Copositive Optimization

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Motivation

Standard quadratic program

$$
\begin{array}{rrl}
\text{(StQ)} & \min & x^T Q x \\
 & \text{s.t.} & e^T x = 1 \\
 & & x \geq 0 \\
\text{(StQCp)} & \min & \langle Q, X \rangle \\
 & \text{s.t.} & \langle E, X \rangle = 1 \\
 & & X \in \left\{ X \in \mathcal{M}_n : X = YY^T, Y \in \mathbb{R}^{n \times k},\ Y \geq O \right\} = \mathcal{C}^* \\
\text{(StQCo)} & \max & y \\
 & \text{s.t.} & Q - yE \in \left\{ X \in \mathcal{M}_n : y^T X y \geq 0 \text{ for all } y \in \Re^n_+ \right\} = \mathcal{C} \\
 & & y \in \mathbb{R}
\end{array}
$$

Copositive Optimization

$$
\begin{aligned}
\min \quad & \langle C, X \rangle \\
\text{s.t.} \quad & \langle A_{i\cdot}, X \rangle \quad = b_i, i \in \{1, \dots, m\} \\
& X \in \mathcal{K}
\end{aligned}
$$

$\mathcal{K} = \mathcal{C}$ Copositive Cone or $\mathcal{K} = \mathcal{C}^*$ Completely Positive Cone

$$\langle X, Y \rangle = trace(Y^T X) = \sum_{i,j=1}^{n} X_{ij} Y_{ij}$$

Lower Bounds

Copositive Relaxation

$$
\begin{aligned}
\min \quad & \langle C, X \rangle \\
\text{s.t.} \quad & \langle A_{i\cdot}, X \rangle = b_i, i \in \{1, \ldots, m\} \\
& \cancel{X \in \mathcal{C}^*} \\
& X \in \mathcal{K} \supset \mathcal{C}^*
\end{aligned}
$$

Outline

Cones

Definition (Cone)

A set $\mathcal{K} \in \Re^n$ is a cone if $\lambda \geq 0, A \in \mathcal{K} \Rightarrow \lambda A \in \mathcal{K}$.

Definition (Pointed Cone)

A cone $\mathcal{K}$ is pointed if $\mathcal{K} \cap -\mathcal{K} = \{0\}$.

Definition (Convex Cone)

A cone $\mathcal{K}$ is convex if for $A, B \in \mathcal{K}$ and $\alpha, \beta \in \Re^+$, $\alpha A + \beta B \in \mathcal{K}$.

Definition (Closed Cone)

A cone $\mathcal{K}$ is closed if it contains its boundary.

Definition (Cone of Symmetric matrices)

$$\mathcal{M}_n = \left\{X \text{ an } n \times n \text{ matrix} : X^T = X\right\}$$

Definition (Cone of Nonnegative symmetric matrices)

$$\mathcal{N}_n = \{X \in \mathcal{M}_n : X_{ij} \geq 0 \text{ for } i,j = 1,\ldots,n\}$$

Definition (Cone of the Positive Semidefinite matrices)

$$\mathcal{S}_n = \left\{X \in \mathcal{M}_n : y^T X y \geq 0 \text{ for all } y \in \Re^n\right\}$$

Definition (Cone of the Positive Definite matrices)

$$\mathcal{S}_n^+ = \left\{X \in \mathcal{M}_n : y^T X y > 0 \text{ for all } y \in \Re^n \setminus \{0\}\right\}$$

Definition (Cone of Doubly Nonnegative matrices)

$$\mathcal{D}_n = \{X \in \mathcal{M}_n : X = D_0 \cap S_0 \text{ with } D_0 \in \mathcal{N}_n \text{ and } S_0 \in \mathcal{S}_n\}$$

Definition (Dual of the Cone of Doubly Nonnegative matrices)

$$\mathcal{D}_n^* = \{X \in \mathcal{M}_n : X = D_0 + S_0 \text{ with } D_0 \in \mathcal{N}_n \text{ and } S_0 \in \mathcal{S}_n\}$$

Definition (Cone of the Copositive matrices)

$$\mathcal{C}_n = \left\{ X \in \mathcal{M}_n : y^T X y \geq 0 \text{ for all } y \in \Re^n_+ \right\}$$

Definition (Cone of the Strict Copositive matrices)

$$\mathcal{C}_n^+ = \left\{ X \in \mathcal{M}_n : y^T X y > 0 \text{ for all } y \in \Re^n_+ \setminus \{0\} \right\}$$

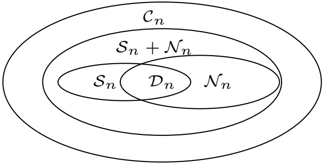
Definition (Cone of the $\mathcal{D}-$Copositive matrices)

$$\mathcal{CD}_n = \left\{ X \in \mathcal{M}_n : y^T X y \geq 0 \text{ for all } y \in \mathcal{D} \subseteq \Re^n_+ \right\}$$

Properties of Copositive Matrices and Copositive Cone

[Diananda(1962)], [Hall and Newman(1963)], [Baston(1968/1969)]

- Nonnegative $(X \in \mathcal{N}_n) \Rightarrow$ Copositive $(X \in \mathcal{C}_n)$
- Semidefinite $(X \in \mathcal{S}_n) \Rightarrow$ Copositive $(X \in \mathcal{C}_n)$



- For $n = 2$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = X_{11} y_1^2 + 2 X_{12} y_1 y_2 + X_{22} y_2^2 \geq 0$$

$$(X_{11} \geq 0) \wedge (X_{22} \geq 0) \wedge (\underbrace{(X_{12} \geq 0)}_{\text{Nonnegative - } \mathcal{N}_n} \vee \underbrace{(X_{12}^2 - X_{11} X_{22} \leq 0)}_{\text{Semidefinite - } \mathcal{S}_n})$$

- $\mathcal{C}_n = \mathcal{N}_n + \mathcal{S}_n$ for $n = 3, 4$.

- Example (Horn)

$$H = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$x^T H x = (x_1 - x_2 + x_3 + x_4 - x_5)^2 + 4x_2x_4 + 4x_3(x_5 - x_4)$$

$$x^T H x = (x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_3(x_4 - x_5)$$

- $X_{ii} \geq 0.$

$$e_i^T = \begin{bmatrix} 0 & \ldots & \underbrace{1}_{i} & \ldots & 0 \end{bmatrix} \text{ then } e_i^T X e_i = X_{ii}$$

- $X_{ii} = 0 \Rightarrow X_{ij} \geq 0$.

$$(\alpha e_i + e_j)^T X (\alpha e_i + e_j) = \alpha^2 X_{ii} + 2\alpha X_{ij} + X_{jj}$$
$$\text{if } X_{ii} = 0 \text{ and } \alpha \to +\infty \text{ then } X_{ij} \geq 0$$

- Not invariant under basis transformations.
- Is invariant under permutation and scaling transformations.
- $\mathcal{C}_n$ is closed, convex, pointed and full dimensional.
- $\mathcal{C}_n$ is nonpolyhedral.
- The interior of $\mathcal{C}_n$ is the set of strictly copositive matrices, $\mathcal{C}_n^+$.
- If there exists a strictly positive vector v such that $v^T A v = 0$ then $A \in \mathcal{S}_n$.

- It is co-NP-complete to check that a matrix is copositive ([Murty and Kabadi(1987)])

Dual Cone

Definition (Dual Cone)

Consider the cone $\mathcal{K} \subseteq \mathbb{R}^{n\times n}$. The dual cone of $\mathcal{K}$ is,

$$\mathcal{K}^* = \left\{Y \in \mathbb{R}^{n\times n} : \forall X \in \mathcal{K}, \langle X, Y\rangle \geq 0\right\}$$

Definition (Self Dual)

A cone $\mathcal{K}$ is self-dual if $\mathcal{K} = \mathcal{K}^*$.

Example: $\mathcal{S}_n^* = \mathcal{S}_n$.

Properties of the Dual Cone

Let $\mathcal{K}$ be a cone,

- $\mathcal{K}^*$ is closed and convex.
- $\mathcal{K}^{**} = \overline{\text{conv}(\mathcal{K})}$.
- $\mathcal{K}$ closed and convex $\Rightarrow \mathcal{K}^{**} = \mathcal{K}$.
- Lemma $\hat{\mathcal{K}} \subseteq \mathcal{K} \Rightarrow \hat{\mathcal{K}}^* \supseteq \mathcal{K}^*$.

Completely Positive Cone - Dual Copositive Cone

Definition (Cone of Completely Positive matrices)

$$
\begin{aligned}
\mathcal{CP}_n &= \left\{ X \in \mathcal{M}_n : X = \sum_{i=1}^{k} z^i (z^i)^T : k \in \mathbb{N}, z^i \geq 0 \right\} \\
&= \left\{ X \in \mathcal{M}_n : X = YY^T, Y \in \mathbb{R}^{n \times k},\ Y \geq O \right\}
\end{aligned}
$$

Theorem

The dual of $\mathcal{C}_n$ is the cone of **Completely Positive matrices**.

Theorem

$$\mathcal{CP}_n = \mathcal{C}_n^*$$
$$\mathcal{CP}_n^* = \mathcal{C}_n$$
$$\mathcal{C}_n^{**} = \mathcal{C}_n$$

Proof

$$\text{Any } A \in \mathcal{C} \text{ and } B = \sum_{i=1}^{k} z^i (z^i)^T,\ z^i \geq 0 \in \mathcal{CP}$$

$$\langle A, B \rangle = \left\langle A, \sum_{i=1}^{k} z^i (z^i)^T \right\rangle = \sum_{i=1}^{k} (z^i)^T A z^i \geq 0 (\text{because } A \in \mathcal{C})$$

$$B \in \mathcal{C}^* \Rightarrow \boxed{\mathcal{CP} \subseteq \mathcal{C}^*}$$

$$\text{Any } A \in \mathcal{CP}^* \text{ then } \langle A, B \rangle \geq 0 \text{ in particular } B = vv^T\ (v \geq 0) \text{ we have}$$

$$\left\langle A, vv^T \right\rangle = v^T A v \geq 0 \text{ and so } A \in \mathcal{C} \text{ and } \mathcal{CP}^* \subseteq \mathcal{C}$$

From the previous result we have that $\boxed{\mathcal{CP} \supseteq \mathcal{C}^*}$ so

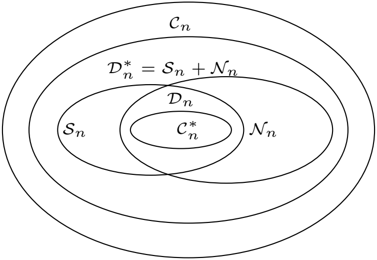
$$\boxed{\mathcal{C}^* = \mathcal{CP}}.$$

□

- $\mathcal{C}_n^*$ is closed, convex, pointed and full dimensional.
- The extremal rays of $\mathcal{C}_n^*$ are the rank-one matrices $X = xx^T$ with $x \geq 0$ and $x \neq 0$.
- Characterization of the interior of the completely positive cone. [Dür and Still(2008)]

$$\text{int}(\mathcal{C}^*) = \{AA^T : A = [A_1|A_2], \text{ with } A_1 > 0 \text{ nonsingular}, \ A_2 \geq 0\}$$

- Checking that a matrix is in $\mathcal{C}_n^*$ is NP-hard. [Dickinson and Gijben(2014)]


$\mathcal{C}_n$
$\mathcal{D}_n^* = \mathcal{S}_n + \mathcal{N}_n$
$\mathcal{D}_n$
$\mathcal{S}_n$
$\mathcal{C}_n^*$
$\mathcal{N}_n$

Detecting Copositivity

Based on Submatrices

A principal submatrix of A is a matrix which is constructed by selecting some of the rows and columns of A simultaneously. Given $I = 1, \ldots, n$, $A_{II} = [A_{ij}]$ for $i, j \in I$.

Eigenvector and eigenvalues

[Kaplan(2001)]

The matrix A is copositive if and only if all principal submatrices of A have no positive eigenvector with negative eigenvalue.

$$A_{II}v = \lambda v \text{ if } v > 0 \Rightarrow \lambda \geq 0$$

Similar to the Schur Complement

$$\begin{bmatrix} a & b^T \\ b & C \end{bmatrix}$$

The matrix A is copositive $(a \geq 0)$ if and only one of the following conditions hold.

- $C \in \mathcal{C} \wedge (aC - bb^T) \in \mathcal{C}_{\mathcal{D}}$ with $D = \left\{ y : b^T y \leq 0,\ y \geq 0 \right\}$ Remember?
- $b \geq 0 \wedge C \in \mathcal{C}$
- $b \leq 0 \wedge (aC - bb^T) \in \mathcal{C}$

Theorem
$A \in \mathcal{M}, D \subseteq \mathbb{R}^n$ a polyhedral cone and R a matrix whose columns are representatives of the extremal rays of D then $A \in \mathcal{C}_\mathcal{D}$ iif $R^T AR \in \mathcal{C}$.

Checking copositivity in polynomial time,

- $\{-1, +1\}^{n \times n}$,
- diagonal matrices,
- tridiagonal matrices,
- acyclic matrices.

Based on Simplicial Partitions

[Sponsel et al.(2012)Sponsel, Bundfuss, and Dür], [Bundfuss(2009)]

Lemma

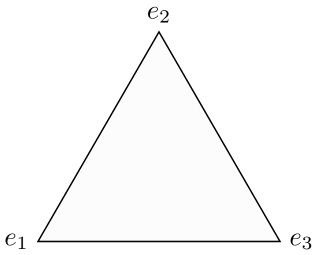
Let $A \in \mathcal{M}_n$.

$$A \in \mathcal{C}_n \quad \Leftrightarrow \quad x^T Ax \geq 0, \forall x \in \mathcal{R}_+^n, \text{ with } \|x\| = 1$$
$$A \in \mathcal{C}_n^+ \quad \Leftrightarrow \quad x^T Ax > 0, \forall x \in \mathcal{R}_+^n, \text{ with } \|x\| = 1$$

Proof $\Leftarrow$ Let $x \in \mathcal{R}_+^n$, $\tilde{x} = \frac{x}{\|x\|}$, such that $\|\tilde{x}\| = 1$ so $\tilde{x}^T A\tilde{x} \geq 0$ but since $\tilde{x}^T A\tilde{x} = \frac{1}{\|x\|^2} x^T Ax$ we have that $x^T Ax \geq 0$. □

Choose the 1-norm, $\|x\|_1$, define the standard simplex

$$\Delta^S = \{x \in \mathcal{R}^n_+ : \|x\|_1 = 1\} = conv\{e_1, e_2, \ldots, e_n\}$$

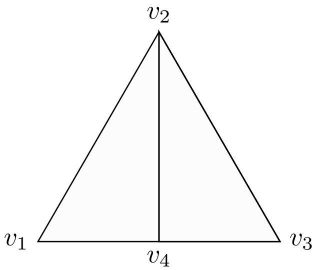


For all $x \in \Delta^S$, there are unique $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ with $\lambda \geq 0$ such that

$$x = \sum_{i=1}^{n} \lambda_i e_i \text{ with } \sum_{i=1}^{n} \lambda_i = 1.$$

$$x^T A x = \left(\sum_{i=1}^{n} \lambda_i e_i^T \right) A \left(\sum_{i=1}^{n} \lambda_i e_i^T \right) = \sum_{i,j=1}^{n} \lambda_i \lambda_i e_i^T A e_j$$

Sufficient condition $e_i^T A e_j \geq 0 \Leftrightarrow A(i,j) \geq 0, \ \forall i,j \Leftrightarrow A \in \mathcal{N}_n$



A family of Ps of simplices $\{\Delta_1, \ldots, \Delta_m\}$ satisfying

$$\bigcup_{i=1}^{m} \Delta_i = \Delta_S \text{ and } \operatorname{int}(\Delta_i) \cap \operatorname{int}(\Delta_j) = \emptyset, \;\; i \neq j$$

is called a *simplicial partition of* Δ_S.

Δ is the convex hull of n affinely independent points (vertices) $\Delta = \text{conv}\{v_1, \dots, v_n\}$
For all $x \in \Delta = \text{conv}\{v_1, \dots, v_n\}$, there are unique $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ with $\lambda \geq 0$ such that (barycentric coordinates with respect to Δ):

$$x = \sum_{i=1}^{n} \lambda_i v_i \text{ with } \sum_{i=1}^{n} \lambda_i = 1.$$

As a simplex Δ is determined by its vertices, it can be represented by a matrix V_Δ whose columns are these vertices. $V_\Delta = [v_1\ v_2\ \dots\ v_n]$

$$x^T A x = \left(\sum_{i=1}^{n} \lambda_i v_i^T \right) A \left(\sum_{i=1}^{n} \lambda_i v_i^T \right) = \sum_{i,j=1}^{n} \lambda_i \lambda_i v_i^T A v_j$$

Sufficient condition $v_i^T A v_j \geq 0\ \forall i, j$
Necessary condition $v_i^T A v_i \geq 0\ \forall i$

Theorem

Let $A \in \mathcal{M}_n$, and let P be a simplicial partition of Δ_S. If

$$(v_i^k)^T A(v_j^k) \geq 0, \; \forall \Delta_k = \text{conv}\{v_1^k, \ldots, v_2^k\} \in P$$

then A is copositive.

Proof

$$V^k = [v_1^k, \ldots, v_n^k]$$

$$x \in \Delta_k$$

$$x^T Ax = (V^k \lambda)^T A(V^k \lambda) = \lambda^T \left(V^{kT} AV^k\right) \lambda \geq 0$$

□

Data: $A \in \mathcal{M}_n$,
Result: Copositive certificate = "Yes" or "No"
$Ps = \{\Delta_S\}$;
while $Ps \neq \emptyset$ **do**
    choose $\Delta = \text{conv}\{v_1, \ldots, v_n\} \in Ps$;
    if $\exists v_i \in \{v_1, \ldots, v_n\} : v_i^T A v_i < 0$ **then**
        return "No"
    else
        if $v_i^T A v_j \geq 0$ for all $i, j = 1, \ldots, n$ **then**
            $Ps \leftarrow Ps \setminus \Delta$;
        else
            $Ps \leftarrow Ps \setminus \Delta$;
            partition Δ into Δ_1 and Δ_2 ;
            $Ps \leftarrow Ps \setminus \Delta \cup \{\Delta_1, \Delta_2\}$
        end
    end
end

$$Ps = \{\Delta_1, \ldots, \Delta_m\}$$

$$\text{(Fineness of a Partition } Ps) \mapsto \delta(Ps) = \max_{\Delta \in Ps} \max_{u,v \in V(\Delta)} \|u - v\|$$

Theorem

Let $A \in \mathcal{M}_n$. The following assertions are equivalent

- A is not copositive,
- There exists $\epsilon > 0$ such that for all partitions Ps of Δ^S with $\delta(Ps) < \epsilon$ there exists a $v \in V(Ps)$ with $v^T Av < 0$.

Theorem

Let $A \in \mathcal{M}_n$, strict-copositive, $A \in \mathcal{C}^+$ then there exits $\epsilon > 0$ such that for all partitions Ps of Δ^S with $\delta(Ps) < \epsilon$, $\quad v^T Au > 0$ for all $(u, v) \in V(Ps)$.

Algorithm may not terminate

Theorem

Let $A \in \mathcal{M}_n$, be copositive, and $\Delta = \text{conv}\{v_1, \ldots, v_n\}$, with $v_i^T A v_i > 0$. If $\exists x \in \Delta \setminus \{v_1, \ldots, v_n\}$ such that $x^T A x = 0$ then there $\exists i, j \ \in \{1, 2, \ldots, n\}$ such that $v_i^T A v_j < 0$.

Proof By contradition $v_i^T A v_j \geq 0$.

$$\begin{aligned} x^T A x &= \left(\sum_{i=1}^n \lambda_i v_i^T \right) A \left(\sum_{i=1}^n \lambda_i v_i^T \right) = \sum_{i,j=1}^n \lambda_i \lambda_i \overbrace{v_i^T A v_j}^{>0} \\ &\geq \sum_{i=1}^n \lambda_i^2 v_i^T A v_i > 0. \end{aligned}$$

□

Require only that A is $\epsilon-$copositive, $x^T A x \geq -\epsilon$

Subdivision

$\Delta = \text{conv}\{v_1, \ldots, v_n\}$

- bisection of the simplex along the longest edge
- $\delta(P) \to 0$
- $v^T Au < 0$, $w = \lambda v + (1 - \lambda)u$ such that $v^T Aw \geq 0$ and $u^T Aw \geq 0$

Polyhedral inner approximations of the copositive cone

$$\mathcal{P} = \{\Delta_1, \dots, \Delta_m\}$$

$$\Delta_k = \text{conv}\{v_1^k, v_2^k, \dots, v_n^k\}$$

$$\mathcal{I}_{\mathcal{P}} = \left\{ A \in \mathcal{M} : \underbrace{(v_i^k)^T A v_j^k}_{\text{linear}} \geq 0, \ \forall k = 1, \dots, m, \ \forall i, j \in \{1, \dots, n\} \right\}$$

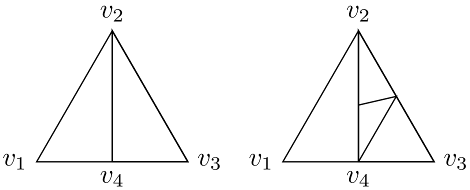
$$\mathcal{I}_{\Delta_S} = \{A \in \mathcal{M} : A_{ij} \geq 0, \ \forall i, j \in \{1, \dots, n\}\} = \mathcal{N}_n$$

Lemma

Let $\mathcal{P}$,$\mathcal{P}_1$,$\mathcal{P}_2$ denote two simplicial partitions of Δ_S. Then

- $\mathcal{I}_{\mathcal{P}}$ is a closed convex polyhedral cone,

- If $\mathcal{I}_{\mathcal{P}} \subseteq \mathcal{C}$ ($\mathcal{I}_{\mathcal{P}}$ is an inner approximation of $\mathcal{C}$),
- if $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, then $I_{\mathcal{P}_1} \subset I_{\mathcal{P}_2}$.



Theorem
Let $\mathcal{P}_r$ be a sequence of simplicial partitions of Δ_S with $\delta(\mathcal{P}_r) \to 0$. Then we have

$$\mathcal{C} = \overline{\bigcup_{r \in \mathcal{N}} \mathcal{I}_{\mathcal{P}_r}}$$

$M \in W \subseteq \mathcal{C}$

Sufficient condition $v_i^T A v_j \geq 0 \; \forall i, j$

$$V_\Delta^T A V_\Delta \in \mathcal{N}$$
$$V_\Delta^T A V_\Delta \in \mathcal{W} \subseteq \mathcal{C}$$

The choice $M = \mathcal{N}$ is not always desirable. To check whether a matrix is non negative does not take much effort but the non negative cone is a poor approximation of the copositive cone.

- the choice of the set M influences the number of iterations and the runtime
- the set M should be a good approximation of $\mathcal{C}$
- checking membership of M should be cheap

Data: $A \in \mathcal{M}_n,\ \mathcal{W} \in \mathcal{C}$
Result: Copositive certificate = "Yes" or "No"
$Ps = \{\Delta_S\}$;
while $Ps \neq \emptyset$ **do**
    choose $\Delta \in Ps$;
    if $\exists v \in V_\Delta^T : v^T A v < 0$ **then**
        return "No" ;
        $Ps = \emptyset$
    else
        if $V_\Delta^T A V_\Delta \in \mathcal{W}$ **then**
            $Ps \leftarrow Ps \setminus \Delta$;
        else
            $Ps \leftarrow Ps \setminus \Delta$;
            partition Δ into Δ_1 and Δ_2 ;
            $Ps \leftarrow Ps \setminus \Delta \cup \{\Delta_1, \Delta_2\}$
        end
    end
end

Based on Polinomials [Parrilo(2000)], [Bomze and de Klerk(2002)], [Peña et al.(2007)Peña, Vera, and Zuluaga], [Lasserre(2000/01)]

$x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n_+$ can be written as $xox = [x_1^2, \ldots, x_n^2]^T \in \mathcal{R}^n$

$x^T Ax \geq 0, x \geq 0$ replacing x_i by x_i^2 we have $P(x) = (xox)^T A(xox) \geq 0$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\langle \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix} \right\rangle$$

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

$$\left\langle \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \begin{bmatrix} x_1^4 & x_1^2x_2^2 & x_1^2x_3^2 \\ x_1^2x_2^2 & x_2^4 & x_2^2x_3^2 \\ x_1^2x_3^2 & x_2^2x_3^2 & x_3^4 \end{bmatrix} \right\rangle =$$

$$a_{11}x_1^4 + a_{22}x_2^4 + a_{33}x_3^4 + 2a_{12}x_1^2x_2^2 + 2a_{13}x_1^2x_3^2 + 2a_{23}x_2^2x_3^2$$

$$w_x^T = \begin{bmatrix} x_1^2 & x_2^2 & \dots & x_n^2 & x_1x_2 & \dots & x_1x_n & \dots & x_{n-1}x_n \end{bmatrix}$$

$$w_x^T M w_x = \left\langle M, w_x w_x^T \right\rangle$$

and M is of order $n + \dfrac{1}{2}n(n-1)$

$$\begin{bmatrix} x_1^4 & x_1^2x_2^2 & x_1^2x_3^2 & x_1^3x_2 & x_1^3x_3 & x_1^2x_2x_3 \\ x_1^2x_2^2 & x_2^4 & x_2^2x_3^2 & x_2^3x_1 & x_2^2x_1x_3 & x_2^3x_3 \\ x_1^2x_3^2 & x_2^2x_3^2 & x_3^4 & x_3^2x_1x_2 & x_3^3x_1 & x_3^3x_2 \\ x_1^3x_2 & x_1^3x_3 & x_3^2x_1x_2 & x_1^2x_2^2 & x_1^2x_2x_3 & x_2^2x_1x_3 \\ x_1^3x_3 & x_2^2x_1x_3 & x_3^3x_2 & x_1^2x_2x_3 & x_1^2x_3^2 & x_3^2x_1x_2 \\ x_1^2x_2x_3 & x_2^3x_3 & x_3^3x_2 & x_2^2x_1x_3 & x_3^2x_1x_2 & x_2^2x_3^2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \mu_{12} & \mu_{13} & 0 & 0 & \eta_{123} \\ \mu_{12} & \alpha_2 & \mu_{23} & 0 & \eta_{213} & 0 \\ \mu_{13} & \mu_{23} & \alpha_3 & \eta_{312} & 0 & 0 \\ 0 & 0 & \eta_{312} & \nu_{12} & \delta_{123} & \delta_{213} \\ 0 & \eta_{213} & 0 & \delta_{123} & \nu_{13} & \delta_{312} \\ \eta_{123} & 0 & 0 & \delta_{213} & \delta_{312} & \nu_{23} \end{bmatrix}$$

$$\begin{aligned} x_i^4 &\quad \rightarrow \quad \alpha_i = a_{ii},\ i = 1, \dots, n \\ x_i^3 x_k &\quad \rightarrow \quad \beta_{ik} = 0, \\ x_i^2 x_j x_k &\quad \rightarrow \quad 2\eta_{ijk} + 2\delta_{ijk} = 0 \\ x_i^2 x_j^2 &\quad \rightarrow \quad 2\mu^{ij} + \nu^{ij} = 2a_{ij} \\ x_i x_j x_k x_s &\quad \rightarrow \quad \pi_{ijks} = 0 \end{aligned}$$

$$L_A^0 = \left\{ M \in \mathcal{M}_d : (xox)^T A(xox) = w_x^T M w_x \right\}$$

Theorem

The matrix A is copositive if there is a matrix $M \in L_A^0$ nonnegative or positive semidefinite.

Lemma

Condition $(xox)^T A(xox) \geq 0$ hold if the polynomial $w_x^T M w_x$ can be written as a sum of squares $\sum_{i=1}^r f_i(x)^2$, for some polynomial functions f_i. A sum of squares decomposition is possible if and only if a representation of $w_x^T M w_x$ exists where $M = \tilde{S} + \tilde{N}$ where $\tilde{S} \in \mathcal{S}_d$ and $\tilde{N} \in \mathcal{N}_d$.

Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} x^T A x &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3 \\ (xox)^T A (xox) &= x_1^4 + x_2^4 + x_3^4 + 2x_1^2x_2^2 + 2x_1^2x_3^2 - 2x_2^2x_3^2 \end{aligned}$$

$$w_x^T = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 & x_1x_2 & x_1x_3 & x_2x_3 \end{bmatrix}$$

$$w_x^T M w_x = w_x^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} w_x =$$

$$= (x_1^2)^2 + (\sqrt{2}x_1 2x_2)^2 + (\sqrt{2}x_1x_3)^2 + (x_2^2 - x_3^2)^2$$

Lemma

$$L_A^0 \cap \mathcal{N}_d \neq 0 \Leftrightarrow A \in \mathcal{N}_n$$
$$L_A^0 \cap \mathcal{S}_d \neq 0 \Leftrightarrow A \in (\mathcal{N}_n + \mathcal{S}_n)$$

How to obtain higher order sufficient conditions?

$$\begin{aligned}
P(x) &= (xox)A(xox) = w_x^T M w_x \\
P^r(x) &= P(x)\left(\sum_{k=1}^{n} x_k^2\right)^r \\
P(x) \geq 0 &\Leftrightarrow P^r(x) \geq 0 \\
P^r(x) \geq 0 &\Leftarrow P^r(x) = \sum_{i=1}^{s} f_i(x)^2
\end{aligned}$$

$$L_A^0 = \left\{ M \in \mathcal{M}_d : P(x) = (xox)^T A(xox) = w_x^T M w_x \right\}$$

$$L_A^r = \left\{ M \in \mathcal{M}_{d_r} : P^r(x) = P(x) \left(\sum_{k=1}^{n} x_k^2 \right)^r = w_{x^r}^T M w_{x^r} \right\}$$

Lemma

$$L_A^r \cap \mathcal{S}_d \neq 0 \Rightarrow A \in \mathcal{C}_n$$

$$
\begin{aligned}
P(x) &= (xox)A(xox) = w_x^T M w_x \\
P^r(x) &= P(x)\left(\sum_{k=1}^{n} x_k^2\right)^r \\
P^r(x) &= \sum_{i=1}^{s} f_i(x)^2
\end{aligned}
$$

Definition

The convex cone $\mathcal{K}_n^r$ consists of the matrices in $\mathcal{M}_n$ for which $P^r(x)$ allows a polynomial sum of squares decomposition (sos). $\mathcal{K}_n^0 = \mathcal{N}_n + \mathcal{S}_n$.

Lemma

$$\mathcal{K}_n^r \subseteq \mathcal{K}_n^{r+1} \text{ for all } r$$

Proof

$$\begin{aligned}
P^{r+1}(x) &= P(x)\left(\sum_{k=1}^{n} x_k^2\right)^{r+1} = \\
&= P(x)\left(\sum_{k=1}^{n} x_k^2\right)^{r}\left(\sum_{k=1}^{n} x_k^2\right) = \\
&= P^r(x)\left(\sum_{k=1}^{n} x_k^2\right) \\
&= \sum_{i=1}^{l} f_i(x)^2\left(\sum_{k=1}^{n} x_k^2\right) = \sum_{ik} (x_k f_i)^2
\end{aligned}$$

$$A \in \mathcal{K}_n^r ?$$

The copositive cone can be approximate to a given accuracy by a sufficiently large set of linear matrix inequalities. Each copositive programming problem can be approximated to a given accuracy by a sufficiently large SDP.

$d = \mathcal{O}(n^{r+2})$. In practice we are restricted to $r = 1$. Degree 6.

For $r > 2$ the resulting problems become too large for current SDP solvers even for small values of n.

Also possible to have LP approximations of the copositive cone, that are weaker than the SDP approximations but are easier to solved.

Definition

The convex cone $\mathcal{P}_n^r$ consists of the matrices in $\mathcal{M}_n$ for which $P^r(x)$ has no negative coefficients. $\mathcal{P}_n^0 = \mathcal{N}_n$ and $\mathcal{P}_n^r \subseteq \mathcal{K}_n^r$ and $\mathcal{P}_n^r \subseteq \mathcal{P}_n^{r+1}$.

$$A \in \mathcal{P}_n^r?$$

The copositive cone can be approximate to a given accuracy by a sufficiently large set of linear inequalities. Each copositive programming problem can be approximated to a given accuracy by a sufficiently large LP.

Theorem

Let $A \in \mathcal{C}_n^+$ such that $A \notin \mathcal{N}_n + \mathcal{S}_n$. Then there are integers $r_{\mathcal{K}}$ and $r_{\mathcal{P}}$ with $1 \leq r_{\mathcal{K}} \leq r_{\mathcal{P}} \leq +\infty$ such that

$$\mathcal{N}_n = \mathcal{P}_n^0 \subseteq \mathcal{P}_n^1 \subseteq \cdots \subseteq \mathcal{P}_n^r$$

$$A \in \mathcal{P}_n^r \text{ for all } r \geq r_{\mathcal{P}} \text{ but } A \notin \mathcal{P}_n^{r_{\mathcal{P}}-1}$$

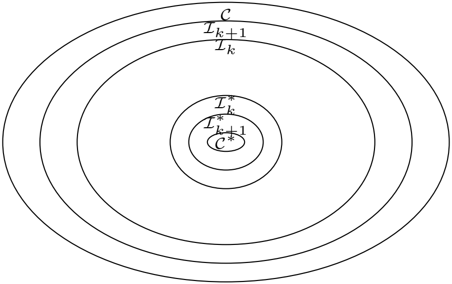
and

$$\mathcal{N}_n + \mathcal{S}_n = \mathcal{K}_n^0 \subseteq \mathcal{K}_n^1 \subseteq \cdots \subseteq \mathcal{K}_n^r$$

$$A \in \mathcal{K}_n^r \text{ for all } r \geq r_{\mathcal{K}} \text{ but } A \notin \mathcal{K}_n^{r_{\mathcal{K}}-1}$$

Approximations for the $\mathcal{C}^*$

The dual cone of $\mathcal{C}$ is the cone $\mathcal{C}^*$ of completely positive matrices. By duality, the dual cone of an inner (resp. outer) approximation of $\mathcal{C}$ is an outer (resp. inner) approximation of $\mathcal{C}^*$.



Duality

Definition (Dual)
The dual of conic problem P

$$\begin{array}{lll} v_P^* & \leftarrow & \inf \langle C, X \rangle \\ \text{s.t.} & & \langle A_i, X \rangle = b_i, i \in \{1, \dots, m\} \\ & & X \in \mathcal{K} \end{array}$$

is the conic problem D

$$\begin{array}{lll} v_D^* & \leftarrow & \sup b^T y \\ \text{s.t.} & & C - \sum_{i=1}^{m} y_i A_i \in \mathcal{K}^* \\ & & y \in \mathcal{R}^m \end{array}$$

Attainability

Definition (Conic duality theorem)

If there exists an interior feasible solution of (P) ($X^0 \in \text{int}(\mathcal{K})$), and a feasible solution of (D) then $v_P^* = v_D^*$ and the supremum in (D) is attained. Similarly, if there exist $y^0 \in \mathcal{R}^m$ such that $C - \sum_{i=1}^m y_i^0 A_i \in \text{int}(\mathcal{K}^*)$ and a feasible solution of (P), then $v_P^* = v_D^*$ and the infimum in (P) is attained.

Dual of a Copositive Program - Completely Positive Program

Definition (Dual)

The dual of conic problem P

$$\begin{aligned} v_P^* \quad &\leftarrow \quad \inf \langle C, X \rangle \\ \text{s.t.} \quad & \quad \langle A_i, X \rangle = b_i, i \in \{1, \ldots, m\} \\ & \quad X \in \mathcal{C} \end{aligned}$$

is the conic problem D

$$\begin{aligned} v_D^* \quad &\leftarrow \quad \sup b^T y \\ \text{s.t.} \quad & \quad C - \sum_{i=1}^{m} y_i A_i \in \mathcal{C}^* \\ & \quad y \in \mathcal{R}^m \end{aligned}$$

Formulation of Problems as Conic Programs

- Single Quadratic Constraint Quadratic Programs [Preisig(1996)]
- Standard Quadratic Program (maximum clique) [Bomze et al.(2000)Bomze, Dür, de Klerk, Roos, Quist, and Terlaky], [Bomze and de Klerk(2002)]
- Binary and continuous nonconvex quadratic programs [Burer(2009)]
- mixed-integer fractional quadratic [Amaral and Bomze(2015)]
- binary and ternary fractional quadratic [Amaral and Bomze(2015)]
- fractional quadratic programs. [Preisig(1996)], [Amaral et al.(2014)Amaral, Bomze, and Júdice]

The pioneer work of Preisig

[Preisig(1996)]

$$
\text{(SQC)} \quad \min \quad x^T Q x \\
\text{s.t.} \quad x^T A x = b \\
x \geq 0
$$

Without loss of generality $b = 1$. Consider $y = x/\sqrt{b}$.

$$
\text{(SQC1)} \quad \min \quad x^T Q x \\
\text{s.t.} \quad x^T A x = 1 \\
x \geq 0
$$

Lemma

$A \in \mathcal{C}^+$ then $\{x : x^T A x = 1,\ x \geq 0\}$ is compact.

Lemma
$A \in \mathcal{C}^+$, $Q \in \mathcal{M}$, then $\exists y_0$ such that

$$\begin{array}{rcl} (Q - yA) & \in & \mathcal{C} \setminus \mathcal{C}^+ \text{ for } y = y_0 \\ (Q - yA) & \in & \mathcal{C}^+, \ \forall y < y_0 \\ (Q - yA) & \notin & \mathcal{C}, \ \forall y > y_0 \end{array} \quad (1)$$

Lemma
$A \in \mathcal{C}^+$, $Q \in \mathcal{M}$, then $\exists x_0 \geq 0,$ and $x_0 \neq 0$, such that

$$x_0^T (Q - y_0 A) x_0 = 0$$

and

$$x_0 = \arg \min_{\substack{x \geq 0 \\ e^T x = 1}} x^T (Q - y_0 A) x$$

where y_0 is as defined in 1.

Lemma

$A \in \mathcal{C}^+$, $Q \in \mathcal{M}$, then $\exists x_0 \geq 0$, and $x_0 \neq 0$, such that

$$\min_{\substack{x \geq 0 \\ e^T x = 1}} x^T (Q - y_0 A) x \quad > 0 \quad \forall y < y_0$$

$$\min_{\substack{x \geq 0 \\ e^T x = 1}} x^T (Q - y_0 A) x \quad < 0 \quad \forall y > y_0$$

where y_0 is as defined in 1.

Theorem

$A \in \mathcal{C}^+$, $Q \in \mathcal{M}$

$$x^* = \arg \min_{\substack{x \geq 0 \\ x^T A x = 1}} x^T Q x$$

$$y^* = \min_{\substack{x \geq 0 \\ x^T A x = 1}} x^T Q x$$

and y_0 is as defined in 1, then $y_0 = y^*$.

Relationship to fractional programming

Theorem

$A \in \mathcal{C}^+$, $Q \in \mathcal{M}$

$$\begin{aligned} y^* &= \min_{\substack{x \geq 0 \\ x^T A x = 1}} x^T Q x \\ y_1^* &= \min_{\substack{x \geq 0 \\ e^T x = 1}} \frac{x^T Q x}{x^T A x} \end{aligned}$$

then $y^* = y_1^*$.

Single Quad. Constrained Quad. Programs ($A \in \mathcal{C}^+$ and $b > 0$)

[Preisig(1996)]

$$
\text{(SQC)} \quad \begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & x^T A x = b \\ & x \geq 0 \end{aligned}
$$

Completely Positive Formulation

$$
\text{(SQCCp)} \quad \begin{aligned} \min \quad & \langle Q, X \rangle \\ \text{s.t.} \quad & \langle A, X \rangle = b \\ & X \in \mathcal{C}^* \end{aligned}
$$

Copositive Formulation

$$
\text{(SQCCo)} \quad \begin{aligned} \max \quad & by \\ \text{s.t.} \quad & Q - yA \in \mathcal{C} \\ & y \in \mathbb{R} \end{aligned}
$$

$A \in \mathcal{C}^+$ and $b > 0$.

$$\text{(SQC)} \quad \min \quad x^T Q x \quad \text{s.t.} \quad x^T A x = b, \quad x \geq 0$$

$x^T Q x = \left\langle Q, xx^T \right\rangle$ and $x^T A x = \left\langle A, xx^T \right\rangle$. Also $X = xx^T$ then $X \in \mathcal{C}^*$ and $rank(X) = 1$.

$$\text{(SQCCpR1)} \quad \min \quad \langle Q, X \rangle \quad \text{s.t.} \quad \langle A, X \rangle = b, \quad X \text{ has rank one}, \quad X \in \mathcal{C}^*$$

Theorem

The extremal points of $\{X : \langle A, X\rangle = b,\ X \in \mathcal{C}^*\}$ are rank-one matrices $X = xx^T$ with $x^T Ax = b$ and $x \geq 0$.

Proof

$Fea(SQC) = \{x \in \mathcal{R}^n : x^T Ax = b\,, x \geq 0\}$
$Fea(SQCCp) = \{X \in \mathcal{M}^n : \langle A, X\rangle = b\,, X \in \mathcal{C}^*\}$

Let $x \in Fea(SQC)$ and consider $X = xx^T$. Then $X \in Fea(SQCCp)$. Now suppose that

$$X = \lambda X_1 + (1 - \lambda)X_2$$

with X_1 and X_2 in $Fea(SQCCp)$. We know that the extreme rays of the Completely Positive cone are the rank-one matrices. If X is an extreme ray of the cone then $X = D_1 + D_2$ implies that $X = \nu_1 D_1$ and $X = \nu_2 D_2$. In this case, from

$X = \lambda X_1 + (1-\lambda)X_2$ there are μ_1 and μ_1 such that $X = \mu_1 X_1$ and $X = \mu_2 X_2$. But since X_1 and X_2 in $Fea(SQCCp)$ we have:

$$b = \langle A, X\rangle = \mu_1 \underbrace{\langle A, X_1\rangle}_{b} = \mu_2 \underbrace{\langle A, X_2\rangle}_{b}$$

so

$$\mu_1 = \mu_2 = 1$$

then from $X = \mu_1 X_1$ and $X = \mu_2 X_2$ we obtain $X = X_1$ and $X = X_2$, and X is an extreme point of $Fea(SQCCp)$.

Now let X be an extreme point of $Fea(SQCCp)$ and suppose that

$$X = \sum_{i=1}^{d} x_i (x_i)^T \text{ with } x_i \geq 0 \text{ and } x_i \neq 0$$

Consider $u_i = \sqrt{\frac{b}{x_i^T Axi}} x_i$ then $u_i A u_i = \sqrt{\frac{b}{x_i^T Axi}} \sqrt{\frac{b}{x_i^T Axi}} x_i^T A x_i = b$

since $x_i = \sqrt{\frac{x_i^T Axi}{b}} u_i$, considering $U_i = u_i(u_i)^T$

$$X = \sum_{i=1}^{d} x_i(x_i)^T = \sum_{i=1}^{d} \left(\sqrt{\frac{x_i^T Axi}{b}} u_i \right) \left(\sqrt{\frac{x_i^T Axi}{b}} u_i \right)^T$$

$$= \sum_{i=1}^{d} \left(\frac{x_i^T Axi}{b} \right) u_i u_i^T = \sum_{i=1}^{d} \left(\frac{x_i^T Ax_i}{b} \right) U_i$$

since $\langle A, X \rangle = b$, $\left\langle A, \sum_{i=1}^{d} x_i(x_i)^T \right\rangle = \sum_{i=1}^{d} x_i^T Ax_i = b$, then $\sum_{i=1}^{d} \frac{x_i^T Ax_i}{b} = 1$ and $\frac{x_i^T Axi}{b} > 0$ then X is a convex combination of $U_1, \ldots, U_d$ but since X is a extreme point of $Fea(SQCCp)$, we have $U_1 = \cdots = U_d$. In that case

$$X = U_1 \sum_{i=1}^{d} \left(\frac{x_i^T Axi}{b} \right)$$

$$X = U_1 = u_1 u_1^T$$

Standard Quadratic Programs

[Bomze et al.(2000)Bomze, Dür, de Klerk, Roos, Quist, and Terlaky], [Bomze and de Kler

$$\text{(StQ)} \quad \min \quad x^T Q x \quad \text{s.t.} \quad e^T x = 1, \quad x \geq 0$$

$$\text{Completely Positive Formulation : (StQCp)} \quad \min \quad \langle Q, X \rangle \quad \text{s.t.} \quad \langle E, X \rangle = 1, \quad X \in \mathcal{C}^*$$

$$\text{Copositive Formulation : (StQCo)} \quad \max \quad y \quad \text{s.t.} \quad Q - yE \in \mathcal{C}, \quad y \in \mathbb{R}$$

Binary and continuous nonconvex quadratic programs

[Burer(2009)]

$$
\begin{aligned}
\text{(MBQ)} \quad \min \quad & x^T Q x + 2c^T x \\
\text{s.t.} \quad & a_i^T x = b_i \text{ for } i = 1, \ldots, \\
& x_j \in \{0,1\} \; \forall j \in B \\
& x \geq 0
\end{aligned}
$$

$$L = \{x \geq 0 : a_i^T x = b_i, \forall i = 1, \ldots, m\}$$

Key assumption: $x \in L \Rightarrow 0 \leq x_j \leq 1 \; \forall j \in B$

$$
\begin{array}{lll}
\text{(MBQ)} & \min & x^T Q x + 2c^T x \\
 & \text{s.t.} & a_i^T x = b_i \text{ for } i = 1, \ldots, m \\
 & & x_j \in \{0, 1\} \; \forall j \in B \\
 & & x \geq 0
\end{array}
$$

$$
\begin{array}{lll}
\text{(MBQ)} & \min & \langle Q, X \rangle + 2c^T x \\
 & \text{s.t.} & a_i^T x = b_i \text{ for } i = 1, \ldots, m \\
 & & x_j = X_{jj} \; \forall j \in B \\
 & & x \geq 0 \\
 & & X = xx^T
\end{array}
$$

$$
\begin{aligned}
\text{(MBQ)} \quad \min \quad & \langle Q, X \rangle + 2c^T x \\
\text{s.t.} \quad & a_i^T x = b_i \text{ for } i = 1, \ldots, \\
& x_j = X_{jj} \ \forall j \in B \\
& x \geq 0 \\
& X = xx^T
\end{aligned}
$$

$$
\begin{aligned}
\text{(MBQ}C^*\text{)} \quad \min \quad & \langle Q, X \rangle + 2c^T x \\
\text{s.t.} \quad & a_i^T x = b_i \text{ for } i = 1, \ldots, \\
& a_i^T X a_i = b_i^2 \text{ for } i = 1, \ldots, \\
& x_j = X_{jj} \ \forall j \in B \\
& \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathcal{C}^*
\end{aligned}
$$

Theorem
(MBQ) $\Leftrightarrow$ (MBQC^*)

Eliminate x from the formulation

$$\exists y \in \mathcal{R}^m \text{ s.t. } \alpha = \sum_{i=1}^{m} y_i a_i \geq 0, \; \sum_{i=1}^{m} y_i b_i = 1$$

Binary and ternary fractional quadratic

[Amaral and Bomze(2015)]

$$\tau_{MI}^* := \inf \left\{ \frac{f(x)}{g(x)} : x \in \mathcal{R}_+^n,\ \widehat{C}x = \widehat{c},\ x_i \in [L_i, U_i] \text{ for all } i \in I \right\}$$

$x_i = L_i + \sum_{j=0}^{l_i} z_i^{(j)} 2^j$, $i \in I$., $z_i^{(j)} \in \{0,1\}$, $j \in [0, l_i]$, where $l_i = \lfloor \log_2(U_i - L_i) \rfloor$,

Example: $x \in [2, 17]$

$$\begin{aligned} x &= 2 + z^{(0)}2^0 + z^{(1)}2^1 + z^{(2)}2^2 + z^{(3)}2^3 \\ &= 2 + z^{(0)} + z^{(1)}2 + z^{(2)}4 + z^{(3)}8 \text{ with } z^{(0)}, \dots, z^{(3)} \in \{0,1\} \end{aligned}$$

$$B := \bigcup_{i \in I} \{i\} \times [0 : l_i] \,.$$

Replace x by $v \in \mathcal{R}^d$ with $d = n + \sum_{i \in I} l_i$

$$v = [x_1, x_2, \ldots, x_r, \ldots \ldots z_i^{(j)} \ldots]$$

$$\tau^*_{MB} := \inf \left\{ \frac{f(v)}{g(v)} : v \in \mathcal{R}^d_+,\, Cv = c\,,\; v_i \in \{0,1\} \text{ for all } i \in B \right\}$$

Homogenize a general quadratic constraint $v^T Qv + q^T v + \gamma$ considering new variables $w = [1, v^T]^T$

$$\overline{Q} = \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \in \mathcal{M}_{d+1}$$

as well as

$$Y = ww^T = \begin{bmatrix} 1 & v^T \\ v & vv^T \end{bmatrix} \in \mathcal{C}^*_{d+1} \, .$$

$$v^T Qv + q^T v + \gamma = \overline{Q} \bullet Y \, .$$

$$Cv = c \Leftrightarrow \|Cv - c\|^2 = 0$$

$$\overline{C_c} = [-c^T|C^T]^T[-c|C] = \left[\begin{array}{cc} c^T c & -c^T C \\ -C^T c^T & C^T C \end{array}\right] \in \mathcal{S}_{d+1} \,,$$

$$Y = ww^T = \left[\begin{array}{cc} 1 & v^T \\ v & vv^T \end{array}\right]$$

$$\|Cv - c\|^2 = 0 \to \overline{C_c} \bullet Y = 0$$

$$Y_{00} = 1$$

$Y_{0i} = Y_{ii}$ ensure that $v_i = v_i^2$ for all $i \in B$, which in turn is equivalent to $v_i \in [0,1]$, so that we arrive at

$$\tau^*_{MB} := \inf \left\{ \frac{f(v)}{g(v)} : v \in \mathcal{R}^d_+,\, Cv = c\,,\; v_i \in \{0,1\} \text{ for all } i \in B \right\}$$

$$\tau^*_{rk\,1} := \inf \left\{ \frac{\overline{A} \bullet Y}{\overline{B} \bullet Y} : \overline{C_c} \bullet Y = 0,\, Y_{0i} - Y_{ii} = 0\,,\, Y_{00} = 1,\; \text{all } i \in B\,,\, Y \in \mathcal{C}^{*,rk\,1}_{d+1} \right\},$$

where $\mathcal{C}^{*,rk\,1}_d$ denotes the (non-convex, not closed) subcone of all completely positive $d \times d$ matrices Y of rank one.

Under conditions

$$\left\{x \in \mathcal{R}_+^d : Cx = 0\right\} = 0 \qquad (\left\{x \in \mathcal{R}_+^d : Cx = 0\right\} \text{ is bounded })$$
$$w^T \overline{B} w > 0 \qquad \text{if } \overline{C} w = 0 \text{ for } w \in \mathcal{R}_x^{d+1} \setminus 0$$

we have $Y_{00} > 0$ and $\overline{B} \bullet Y > 0$ and we replace Y rank-one by $Y \neq 0$.
So !!

$$\tau_{rk\,1}^* := \inf \left\{ \frac{\overline{A} \bullet Y}{\overline{B} \bullet Y} : \overline{C_c} \bullet Y = 0,\ Y_{0i} - Y_{ii} = 0\,,\ Y_{00} = 1,\ \text{all } i \in B\,,\ Y \in C_{d+1}^{*,rk\,1} \right\}\,,$$

Under previous conditions and *Burer's key condition* we have an equivalent formulation $\overline{B} \bullet Y = 1$.

$$\tau_{COP}^* := \inf \left\{ \overline{A} \bullet Y :, \overline{B} \bullet Y = 1\,, \overline{C_c} \bullet Y = 0,\ Y_{0i} - Y_{ii} = 0\,,\ \text{all } i \in B\,,\ Y \in C_{d+1}^* \right\}\,,$$

Fractional quadratic programs

. [Amaral et al.(2014)Amaral, Bomze, and Júdice]

TO BE CONTINUED ....

Infeasibility, Fractional Quadratic Problems and Copositivity

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