{-5}$;

d) $\left(\dfrac{7}{12}\right)^{-5}$;

e) $(2z^{2})^{-9}$;

f) $(24a^{-2})^{4}$.

Solution:

a) $7^{-2} \cdot 7^{-3} = 7^{-2+(-3)} = 7^{-5}$

b) $\dfrac{x^{-6}}{x^{-9}} = x^{-6-(-9)} = x^{3}$

c) $(37^{-2})^{-5} = 37^{-2(-5)} = 37^{10}$

d) $\left(\dfrac{7}{12}\right)^{-5} = \dfrac{7^{-5}}{12^{-5}}$

e) $(2z^{2})^{-9} = 2^{-9} \cdot z^{-18}$

f) $(24a^{-2})^{4} = 24^{4}a^{-8}$

a) $\left(\dfrac{3^{-1}L}{L^{-2}}\right)^{-3}$;

b) $(2w^2y^3)^{-1}\cdot\dfrac{1}{y^{-8}}$;

c) $\dfrac{a^{-5}(b^3c^2)^{-7}}{(a^{-2}b^{-4})^3\,c^{-1}}$;

d) $\left(\dfrac{4^{-2}a^{-5}b^2}{8^{-1}a^{-8}b^{-4}}\right)^{-1}$;

e) $\left(\dfrac{7^2x^0z^3}{x^{-5}y^{-1}z^{-2}}\right)^{-4}$;

f) $\left(\dfrac{16^0x^3y^{-2}z^{-10}}{4(x^{-3}y^{-2})^{-1}z^{-5}}\right)^2$.

Solution:

a) $\left(\frac{3^{-1}L}{L^{-2}}\right)^{-3} = \frac{(3^{-1}L)^{-3}}{(L^{-2})^{-3}} = \frac{3^3\,L^{-3}}{L^6} = 3^3\,L^{-3-6} = 27\,L^{-9}$.

b) $(2w^2y^3)^{-1}\cdot\frac{1}{y^{-8}} =$

$2^{-1}(w^2)^{-1}(y^3)^{-1}y^8 = \frac{1}{2}\,w^{-2}y^{-3}y^8$

$\frac{1}{2}\,w^{-2}y^{-3+8} = \frac{1}{2}\,w^{-2}y^5$.

c) $\frac{a^{-5}(b^3c^2)^{-7}}{(a^{-2}b^{-4})^3\,c^{-1}} = \frac{a^{-5}(b^3)^{-7}(c^2)^{-7}}{(a^{-2})^3(b^{-4})^3\,c^{-1}} = \frac{a^{-5}b^{-21}c^{-14}}{a^{-6}b^{-12}\,c^{-1}}$

$= a^{-5+6}b^{-21+12}c^{-14+1} = ab^{-9}c^{-13}$.

d) $$\left(\frac{4^{-2}a^{-5}b^{2}}{8^{-1}a^{-8}b^{-4}}\right)^{-1} = \frac{\left(4^{-2}a^{-5}b^{2}\right)^{-1}}{\left(8^{-1}a^{-8}b^{-4}\right)^{-1}} = \frac{\left(4^{-2}\right)^{-1}\left(a^{-5}\right)}{\left(8^{-1}\right)^{-1}\left(a^{-8}\right)}$$

$$= \frac{4^{2}a^{5}b^{-2}}{8\,a^{8}b^{4}} = 2a^{5-8}b^{-2-4} = 2a^{-3}b^{-}$$

e) $$\left(\frac{7^{2}x^{0}z^{3}}{x^{-5}y^{-1}z^{-2}}\right)^{-4} = \frac{\left(7^{2}x^{0}z^{3}\right)^{-4}}{\left(x^{-5}y^{-1}z^{-2}\right)^{-4}} = \frac{\left(7^{2}\right)^{-4}}{\left(x^{-5}\right)^{-4}\left(y^{-1}\right.}$$

$$= \frac{7^{-8}z^{-12}}{x^{20}y^{4}z^{8}} = \frac{1}{7^{8}x^{20}y^{4}z^{12+8}} = \frac{}{7^{8}x}$$

f) $$\left(\frac{16^0 x^3 y^{-2} z^{-10}}{4(x^{-3}y^{-2})^{-1} z^{-5}}\right)^2 = \frac{\left(x^3 y^{-2} z^{-10}\right)^2}{\left(4x^3 y^2 z^{-5}\right)^2}$$

$$= \frac{\left(x^3\right)^2 \left(y^{-2}\right)^2 \left(z^{-10}\right)^2}{4^2 \left(x^3\right)^2 \left(y^2\right)^2 \left(z^{-5}\right)^2}$$

$$= \frac{x^6 y^{-4} z^{-20}}{16\, x^6 y^4 z^{-10}} = \frac{1}{16\, y^8 z^{10}}.$$

Radicals

Let a be a real number, m and n natural numbers

1. $a^{1/n} = \sqrt[n]{a}$ (If n is even then $a \geq 0$).
2. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ (m not a multiple of n e, If n is even, then we must have $a \geq 0$).

Rules are similar to integer exponents

1. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$.
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.
4. $\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even.} \end{cases}$

$$\sqrt{a^2} = |a|$$

If $a \geq 0$ then $\sqrt[n]{a^n} = (a^n)^{1/n} = a^1 = a$.

If $a < 0$ and n is odd then $\sqrt[n]{a^n} = a$. For instance if $n = 3$ and $a = -2$: $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$.

If $a < 0$ and n is even then $\sqrt[n]{a^n} = |a|$. For instance $n = 2$ and $a = -2$: $\sqrt{(-2)^2} = \sqrt{4} = 2 = |-2|$.

Simplify

a) $4\sqrt{5} \cdot 2\sqrt{7}$;

b) $2\sqrt{6} \cdot 3\sqrt[3]{11}$;

c) $\dfrac{4\sqrt{3}}{3\sqrt{2}}$;

d) $(6\sqrt{3} - 5\sqrt{2})(2\sqrt{3} + 3\sqrt{2})$;

e) $\sqrt[3]{8w^{18}}$;

f) $\sqrt{\dfrac{64k^2}{9T^9}}$.

Solution:

a) $4\sqrt{5}\cdot 2\sqrt{7} = 8\sqrt{5}\sqrt{7} = 8\sqrt{5\cdot 7} = 8\sqrt{35}$.

b) Since $\sqrt{6} = \sqrt[6]{6^3}$ and $\sqrt[3]{11} = \sqrt[6]{11^2}$ we have

$$2\sqrt{6}\cdot 3\sqrt[3]{11} = 2\sqrt[6]{6^3}\cdot 3\sqrt[6]{11^2} = 6\sqrt[6]{6^3\cdot 11^2}.$$

c) $\dfrac{4\sqrt{3}}{3\sqrt{2}} = \dfrac{4}{3}\sqrt{\dfrac{3}{2}}$.

d)
$$(6\sqrt{3} - 5\sqrt{2})(2\sqrt{3} + 3\sqrt{2})$$
$$= 12\sqrt{3}\sqrt{3} + 18\sqrt{3}\sqrt{2} - 10\sqrt{3}\sqrt{2} - 15\sqrt{2}\sqrt{2}$$
$$= 36 + 18\sqrt{6} - 10\sqrt{6} - 30 = 6 + 8\sqrt{6}.$$

e) $\sqrt[3]{8w^{18}} = \sqrt[3]{2^3} \cdot \sqrt[3]{w^{18}} = 2w^6$.

f) $\sqrt{\dfrac{64k^2}{9T^9}} = \dfrac{\sqrt{64k^2}}{\sqrt{9T^9}} = \dfrac{\sqrt{64}\sqrt{k^2}}{\sqrt{9}\sqrt{T^9}} = \dfrac{8|k|}{3T^{9/2}}$.