Infeasibility, Fractional Quadratic Problems and Copositivity

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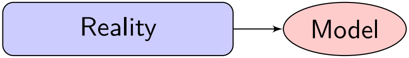
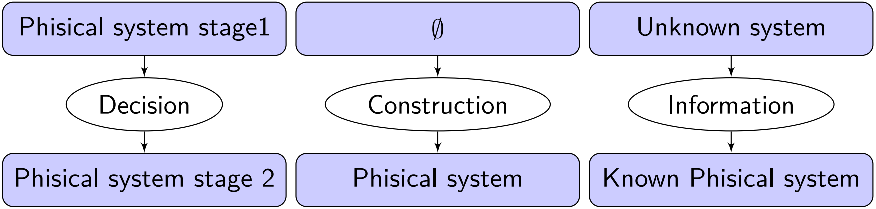
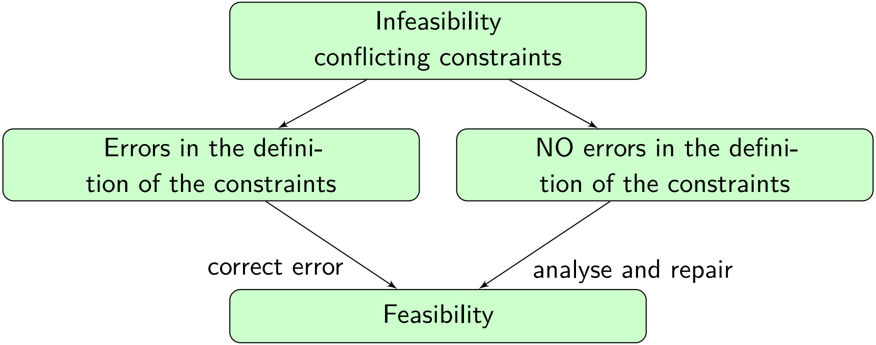
Infeasibility





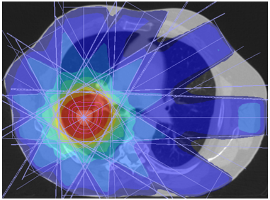
Restrictions $\hookrightarrow$ constraints

Performance $\hookrightarrow$ objective function

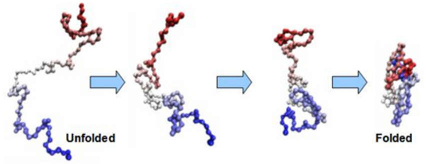


Radiation Treatment Planning IMRT (Intensity-modulated rad. Therapy)

- small individual beams of radiation of adjustable angles and intensity are directed through the body.
- Body modeled as a collection of voxels (volume pixels).
- Constraints $\underbrace{A_1x \leq u,\ A_2x \geq u}_{infeasible},\ x \geq 0.$



Prediction of the three-dimensional protein folding pattern from its amino acid sequence



- protein energy models. (hundred of millions of constraints).
- energy function is a linear combination of basis functions.
- comparing the energy of a misfolded shape to the energy of the native shape.
- native shapes always have a lower energy than a misfolded shape.

- given a set of similar decoy structures to compare against, energy inequalities can be constructed.
- constraints $E_{misfolded}(x) - E_{native}(x) \geq \epsilon,\ x \geq 0.$
- x - parameters (hundred)
- MAXFS helps defining the region for the correct native structure.

Digital Video Broadcasting (Amaldi 2005)

Given m square areas, representing test points for signal reception, and n transmitters, determine the transmission power of each transmitter so that the signal reception at each test point is acceptable.

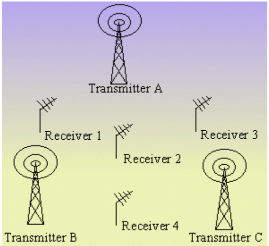


Figure 1: Example of a DVB-T network

- there may be interference in each test point due large delays is signal arrival from different transmitters.
- x_j is the unknown power of the transmitter j.
- constraints $\sum_j a_{ij}x_j \geq b_i$ representing the total signal strength at point j
- a_{ij} is positive if the signal is useful and negative if it interferes.
- b_i is the minimum signal streght needed at test point i to provide adequate signal strength with 95% probability.
- complete coverage of a large number of test points is not usually possible.
- find maximal number of constraints satisfied.

Post-infeasibiity analysis

- detecting feasibility status
- retrieve valuable information regarding the inconsistency
  - identication of conficting sets of constraints
  - irreducible inconsistent sets (IIS)
- achieving feasibility
  - removing constraints
  - changing the coefficients of the constraints

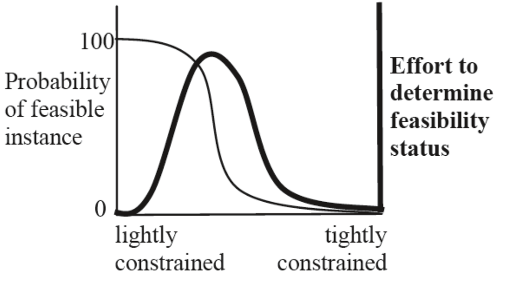


Fig. P1.1. Typical easy-hard-easy pattern for determination of feasibility status

Figure 1: Picture from - Feasibility and infeasibility in Optimization: algorithms and computational methods, John W Chinneck.

Optimal correction of an infeasible system

Given linear system of inequalities

$$Ax \leq b \rightarrow (A+H)x \leq (b+p)$$

Optimal correction p and H, respectively, of the vector b and the matrix A:

$$\begin{aligned} \text{Minimize} \quad & \varphi(H,p) && (1) \\ \text{subject to} \quad & (A+H)x \leq b+p && (2) \\ & x \in \mathcal{X}, H \in \mathcal{R}^{m\times n}, p \in \mathcal{R}^m, && (3) \end{aligned}$$

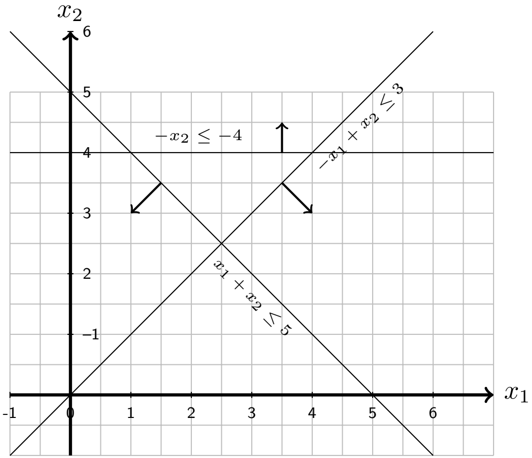
where $\mathcal{X} \subseteq \mathcal{R}^n$ is a convex set and φ is an appropriate matrix norm.

For

- $\varphi = \| \cdot \|_{l_1}$, $(\|A\|_{l_1} = \sum_{ij} |a_{ij}|)$
- $\varphi = \| \cdot \|_{l_\infty}$, $(\|A\|_{l_\infty} = \max_{ij} |a_{ij}|)$
- $\varphi = \| \cdot \|_{\infty}$, $(\| A \|_{\infty} = \max_i \sum_j |a_{ij}|)$

Vatolin (1980) find an optimal correction by solving a set of linear programming problems and that the number of linear programming problems to be solved is

- $l_1 \to 2n + 1$
- $\infty \to 2n + 1$
- $l_\infty \to 2^n$



x_2
x_1
6
5
4
3
2
−1
-1
0
1
2
3
4
5
6
$-x_2 \leq -4$
$-x_1 + x_2 \leq 3$
$x_1 + x_2 \leq 5$

$$\begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$$

$$\| \cdot \|_{l_\infty}, (\|A\|_{l_\infty} = \max_{ij} |a_{ij}|) \rightarrow \begin{bmatrix} -0.1579 & -0.1579 & 0.1579 \\ -0.1579 & -0.1579 & 0.1579 \\ -0.1579 & -0.1579 & 0.1579 \end{bmatrix}$$

$$\| \cdot \|_{l_1}, (\|A\|_{l_1} = \sum_{ij} |a_{ij}|) \rightarrow \begin{bmatrix} -0.6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -0.6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\| \cdot \|_{\infty}, (\| A \|_{\infty} = \max_{i} \sum_{j} |a_{ij}|) \rightarrow \begin{bmatrix} 0 & -0.2308 & 0 \\ 0 & -0.2308 & 0 \\ 0 & -0.2308 & 0 \end{bmatrix}$$

$$\text{Frobenius norm } \|\cdot\|_F \rightarrow \begin{bmatrix} -0.1365 & -0.1613 & 0.0522 \\ -0.0714 & -0.0844 & 0.0273 \\ -0.1065 & -0.1259 & 0.0407 \end{bmatrix}$$

$$\begin{aligned} (P) \quad & \text{Minimize } \|[H,p]\|_F^2 \\ \text{subject to} \quad & (A+H)x \le b+p \\ & H \in \mathcal{R}^{m\times n}, \quad p \in \mathcal{R}^m, \quad x \in \mathbf{X}. \end{aligned} \tag{4}$$

Fractional Quadratic Problem

$$L(h,p,x,\lambda) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} h_{ij}^2 + \frac{1}{2}\sum_{i=1}^{m} p_i^2 + \sum_{i=1}^{m}\lambda_i \sum_{j=1}^{n}(a_{ij}+h_{ij})\,x_j + p_i - b_i \quad (5)$$

From the KKT conditions we know we must have

$$\frac{\delta L(h,p,x,\lambda)}{h_{ij}} = 0 \quad \Leftrightarrow \quad h_{ij} + \lambda_i x_j = 0, \quad (6)$$

$$\frac{\delta L(h,p,x,\lambda)}{p_i} = 0 \quad \Leftrightarrow \quad p_i + \lambda_i = 0, \quad (7)$$

$$\frac{\delta L(h,p,x,\lambda)}{x_j} = 0 \quad \Leftrightarrow \quad \Sigma_{i=1}^{m}\lambda_i(a_{ij}+h_{ij}) = 0, \quad (8)$$

together with

$$\lambda^T((A+H)x - b + p) = 0 \text{ (complementarity slackness conditions)}, \quad (9)$$
$$\lambda \geq \mathbf{0}. \quad (10)$$

We can express conditions (6) to (8) by the following relations between variables

$$\begin{aligned} H &= -\lambda x^T, & (11) \\ p &= -\lambda, & (12) \\ \lambda^T(A+H) &= \mathbf{0}. & (13) \end{aligned}$$

$$\begin{aligned} \min \quad & \lambda^T\lambda(x^Tx+1) \\ \text{s.t.} \quad & Ax - \lambda x^T x \leq b + \lambda \quad (14) \\ & \lambda \geq \mathbf{0}. \end{aligned}$$

Through the complementarity conditions (9) we can further reduce the number of variables, since

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - \lambda_i \sum_{j=1}^{n} x_j^2 - b_i - \lambda_i \right) = 0 \Leftrightarrow$$

$$\lambda_i = 0 \text{ or } \lambda_i = \frac{\sum_{j=1}^{n} a_{ij} x_j - b_i}{\sum_{j=1}^{n} x_j^2 + 1} \quad \forall i = 1, \ldots, m.$$

$$(.)^{+} = \max(0, .)$$

The set of Lagrange multipliers depends on x alone

$$\lambda^* = \frac{(Ax^* - b)^{+}}{x^{*T} x^* + 1}. \tag{15}$$

Replacing λ in formulation (14) by (15) we obtain

$$\begin{array}{ll} \min & \dfrac{(Ax-b)^{+T}(Ax-b)^{+}}{(x^Tx+1)^2}(x^Tx+1) \\ \text{s.t.} & \dfrac{Ax-b}{x^Tx+1} \leq \dfrac{(Ax-b)^{+}}{x^Tx+1} \\ & \dfrac{(Ax-b)^{+}}{x^Tx+1} \geq \mathbf{0}. \end{array}$$

Thus, given that the constraints are trivially satisfied, this result proves that (4) can be formulated as the unconstrained problem

$$\min \frac{(Ax-b)^{+T}(Ax-b)^{+}}{x^Tx+1}.$$

$$\begin{array}{rrrcr} x_1 & +2x_2 & & \leq & 6 \\ x_1 & & +x_3 & \leq & -7 \\ 2x_1 & +x_2 & +x_3 & \leq & -5 \\ x_1 & & +2x_3 & = & 3 \end{array}$$

$$(x_1, x_2, x_3) \in \Omega.$$

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}^2 + \sum_{i=1}^{n} p_i^2$$

$$\begin{array}{rrrcr} (1+h_{11})x_1 & +(2+h_{12})x_2 & +(0+h_{13})x_3 & \leq & 6+p_1 \\ (1+h_{21})x_1 & +(0+h_{22})x_2 & +(1+h_{23})x_3 & \leq & -7+p_2 \\ (2+h_{31})x_1 & +(1+h_{32})x_2 & +(1+h_{33})x_3 & \leq & -5+p_3 \\ (1+h_{41})x_1 & +(0+h_{42})x_2 & +(2+h_{34})x_3 & = & 3+p_4 \end{array}$$

$$(x_1, x_2, x_3) \in \Omega.$$

$$\min_{x \in \Omega} \widetilde{\varphi}(x) = \frac{1}{2} \frac{(x_1 + 2x_2 - 6)^{+2} + (x_1 + x_3 + 7)^{+2} + (2x_1 + x_2 + x_3 + 5)^{+2} + (x_1 + 2x_3 - 3)^2}{1 + x_1^2 + x_2^2 + x_3^2}$$

$$(P) \qquad \min_{x \in \mathbf{X}} \frac{\|(Ax-b)^+\|^2}{1+\|x\|^2}, \tag{16}$$

Equivalent to the following CFQP, where without loss of generality we assume that $\left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right)$ represents $m-r$ initial equalities, followed by r inequalities.

$$(\mathcal{P}_F): \quad \phi = \min \quad \frac{\|v\|^2}{1+\|x\|^2} \tag{17}$$

$$\text{subject to} \quad Ax - v \left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right) b \tag{18}$$

$$v_i \geq 0 \text{ for } i = m-r+1, \ldots, m \tag{19}$$

$$x \in \mathcal{X}. \tag{20}$$

Copositive Formulation

$$\psi = \min \left\{ f(\mathbf{x}) = \underbrace{\frac{\mathbf{x}^T C \mathbf{x} + 2\mathbf{c}^T \mathbf{x} + \gamma}{\mathbf{x}^T B \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + \beta}}_{p(x)} : \underbrace{\mathbf{x} \geq 0, A\mathbf{x} = \mathbf{a}}_{\mathcal{T}} \right\}$$

Assumptions $p(x) > 0$ and $\mathcal{T}$ is compact.

Compactness of $\mathcal{T}$ and strict positivity of p over this set implies primal attainability.

$$\psi = \min \left\{ f(\mathbf{x}) = \frac{\mathbf{x}^T C \mathbf{x} + 2\mathbf{c}^T \mathbf{x} + \gamma 1}{\mathbf{x}^T B \mathbf{x} + 2\mathbf{b}^T \mathbf{x} + \beta 1} : \mathbf{x} \geq 0, A\mathbf{x} - \mathbf{a}1 = 0 \right\} \qquad (21)$$

$$\overline{A} = \begin{bmatrix} \mathbf{a}^T\mathbf{a} & -\mathbf{a}^T A \\ -A^T\mathbf{a} & A^T A \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \beta & \mathbf{b}^T \\ \mathbf{b} & B \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \gamma & \mathbf{c}^T \\ \mathbf{c} & C \end{bmatrix}.$$

$$A\mathbf{x} = \mathbf{a} \iff [-\mathbf{a}\,,\, A]\mathbf{z} = \mathbf{o} \iff \mathbf{z}^T \overline{A} \mathbf{z} = 0\,,$$

$$\mathbf{z} = \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} \in \mathbb{R}^{n+1}$$

,

$$\psi = \min \left\{ \frac{\mathbf{z}^T \overline{C} \mathbf{z}}{\mathbf{z}^T \overline{B} \mathbf{z}} : \mathbf{z} \in \mathbb{R}^{n+1}_+,\ \mathbf{z}_1 = 1,\ \mathbf{z}^T \overline{A} \mathbf{z} = 0 \right\}.$$

$$\psi = \min \left\{ \frac{\mathbf{z}^T \overline{C} \mathbf{z}}{\mathbf{z}^T \overline{B} \mathbf{z}} : \mathbf{z} \in \mathbb{R}_+^{n+1},\ \mathbf{z}_1 = 1,\ \mathbf{z}^T \overline{A} \mathbf{z} = 0 \right\} .$$

$Z = \mathbf{z}\mathbf{z}^T$ and $\mathbf{z} \in \mathbb{R}_+^{n+1}$

$$\mathbf{z}^T \overline{A} \mathbf{z} = \overline{A} \bullet Z$$

$\overline{A}$ psd

$$Z_{11} = z_1^2$$

$$\psi = \min \left\{ \frac{\overline{C} \bullet Z}{\overline{B} \bullet Z} : Z_{11} = 1\,,\ \overline{A} \bullet Z = 0\,,\ \operatorname{rank}(Z) = 1\,,\ Z \in \mathcal{C}_{n+1}^* \right\} .$$

$$\psi = \min \left\{ \frac{\overline{C} \bullet Z}{\overline{B} \bullet Z} : Z_{11} = 1\,,\; \overline{A} \bullet Z = 0\,,\; \mathrm{rank}(Z) = 1\,,\; Z \in \mathcal{C}^*_{n+1} \right\}.$$

By homogeneity, for any Z feasible $Z_{11} = 1$ can be replace the by $Z_{11} > 0$.

$$X = \frac{1}{\overline{B} \bullet Z}\, Z \in \mathcal{C}^*_{n+1}$$

X also has rank one with $X_{11} > 0$ and satisfies $\overline{B} \bullet X = 1$

$$\psi = \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1,\, \overline{A} \bullet X = 0,\, \mathrm{rank}(X) = 1, X_{11} > 0,\, X \in \mathcal{C}^*_{n+1} \right\}.$$

$$\psi = \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1,\ \overline{A} \bullet X = 0,\ \mathrm{rank}(X) = 1, X_{11} > 0,\ X \in \mathcal{C}^*_{n+1} \right\}.$$

It was also proved that the strict linear inequality and the (non-convex) rank-one constraint could be droped to obtain the equivalent problem

$$\min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}.$$

$$\begin{aligned} \psi &= \min \left\{ f(\mathbf{x}) = \frac{\mathbf{x}^T C \mathbf{x}}{\mathbf{x}^T B \mathbf{x}} : A\mathbf{x} = \mathbf{a}, \mathbf{x} \in \mathbb{R}^n_+ \right\} \\ &= \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}. \end{aligned}$$

Duality

$$\psi = \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\} . \tag{22}$$

By weak duality

$$\psi \geq \lambda^* = \sup \left\{ \lambda : \overline{C} - \lambda \overline{B} - \mu \overline{A} \in \mathcal{C}_{n+1} \right\} . \tag{23}$$

Slater's condition is always violated. Indeed, if $Z \in \text{int } \mathcal{C}^*_{n+1}$ is feasible to (22), then $Z - \alpha I_{n+1} \in \mathcal{C}^*_{n+1}$ for a small $\alpha > 0$, and in particular this matrix is psd. But

$$\overline{A} \bullet (Z - \alpha I_{n+1}) = 0 - \alpha \, \text{trace}(\overline{A}) < 0 \, ,$$

is a contradiction to the fact of $\overline{A} \in \mathcal{P}_{n+1} \setminus \{O\}$.

- it is not possible to infer strong duality (in particular, dual attainability) from standard arguments.

- the dual problem is strictly feasible, which implies attainability of the primal (this was already established before) and zero duality gap

Lower Bounds

Checking condition $X \in \mathcal{C}^*_{n+1}$ is (co-)NP-hard but using the inclusion $\mathcal{C}^*_{n+1} \subseteq \mathcal{D}_{n+1} = \mathcal{P}_{n+1} \cap \mathcal{N}_{n+1}$ a lower bound for the CFQP was proposed by solving

$$\psi_{\text{cop}} = \min \left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{D}_{n+1} \right\} \tag{24}$$

Size reduction, from n^2 to $(n-m)^2$

A is an $m \times n$ matrix with full row rank $m < n$.

$A^T A$ is psd and $n \times n$

$$\dim \mathsf{ker}\ (A^T A) = k = n - m$$

$$\mathsf{ker}\ (A^T A) = \langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$$

$$\dim \mathsf{ker}\ \overline{A} = k + 1$$

$\overline{\mathbf{u}}_i = [0, \mathbf{u}_i^T]^T$ form an orthonormal system in ker $\overline{A} \subseteq \mathbb{R}^{n+1}$

$$\tilde{\mathbf{u}}_0 = \left[\begin{array}{c} 1 \\ A^T(AA^T)^{-1}\mathbf{a} \end{array} \right] \quad \text{and} \quad \overline{\mathbf{u}}_0 = \frac{1}{\|\tilde{\mathbf{u}}_0\|}\,\tilde{\mathbf{u}}_0\,.$$

$$\text{ker }(\overline{A}) = \langle \overline{\mathbf{u}}_0, \dots, \overline{\mathbf{u}}_k \rangle$$

Q is a $(k+1) \times (n+1)$ matrix, collecting the above system as rows:

$$Q^T = [\overline{\mathbf{u}}_0, \dots, \overline{\mathbf{u}}_k]$$

For any $X \in \mathcal{D}_{n+1}$, we have $\overline{A} \bullet X = 0$ if and only if

$$\overline{A}X = O$$

if and only if

$$X = Q^T Y Q \quad \text{for some } Y \in \mathcal{P}_{k+1} \text{ satisfying } Q^T Y Q \in \mathcal{N}_{n+1} \,. \tag{25}$$

Hence, using

$$\overline{C} \bullet X = (Q\overline{C}Q^T) \bullet Y$$

$$\overline{A} \bullet X = (Q\overline{A}Q^T) \bullet Y$$

$$\overline{B} \bullet X = (Q\overline{B}Q^T) \bullet Y$$

We arrive at the reduced SDP

$$\min \left\{ (Q\overline{C}Q^T) \bullet Y : (Q\overline{B}Q^T) \bullet Y = 1 \, , \, Q^T Y Q \geq O \, , Y \in \mathcal{P}_{k+1} \right\} , \qquad (26)$$

working on smaller psd matrices, but retaining $\mathcal{O}(n^2)$ linear inequalities.

Computational results

Generate Instances

For $\beta = \gamma = 1$ and for selected values of n and $m = \lfloor \frac{n}{2} \rfloor$, we have generated instances of program (21) as follows:

1. a symmetric psd $n \times n$ matrix B is randomly generated, along with a suitably scaled vector $\mathbf{b} \in \mathrm{int}(\mathbb{R}^n_+)$ such that $\overline{B}$ given by () need not be psd, and can have negative entries (but obviously $\overline{B} \in \mathcal{D}^*_{n+1}$). Observe that by construction, $\overline{B}$ is strictly $\mathbb{R}^n_+$-copositive and therefore, for any choice of $\overline{A}$, strictly $\Gamma_{\overline{A}}$-copositive for sure.

2. a (possibly indefinite) symmetric $n \times n$ matrix C is randomly generated with entries of varying sign, along with a randomly drawn vector $\mathbf{c} \in \mathbb{R}^n$ (again, no sign restrictions on the coordinates).

3. an $m \times n$ matrix A with a strictly positive first row, but varying sign of entries elsewhere, is randomly generated;

4. an arbitrary vector $\mathbf{x} \in \Delta$ is drawn at random. Then the choice $\mathbf{a} = A\mathbf{x}$ ensures that $\mathcal{T}$ is compact, so the model assumptions are guaranteed.

5. Finally, based on $(A, \mathbf{a})$, the matrix Q is determined and a solution Y to (26) is calculated. As stated before, $X = Q^T Y Q$ solves (24). The objective $\overline{C} \bullet X = (Q\overline{C}Q^T) \bullet Y$ is used as a relaxation bound.

- Instances of sizes $n \in \{4, 9, 49, 79\}$ were generated, resulting in SDP instances of dimensionality 5, 10, 50 and 80
- The maximum size of 80 was possible due to the size reduction achieved in (26).
- The clear impact of this reduction is depicted in the three last columns of next Figure. For two problems of size 5 (ABJ5_0) and 10 (ABJ10_0), `SeDuMi` output reports the size of the SDP problems, for the GPM approach, the direct copositive relaxation (24) and with (26).

	GPM (Gloptipoly3)	Copositive Relaxation	Copositve Relaxation with Reduction
ABJ5_0	eqs m = 210, order n = 98, dim = 2380, blocks = 7 nnz(A) = 2385 + 0, nnz(ADA) = 44100, nnz(L) = 22155 Detailed timing (sec) Pre IPM Post 7.001E-03 3.920E-01 2.002E-03	eqs m = 15, order n = 33, dim = 53, blocks = 3 nnz(A) = 55 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 5.200E-02 7.001E-02 9.958E-04	eqs m = 6, order n = 27, dim = 33, blocks = 3 nnz(A) = 121 + 0, nnz(ADA) = 36, nnz(L) = 21 Detailed timing (sec) Pre IPM Post 4.003E-03 3.800E-02 9.958E-04
ABJ10_0	eqs m = 5005, order n = 718, dim = 85638, blocks = 12 nnz(A) = 138325 + 0, nnz(ADA) = 25050025, nnz(L) = 12527515 Detailed timing (sec) Pre IPM Post 8.460E-01 4.794E+02 2.800E-02	eqs m = 55, order n = 113, dim = 203, blocks = 3 nnz(A) = 210 + 0, nnz(ADA) = 3025, nnz(L) = 1540 Detailed timing (sec) Pre IPM Post 6.100E-02 6.500E-02 1.006E-03	eqs m = 15, order n = 99, dim = 119, blocks = 3 nnz(A) = 1291 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 2.900E-02 5.301E-02 1.992E-03

Figure 2: Sizes of SDP relaxations

Table 2 reports for each instance the information,

- Instance — Instance name;
- Cop R — Value of the lower bound obtained by the SDP relaxation of the copositive formulation (26);
- Time1(s) — CPU time in seconds to obtain Cop R;
- Gap — The relative gap provided by Cop R, $\left|\frac{\text{Cop R-}\mathtt{BARON}\text{ Optimal value}}{\mathtt{BARON}\text{ Optimal value}}\right|$;
- GPM — Value of the lower bound obtained by `Gloptipoly 3`;
- Time2(s) — CPU time in seconds to obtain the GPM lower bound;
- St — Status of `Gloptipoly 3` solution for the default relaxation order;
- root B — Value of the lower bound obtained at the root node by `BARON`;

Table 1: Copositive Relaxation versus Gloptipoly 3 and BARON

Instance	Cop R	Time1(s)	Gap	GPM R	Time2(s)	St.	root B.
ABJ5_0	-0.7865	2.700e-02	0.4837	-0.5275	1.045e+00	1	-26.1028
ABJ5_1	-0.4923	3.400e-02	1.8293	-0.5414	1.014e+00	1	-11.8308
ABJ5_2	-0.7693	2.700e-02	0.6771	-0.5089	9.672e-01	1	-11.9631
ABJ5_3	-0.3603	2.900e-02	0.9907	-0.2207	1.310e+00	1	-3.9613
ABJ5_4	-1.2562	2.700e-02	0.5467	-0.9428	9.984e-01	1	-0.8123
ABJ5_5	+0.4643	3.000e-02	0.1552	+0.2225	1.108e+00	1	-2.2940
ABJ5_6	-0.5768	3.100e-02	0.5831	-0.3671	9.828e-01	1	-8.6291
ABJ5_7	-0.0815	3.300e-02	15.2108	-0.0657	8.892e-01	1	-5.1034
ABJ5_8	-0.5946	2.600e-02	0.4752	-0.3708	9.516e-01	1	-0.4031
ABJ5_9	-0.8705	3.100e-02	0.9123	-0.5753	6.708e-01	1	-0.4553
ABJ10_0	-0.3095	3.500e-02	0.5090	-0.1962	7.010e+02	1	-23.9325
ABJ10_1	-0.6779	3.100e-02	0.4781	-0.4882	5.737e+02	1	-0.4587
ABJ10_2	+0.4144	3.400e-02	0.0533	+0.4288	6.395e+02	1	-3.4076
ABJ10_3	-0.3105	3.500e-02	1.2843	-0.1840	6.298e+02	1	-12.3357
ABJ10_4	-0.3885	3.900e-02	0.4746	-0.2689	5.122e+02	1	-0.2635
ABJ10_5	-0.7710	4.300e-02	0.2028	-0.6198	6.619e+02	1	-55.5414
ABJ10_6	-1.2861	3.100e-02	0.5562	-0.8749	7.123e+02	1	-0.8265
ABJ10_7	-0.1154	3.900e-02	1.1720	-0.0760	6.219e+02	1	-25.4559
ABJ10_8	-0.6486	3.100e-02	0.2828	-0.4558	6.239e+02	1	-0.5056
ABJ10_9	-0.3070	4.800e-02	0.5997	-0.1794	6.183e+02	1	-0.1919

Table 2: Copositive Relaxation versus Gloptipoly 3 and BARON

Instance	Cop R	Time1(s)	Gap	GPM R	Time2(s)	St.	root B.
ABJ50_0	-0.7435	3.238e+00	0.3552	O of M	-		-502.4740
ABJ50_1	-0.9606	2.731e+00	0.2229	O of M	-		-0.7856
ABJ50_2	-0.7844	3.192e+00	0.2786	O of M	-		-0.6135
ABJ50_3	-0.4022	2.983e+00	0.3630	O of M	-		-1463.1800
ABJ50_4	-0.2677	3.001e+00	0.8199	O of M	-		-451.7790
ABJ50_5	-0.6484	2.981e+00	0.6369	O of M	-		-0.3962
ABJ50_6	-0.5760	3.498e+00	0.3702	O of M	-		-989.5200
ABJ50_7	-0.6486	2.993e+00	0.3201	O of M	-		-0.4914
ABJ50_8	-0.5985	3.221e+00	0.3456	O of M	-		-490.0360
ABJ50_9	-0.3730	3.244e+00	0.3215	O of M	-		-626.8870
ABJ80_0	-0.4427	5.049e+01	0.5019	O of M	-		-1394.8500
ABJ80_1	-0.5806	5.532e+01	0.2984	O of M	-		-0.4472
ABJ80_2	-0.8597	5.532e+01	0.2869	O of M	-		-0.6681
ABJ80_3	-0.4345	5.519e+01	0.3302	O of M	-		-1849.5000
ABJ80_4	-0.8625	5.101e+01	0.3214	O of M	-		-0.6528
ABJ80_5	-0.4670	5.117e+01	0.3301	O of M	-		-0.3511
ABJ80_6	-0.3473	5.539e+01	0.6090	O of M	-		-2488.4700
ABJ80_7	-0.5883	5.105e+01	0.3607	O of M	-		-1487.1000
ABJ80_8	-0.4181	5.532e+01	0.5004	O of M	-		-736.0130
ABJ80_9	-0.7023	5.099e+01	0.3568	O of M	-		-0.5177

All the tests have been performed on a Pentium Intel(R) Core(TM)i7, with CPU E8400, 2.8GHz, 4,00 GB RAM, and 64-bit operating system Windows. A tolerance parameter 10^{-4} was considered for `BARON` and `SeDuMi`.

- The lower bounds provided by solving the SDP relaxation of the Copositive formulation are very good, as the gaps show, and outperforms the initial lower bound of `BARON`, and of the GPM relaxation.
- For problems of size 50 and 80 `Gloptipoly 3` ran out of memory (O of M), as expected given the size of the corresponding SDP problem
- The results show that the reduction proposed in (26) is crucial as the size of the problem increases.

Conclusions?

Think (CO) or (Completely) Positive!

Think co(mpletely)positive ! Matrix properties, examples and a clustered bibliography on copositive optimization
Immanuel M. Bomze, Werner Schachinger , Gabriele Uchida
J Glob Optim (2012)

Questions?