Mathematics
Precalculus

Essentials prior to a Calculus course

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Chapter 1

Sets and Logic

Contents

1.1 Sets

Sets Theory was develop by Cantor in the end of XIX century and it has influenced almost all areas of Mathematics and constitutes a pilar of Modern Mathematics. The term "set" was coined by Bernard Bolzano as the translation of the German "Menge", appearing in his work "The Paradoxes of the Infinite".

Definition 1.1.1 *Sets A* ***set*** *is a collection of distinct and well-defined objects, of any kind. These objects are the* ***elements*** *of the set. Usually we use capital letters to designate a set and small letters for elements.*
To say that x is an element of set X $x \in X$ *meaning that "x belongs to X". To represent that "x is not in X" we write* $x \notin X$ *reading "x does not belong to X".*

In what follows A and B are two arbitrary set.

Definition 1.1.2 *Subsets* A *is a* ***subset*** *of* B *and we say that* A ***is contained*** *in* B*, writing* $A \subseteq B$*, if every element of* A *is also an element of* B*. Otherwise we write* $A \nsubseteq B$*, in which case at least one element from* A *is not an element of* B*.*

$$\boxed{\begin{aligned} &A \subseteq B \leftrightarrow \forall a \in A, a \in B \\ &A \nsubseteq B \leftrightarrow \exists a \in A, a \notin B \\ &A \subseteq A, \forall A \end{aligned}} \tag{1.1}$$

Definition 1.1.3 *Similar sets A and B are similar if they have the same elements and we write that* $A = B$.

$$\boxed{\begin{aligned} A = B \leftrightarrow &\forall a \in A, a \in B \text{ and} \\ &\forall a \in B, a \in A \\ A = B \leftrightarrow &A \subseteq B \text{ and } B \subseteq A. \end{aligned}}$$

Example 1.1.4 *Let* $A = \{1, 2, 3, 5\}$, $B = \{1, 3\}$ *e* $C = \{2, 4, 5\}$.

$$2 \in A, \text{ but } 2 \notin B.$$

$$B \subseteq A, \text{ but } B \subsetneqq A.$$

$$B \nsubseteq C \text{ and } C \nsubseteq B.$$

Definition 1.1.5 *Empty set An* ***empty set*** *is a set without any element represented by* $\{\}$*or* $\varnothing$.

Some common sets are:

$\mathbb{N}$ - Set of natural numbers

$\mathbb{Z}$ -Set of integer numbers

$\mathbb{Q}$ - Set of racional numbers

$\mathbb{R}$ - Set of real numbers

$\mathbb{C}$ - Set of complex numbers.

Definition 1.1.6 *Representation of Sets We can represent a set using*

Tabular Form *Listing all the elements of a set, separated by commas and enclosed within curly brackets* $\{\{$.

Descriptive Form *State in words the elements of the set.*

Set Builder Form *Writing in symbolic form the common characteristics shared by all the elements of the set*

Example 1.1.7 *These are different representations of the same set*

- *Tabular Form* - $A = \{1, 3, 5, 7, 9, \dots\}$.
- *Descriptive Form* - $A =$*Set of positive odd integer.*
- *Set Builder Form* - $A = \{x : \ x = 2n - 1, \ n \in \mathbf{N}\}$

Definition 1.1.8 *Operations with sets Given sets A and B, we m*

Union $A \cup B$ **-** *set that consists of all elements belonging to either set A or set B (or both).*

Intersection $A \cap B$ **-** *set composed of all elements that belong to both A and B.*

Setminus $A \setminus B$ **-** *set composed by all elements in A that are not in B.*

Complemet A^c **-** *set of all elements in the universe U that are not in A. We admit that the admissible elements are restricted to some fixed class of objects U called the universal set (or universe). Also can be described as $U \setminus A$*

Cartesian Product $A \times B$ **-** *set consisting of all ordered pairs (a, b) for which $a \in A$ and $b \in B$.*

Example 1.1.9 *Given $A = \{1, 5\}$, $B = \{1, 4\}$ and $U = \{1, 2, 3, 4, 5\}$*

$$\begin{aligned}
A \cup B &= \{1, 4, 5\} \\
A \cap B &= \{1\} \\
A \setminus B &= \{5\} \\
A^c &= \{2, 3, 4\} \\
A \times B &= \{(1, 1), (1, 4), (5, 1), (5, 4)\} \\
B \times A &= \{(1, 1), (1, 5), (4, 1), (4, 5)\}
\end{aligned}$$

1.2 Logic

Definition 1.2.1 *Propositional logic is a mathematical model that allows us to reason about the truth or falsehood (T,F) of logical expressions.*

Definition 1.2.2 *Truth tables*

Negation (NO)

p	$\sim p$
T	F
F	T

Conjunction (AND)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Disjunction (XOR)

p	q	$p \dot{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (IF THEN)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence (IIF)

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

1.2.1 Properties

Properties	**Conjunction**	**Disjunction**
Comutativity	$(p \wedge q) \Leftrightarrow (q \wedge p)$	$(p \vee q) \Leftrightarrow (q \vee p)$
Associativity	$[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$	$[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$
Idempotence	$(p \wedge p) \Leftrightarrow p$	$(p \vee p) \Leftrightarrow p$
Identity	$(p \wedge V) \Leftrightarrow p \Leftrightarrow (V \wedge p)$	$(p \vee F) \Leftrightarrow p \Leftrightarrow (F \vee p)$
Annihilator	$(p \wedge F) \Leftrightarrow F \Leftrightarrow (F \wedge p)$	$(p \vee V) \Leftrightarrow V \Leftrightarrow (V \vee p)$

Properties Disjunction - Conjunction

Distributivity of $\wedge$ over $\vee$

$$[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$$

$$[(q \vee r) \wedge p] \Leftrightarrow [(q \wedge p) \vee (r \wedge p)].$$

Distributivity of $\vee$ over $\wedge$,

$$[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$$

$$[(q \wedge r) \vee p] \Leftrightarrow [(q \vee p) \wedge (r \vee p)].$$

Chapter 2

Operations with real numbers, solving equations and inequalities

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Chapter 3

Sequences

Contents

3.1 Basics definitions

In this chapter we will study a special case of functions, named sequences.

> **Definition 3.1.1** *(Sequence)*
> *A sequence (infinite) is a function of* $\mathbb{N}$ *in* $\mathbb{R}$

To simplify notation instead of $f(n)$ we use f_n and in general we adopt the letter u, v, w to designate sequences.

Unlike a set, the same elements can appear multiple times at different positions in a sequence, and order matters. The variable n is called an index. The position of an element in a sequence is its rank or index

Example 3.1.2

$$u_n = \frac{n+1}{n+2} \tag{3.1}$$

For u_n *in 3.1, the element of rank 1 is* $u_1 = \frac{1+1}{1+2} = 2/3$. *The element of rank 5 is* $6/7$.

A sequence can be defined by a list of its first elements, $v_n = \{1, 4, 9, 16, 25, \dots\}$ by the general term $v_n = n^2$ or by recursion. In a sequence defined by recursion a term depends on previous terms, like the Fibonacci numbers

Example 3.1.3

$$\begin{cases} w_1 = 0 \\ w_2 = 1 \\ w_{n+2} = w_n + w_{n+1} \end{cases}$$

For the Fibonacci sequence to find the element of rank 5 we have first to find the element of rank 3, $w_3 = w_2 + w_1 = 1$, *of rank 4* $w_4 = w_3 + w_2 = 2$ *and finally* $w_5 = w_4 + w_3 = 2 + 1 = 3$.

Example 3.1.4 *For the sequence defined by recursion:*

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + n + 1, \quad \forall n \in \mathbb{N} \end{cases}$$

the first 7 elements are $1, 3, 6, 10, 15, 21, 28$.

The sequence:

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_{n+2} = 2a_{n+1} + a_n, \quad \forall n \in \mathbb{N} \end{cases}$$

has as first elements $1, 1, 3, 7, 17, 41$.

3.2 Properties

There are several properties that are important to study a sequence.

Definition 3.2.1 *(Increasing and decreasing)*
A sequence u_n *is said to be*

- *monotonically increasing if* $u_{n+1} \geq u_n$, $\forall n \in \mathbb{N}$.
- *strictly monotonically increasing if* $u_{n+1} > u_n$, $\forall n \in \mathbb{N}$.
- *monotonically decreasing if* $u_{n+1} \leq u_n$, $\forall n \in \mathbb{N}$.
- *strictly monotonically decreasing if* $u_{n+1} < u_n$, $\forall n \in \mathbb{N}$.

Example 3.2.2 *Lets study the monotonicity of the sequence*

$$a_n = \frac{n+1}{2^n}.$$

By definition lets study the sign of

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)+1}{2^{n+1}} - \frac{n+1}{2^n} = \frac{n+2}{2^n \times 2} - \frac{n+1}{2^n} \\ &= \frac{n+2-(n+1)\times 2}{2^n \times 2} = \frac{n+2-2n-2}{2^n \times 2} \\ &= \frac{-n}{2^{n+1}}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

its is clear that this difference is alway negative so we may conclude that a_n is strictly monotonically decreasing. Now for

$$b_n = \frac{1}{7-2n}.$$

just looking at the first elements of this sequence,

$$b_1 = \frac{1}{5};\ b_2 = \frac{1}{3};\ b_3 = 1;\ b_4 = -1$$

we see that $b_1 < b_2 < b_3$ but $b_3 > b_4$ so we may conclude that b_n is not monotone.

Definition 3.2.3 *(Bounded)*
A sequence u_n is said to be

- *bounded from above if all the terms are less than some real number M, there is if,*

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : u_n \leq M.$$

- *bounded from below if all the terms are greater than some real number M, there is if,*

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : u_n \geq M.$$

- *bounded if it is both bounded from above and bounded from below,*

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : |u_n| \leq M.$$

3.3 Arithmetic and Geometric Progressions

The sequences (a_n) with elements $1, 4, 7, 10, 13, \ldots$ and (b_n) with elements $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, have some special features. In fact we can easily note that for (a_n)

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 3, \quad \forall n \in \mathbb{N} \end{cases}$$

and for (b_n)

$$\begin{cases} b_1 = 1 \\ b_{n+1} = \frac{1}{2} b_n, \quad \forall n \in \mathbb{N} \end{cases}$$

Sequences with these behavior are known as Progressions.

Definition 3.3.1 *(Progressions)*
A sequence u_n is said to be

- *an Arithmetic Progressions if the difference between the consecutive terms is constant.*

  $$\forall n \in \mathbb{N} : u_{n+1} = u_n + k = u_1 + nk, k \in \mathbb{R}$$

  k is the common difference.

- *a Geometric Progressions if the quotient of any two successive members of the sequence is a constant*

  $$\forall n \in \mathbb{N} : u_{n+1} = ru_n = r^n u_1, r \in \mathbb{R} \setminus \{0\}$$

  $r \neq 0$ is the common ratio and u_1 is a scale factor

We may observe that an arithmetic progression is monotonically

- increasing if the common difference $k > 0$
- decreasing if $k < 0$.
- if $k = 0$ then the sequence is constant.

Regarding the monotonicity of a geometric progression with common ratio r and scale factor u_1 its is

- Increasing if $u_1 > 0$ and $r > 1$ or if $a_1 < 0$ and $0 < r < 1$;
- Decreasing if $a_1 > 0$ and $0 < r < 1$ or if $a_1 < 0$ and $r > 1$;
- Constant if $r = 1$;
- Not monotone if $r < 0$.

The sum S_n of the first n terms of an arithmetic progression (a_n), is given by

$$S_n = \frac{a_1 + a_n}{2} \times n.$$

The sum S_n of the first n terms of a geometric progression (a_n), is given by

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

where r is the common ratio and a_1 the scale factor.

3.4 Limits

Consider (a_n) the sequence $1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, \ldots, 1+\frac{1}{2^n}, \ldots$. This sequence is monotonically decreasing, with elements positive and approaching 1. In fact the distance between the elements of the sequence and 1, given by

$$|a_n - 1|$$

takes the values $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$. No matter how small we consider this distance, say ε, we know that we will find a rank p such that the distance of the elements of the sequence For every real number $\varepsilon > 0$, there is a natural

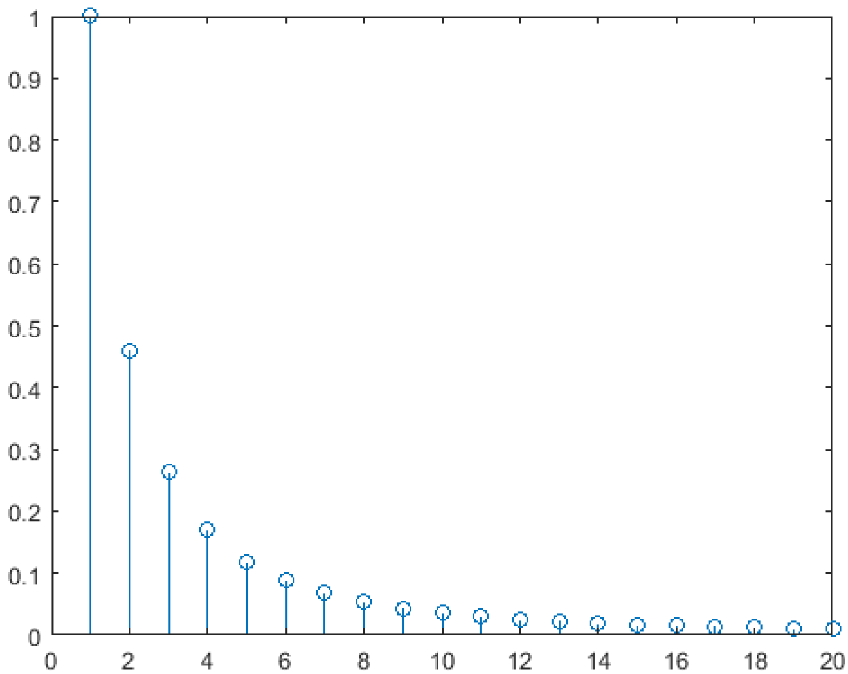


Figure 3.1: Plot $|a_n - 1|$.

number p such that for every natural number $n > p$, we have $|a_n - 1| < \varepsilon$".

Definition 3.4.1 *(Limit)*
A sequence a_n is said to converge to the limit a and we write

$$\lim_{n\to+\infty} a_n = a \text{ or } a_n \to a \text{ if}$$

$$\forall \varepsilon > 0 \;\; \exists p \in \mathbb{N} \;\; \forall n \in \mathbb{N}: \quad n > p \Rightarrow |a_n - a| < \varepsilon.$$

3.4.1 Algebra of limits

We shall introduce some results regarding arithmetic operations on limits.

Theorem 3.4.2 *If* (a_n) *and* (b_n) *are convergent sequences, then the sequence* $(a_n + b_n)$ *is convergent and*

$$\lim\,(a_n + b_n) = \lim\ a_n + \lim\ b_n.$$

Theorem 3.4.3 *If* (a_n) *and* (b_n) *are convergent sequences, then the sequence* $(a_n \times b_n)$ *is convergent and*

$$\lim(a_n \times b_n) = \lim a_n \times \lim b_n.$$

Theorem 3.4.4 *If* (a_n) *is a convergent sequence and* p *is a natural number, then the sequence* $(a_n)^p$ *is convergent and*

$$\lim(a_n)^p = (\lim a_n)^p.$$

Theorem 3.4.5 *If* (a_n) *and* (b_n) *are convergent sequences, then the sequence* $(a_n - b_n)$ *is convergent and*

$$\lim\,(a_n - b_n) = \lim\ a_n - \lim\ b_n.$$

Theorem 3.4.6 *If* (a_n) *and* (b_n) *are convergent sequences,* $b_n \neq 0$, $\forall n \in \mathbb{N}$, *and* $\lim b_n \neq 0$ *then the sequence,* $\left(\dfrac{a_n}{b_n}\right)$ *is convergent and*

$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}.$$

Theorem 3.4.7 *If* p *is a natural number and* (a_n) *is a convergent sequence with non-negative elements, then the sequence* $(\sqrt[p]{a_n})$ *is convergent and*

$$\lim\ \sqrt[p]{a_n} = \sqrt[p]{\lim\ a_n}.$$

3.4.2 Infinite limits

Theorem 3.4.8 *A sequence (a_n) is said to tend to infinity (as n tends to infinity), or to have infinity as its limit, and we write $\lim a_n = +\infty$, if $\forall L > 0 \;\; \exists p \in \mathbb{N} \;\; \forall n \in \mathbb{N}: \;\; n > p \Rightarrow a_n > L$.*

Theorem 3.4.9 *A sequence (a_n) is said to tend to minus infinity (as n tends to minus infinity), or to have $-\infty$ as its limit, and we write $\lim a_n = -\infty$, if $\forall L > 0 \;\; \exists p \in \mathbb{N} \;\; \forall n \in \mathbb{N}: \;\; n > p \Rightarrow a_n < -L$.*

Question: What about $b_n = (-2)^n$?

Show that $\lim a_n = +\infty$ using the definition for

$$a_n = \begin{cases} n+1, & \text{se n é par} \\ n^2 - 10, & \text{se n é ímpar} \end{cases}$$

In $\overline{\mathbb{R}}$:

$$\begin{array}{ll} a \times \infty = \infty & (a \neq 0) \\ \dfrac{a}{0} = \infty & (a \neq 0) \\ \dfrac{a}{\infty} = 0 & (a \neq \infty) \\ \dfrac{\infty}{a} = \infty & (a \neq \infty) \\ \infty^p = \infty & (p \in \mathbb{N}) \\ \sqrt[p]{\infty} = \infty & (p \in \mathbb{N}) \\ \infty^k = 0 & (k < 0) \end{array}$$

3.4.3 Indeterminates

In calculus limits involving an algebraic combination of sequences are evaluated by replacing the sequences by their limits; if the expression obtained after this substitution cannot be evaluates because of lack of information it is said to take on an indeterminate form.

The most common indeterminate forms are:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 1^{\infty}, \infty - \infty, 0^0 \text{ and } \infty^0.$$

3.4.4 Special limits - ratio of polynomial in n

For $k, r \in \mathbb{N}$,

$$\lim \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0}{b_r n^r + b_{r-1} n^{r-1} + \cdots + b_0} = \begin{cases} \infty & \text{if } k > r \\ a_k / b_r & \text{if } k = r \\ 0 & \text{if } k < r \end{cases}$$

Example 3.4.10 $\lim \left(\frac{n^2 - 3}{2n^2 + 1} \right) = 1/2$

Exercices:

1. $\left(\frac{n^2 - 3}{2n^2 + 3n + 1} \right)$;
2. $\left(\frac{n^2 - 3}{n + 1} \right)$;
3. $\left(\frac{n^2 - 3}{4n^3 + n^2 + 1} \right)$;
4. $\left(\frac{4n^4 + n^3 + 2}{2n^4 + 6n + +1} \right)$;

3.4.5 Special limits - Generalization of ratio of polynomials

The previous result cam be generalized to powers of racional exponent, for example:

$$\lim \frac{\sqrt[3]{3n^3 + 3}}{\sqrt[2]{2n^2 + 3}} = \lim \frac{\sqrt[6]{(3n^3 + 3)^2}}{\sqrt[6]{(2n^2 + 3)^3}} = \lim \sqrt[6]{\frac{(3n^3 + 3)^2}{(2n^2 + 3)^3}} = \sqrt[6]{\frac{3^2}{2^3}} = \frac{\sqrt[3]{3}}{\sqrt[2]{2}}$$

For $k, r \in \mathbb{Q}^+$,

$$\lim \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0}{b_r n^r + b_{r-1} n^{r-1} + \cdots + b_0} = \begin{cases} \infty & \text{if } k > r \\ a_k / b_r & \text{if } k = r \\ 0 & \text{if } k < r \end{cases}$$

Example 3.4.11 $\lim \frac{\sqrt[2]{n^2 - 3}}{\sqrt[2]{4n^2 + n + 1}} = \frac{\sqrt[2]{1}}{\sqrt[2]{4}}$

Exercices:

1. $\lim \frac{n^2 \sqrt[2]{n^2 + 1}}{\sqrt[2]{3n^6 + n + 1}}$;
2. $\lim \frac{2n \sqrt[2]{n - 3} + n^2}{\sqrt[6]{4n^6 + n^2 + 1}}$;
3. $\lim \frac{n \sqrt[4]{n^2 - 3} + n^2}{4n^3 + 1}$;
4. $\lim \frac{n^4 + \sqrt[2]{n - 3} + n^2}{\sqrt[5]{4n^{10} + n^2 + 1}}$;

3.4.6 Special limits - Exponential a^n

Value a	Monotony a^n
$a > 1$	increasing
$a = 1$	constant
$0 < a < 1$	decreasing
$a = 0$	constant
$a < 0$	not monotone

Value of a	Limit of a^n
$a > 1$	$+\infty$
$a = 1$	1
$-1 < a < 1$	0
$a = -1$	does not exist
$a < -1$	∞

Example 3.4.12 $\left(\frac{3}{4}\right)^n = 0$

Exercise $\left(\frac{4n}{2n+1}\right)^n$

3.4.7 Special limits - Nepper

$$\lim \left(1+\frac{k}{n}\right)^n = e^k$$

If $u_n \longrightarrow +\infty$

$$\lim \left(1+\frac{k}{u_n}\right)^{u_n} = e^k$$

If $v_n \longrightarrow -\infty$

$$\lim \left(1+\frac{k}{v_n}\right)^{v_n} = e^k$$

Example 3.4.13

$$\left(\frac{n+2}{n}\right)^{n+2} = \left(\frac{n+2}{n}\right)^n \left(\frac{n+2}{n}\right)^2 =$$
$$= \left(1+\frac{2}{n}\right)^n \left(\frac{n+2}{n}\right)^2 = e^2.1 = e^2$$

Exercise

1. $\left(\frac{n-3}{n}\right)^{n+1}$;
2. $\left(1-\frac{1}{n+1}\right)^n$;
3. $\left(1+\frac{2}{3n}\right)^n$;
4. $\left(\frac{2n-1}{3n+2}\right)^n$;
5. $\left(1-\frac{4}{n^2}\right)^{2n}$;
6. $\left(\frac{2n+3}{-3n+5}\right)^{4n}$;
7. $\left(1-\frac{2}{n^2}\right)^{n^3}$.

3.4.8 Special limits - Product of an infinitesimal by a bounded sequence

If $u_n \longrightarrow \infty$ and $v_n \longrightarrow 0$ then $\lim (u_n v_n) = 0$.

Example 3.4.14 *To find* $\lim \left((-1)^n \frac{1}{n^2+1}\right)$ *we cannot apply the algebra of limits because* $\lim(-1)^n$ *does not exists but it is bounded since* $-1 \leq (-1)^n \leq 1$. *Since* $\lim \frac{1}{n^2+1} \longrightarrow 0$ *we may conclude that* $\lim \left((-1)^n \frac{1}{n^2+1}\right) \longrightarrow 0$.

Exercise

1. $\lim \left(\frac{-1}{n+1}\right)^n$;

2. $\left(\sin(n)\frac{1}{n+1}\right)$;

3.5 Exercises

1. Consider the sequence $u_n = \dfrac{2n-1}{n+1}$.
   (a) Find the terms of rank 5, 20 and n+1.
   (b) Given the real numbers $\dfrac{29}{16}, \dfrac{40}{19}$ find it they are elements of u_n.
   (c) Prove that:
      (i) (u_n) is monotonically increasing;
      (ii) $\forall n \in \mathbb{N}$, $\frac{1}{2} \leq u_n < 2$;
      (iii) (u_n) is convergent.
   (d) Find an upper and lower limit.

2. Given $u_n = \dfrac{\sqrt{2n}}{1+\sqrt{n}}$:
   (a) Show that $\lim u_n = \sqrt{2}$
   (b) Find the rank of the first element of the sequence that verifies
   $$|u_n - \sqrt{2}| < 10^{-1}.$$

3. Show that the sequence $b_n = \dfrac{2^n}{(n+1)!}$ é is strictly decreasing.

4. Consider
   $$u_n = -2 \times 3^{n-5}.$$
   (a) Show that u_n is a geometic progression .
   (b) Study its monotonicity.
   (c) Find $\displaystyle\sum_{k=2}^{8} u_k$.

5. In a aritmetic progression with common diffrence 5 we know that the element of rank 10 its three times the element of rank 8. Find the sum of the first 20 elements.

6. Find the limit of
   (a) $\dfrac{4-n^2}{n^3-2}$
   (b) $\dfrac{2}{n^3+5} \times \sqrt{n-3}$
   (c) $\dfrac{5^n + (-7)^{n+1}}{4^{n+2} - 3^n}$
   (d) $\left(\dfrac{n+5}{n+2}\right)^n$

(e) $\left(\frac{n^3-2}{n^3}\right)^{n^2-3}$

7. Let (a_n) be the general term . Write a_{n+1}, a_{2n} and a_{n+p}, $p \in \mathbb{N}$, for the following cases:

   (a) $a_n = \frac{2^n}{n+1}$

   (b) $a_n = \frac{(n+1)!}{(3n-1)!}$

   (c) $a_n = \frac{(n-1)^2}{2n+1}$

   (d) $a_n = \sqrt[n]{\frac{(2n-1)!}{2^{n+1}+\log n}}$

   (e) $a_n = \frac{(n^2+1)!}{(n^2-1)!}$

8. Write the general term of the following sequences and check if they are bounded.

   (a) The sequence formed by the simetrics of the perfect squares.

   (b) The sequence of the powers of base (-2) and natural exponent.

Chapter 4

Study of Functions

Contents

4.1 Basic Definitions

Definition 4.1.1 *(Function)*
Let A e B be two sets. A function f is a rule that assigns to each element x in A exactly one element, $y = f(x)$, in B.

The variable x is the independent variable and y is the dependent variable.

Definition 4.1.2 *(Domain and Range)*
Given a real function f of real variable, the domain of f is the set of values in $\mathbb{R}$ such that $f(x)$ can be algebraically calculated. The range is the set of values $y = f(x)$ for every which x in the Domain of f.

Definition 4.1.3 *(Properties)*
A function f from A to B is

- *Injective if* $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
- *Surjective if* $\forall y \in B, \exists x \in A : y = f(x)$.
- *Bijective if it is injective and surjective.*
- *Even if* $f(x) = f(-x)$.
- *Odd if* $f(x) = -f(-x)$.
- *Increasing if* $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$.
- *Strictly increasing if* $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.
- *Decreasing if* $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$.
- *Strictly decreasing if* $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

Definition 4.1.4 *(Composition of functions)*
Composition of function is a sequence of nested functions, where the input of one function is the output of the previous function. For the composition of two functions we say f after g and write $f \circ g$ and the expression is given by $f \circ g(x) = f(g(x))$.
The domain of $f \circ g$ is given by

$$D_{f \circ g} = \{x \in \mathbb{R} : x \in D_g \wedge y = g(x) \in D_f\}$$

Definition 4.1.5 *(Roots, maximum and minimum)*
We say that x_0 is a root or a zero of f if $f(x) = 0$.
$(f(x_1))$ is a relative or local minimum of f if

Definition 4.1.6 *(Inverse function)*
We say that f and g are inverse functions if $f \circ g = g \circ f = I$ where I is the identity function $I(x) = x$.

Definition 4.1.7 *(Algebric operations on functions)*

- $(f+g)(x) = f(x) + g(x)$ *and*
$$D_{f+g} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(fg)(x) = f(x).g(x)$ *and*
$$D_{fg} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(f-g)(x) = f(x) - g(x)$ *and*
$$D_{f-g} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(f/g)(x) = f(x)/g(x)$ *and*
$$D_{f/g} = \{x \in \mathbb{R} : x \in D_g \wedge g(x) \neq 0 \wedge x \in D_f\}$$

Definition 4.1.8 *(Stepwise function)*
We say that f is a stepwise function if

$$f = \begin{cases} g_1(x) & \text{if } x \in A_1 \\ g_2(x) & \text{if } x \in A_2 \\ \dots & \\ g_k(x) & \text{if } x \in A_k \end{cases}$$

where $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$ *and*

$$D_f = \{x \in \mathbb{R} : (x \in A_1 \wedge x \in D_{g_1}) \vee \cdots \vee (x \in A_k \wedge x \in D_{g_k})\}$$

4.1.1 Exercises

1. Find the domain of the following functions

   (a) $f = \begin{cases} \sqrt{x-1} & \text{if } x \geq 0 \\ \dfrac{x-1}{x+2} & \text{if } x < 0 \end{cases}$

   (b) $g = \begin{cases} x^2+2 & \text{if } x \geq 4 \\ \dfrac{1}{x^2-4} & \text{if } x < 4 \end{cases}$

   (c) $h = \begin{cases} \dfrac{1}{2x^2-8x+6} & \text{if } x \leq 1 \\ \dfrac{1}{x^2} & \text{if } x > 1 \end{cases}$

2. For $f(x) = \frac{1}{x^2}$ and $r(x) = 2x - 1$ write $f \circ r$ and $r \circ f$.

3. For $g(x) = x^3 + 3$ and $h(x) = x + 2$ write $g \circ h$ and $h \circ g$.
4. Find the inverse function of $f(x) = 3x - 7$.
5. For $g(x) = x + 1$ and $s(x) = x^3$ write $g \circ s$. Define the inverse of g, the inverse of s and the inverse of $g \circ s$ and relate $(g \circ s)^{-1}$ with g^{-1} and s^{-1} .
6. Check if the function $f(x) = (x - 1)^3 + 2$ have inverse and in case of a positive answer find the expression of f^{-1}.
7. Check if the function $h(x) = x^2 - 6$ have inverse and in case of a positive answer find the expression of h^{-1}.

4.2 Exponential and logarithmic functions

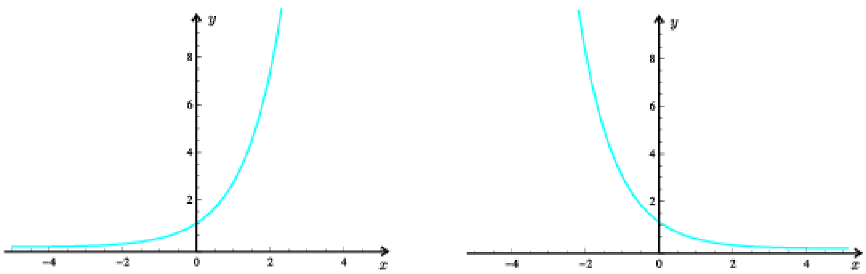
The exponential function is given by the expression

$$f(x) = a^x$$

with $a > 0$. The domain is $\mathbb{R}$, the range is $\mathbb{R}^+$.

- $a^x \cdot a^y = a^{x+y}, \forall a \in \mathbb{R}^+, \ \ \forall x, y \in \mathbb{R}$;
- $\dfrac{a^x}{a^y} = a^{x-y}, \forall a \in \mathbb{R}^+, \ \ \forall x, y \in \mathbb{R}$;
- $(a^x)^y = a^{x \cdot y}, \forall a \in \mathbb{R}^+, \ \ \forall x, y \in \mathbb{R}$;
- $a^x \cdot b^x = (a \cdot b)^x, \forall a, b \in \mathbb{R}^+, \ \ \forall x \in \mathbb{R}$;
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x, \forall a, b \in \mathbb{R}^+, \ \ \forall x \in \mathbb{R}$.

The function has no zeros. It is strictly increasing for $a > 1$ and strictly decreasing for $0 < a < 1$.
Let us remind some properties of exponentials. Among exponential functions



(a) The function a^x, $a > 1$. (b) The function a^x, $0 < a < 1$.

Figure 4.1: Graphical representation of exponential function.

it is relevant, for its practical applications, the function of base $a = e$ where e is the number of Neper. In general we refer to e^x simply as the exponential function. The inverse of the exponential function is the logarithmic function.

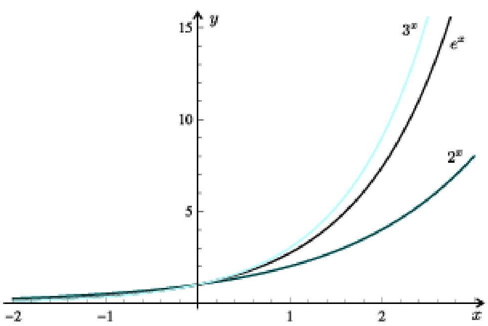


Figure 4.2: Function a^x, with $a = 2$, $a = e$ e $a = 3$.

In fact we have

$$, x = a^y \Leftrightarrow \log_a x = y$$

and

$$\log_a a^y = y \text{ and } a^{\log_a x} = x. \tag{4.1}$$

From the definition of logarithm we have the following properties

$$\log_a a = 1 \text{ (because } a^1 = a)$$

e

$$\log_a 1 = 0 \text{ (because } a^0 = 1).$$

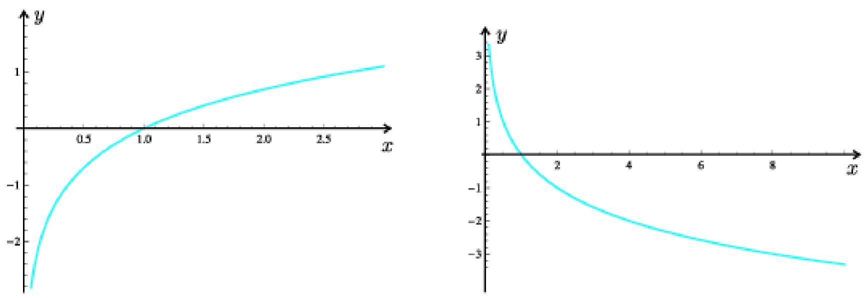
We have also the following properties, in which we consider $a, b \in \mathbb{R}^+ \setminus \{1\}$, $x, y \in \mathbb{R}^+$, $z \in \mathbb{R}$ e $n \in \mathbb{N}$:

- $\log_a(x.y) = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $log_a\,(x^z) = z \cdot \log_a x$
- $log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$
- $\log_b x = \frac{\log_a x}{\log_a b}$

Note: In the case of $a = e$ (Neper number) we will adopt the notation $\log_e x = \log x$.

Properties

- Domain is $\mathbb{R}^+$ and range is $\mathbb{R}$;
- The function has only one root at $x = 1$.
- The function is surjective ans injective, so is bijective;



(a) The function $\log_a x$, $a > 1$.

(b) The function $\log_a x$, $0 < a < 1$.

Figure 4.3: Graphical representation of $log_a(x)$.

4.2.1 Exercises

1. Solve the equations:

   (a) $2^x = 128$;

   (b) $10^x = 100$;

   (c) $15^x = 225$;

   (d) $4^{2x+1} = \dfrac{1}{64}$;

   (e) $3^{2-x} = 81$;

   (f) $5^{3x+2} = \dfrac{1}{125}$.

2. Simplify the expressions:

   (a) $\log_3 9 + \log_3 36 - \log_3 4$;

   (b) $\dfrac{\log_5 \frac{1}{8}}{\log_5 2}$;

   (c) $\log_{10}(x+3) - 4\log_{10} x$;

   (d) $4\log x - 6\log(x+2)$;

   (e) $\log_b y^3 + \log_b y^2 - \log_b y^4$;

   (f) $\dfrac{1}{2}\log_{1/3} x^2 + 5\log_{1/3} x$;

   (g) $\log_{1/5} 16 + \log_{1/5} 20 - \log_{1/5} 4^3$;

   (h) $\dfrac{1}{3}(3\log_2 x - \log_2 \frac{1}{y^3} + 6\log_2 z)$.

3. Solve the equations

   (a) $\log_b 256 = 4$;

   (b) $\log_5 125 = x$;

   (c) $\log_b 8 = \dfrac{3}{2}$;

   (d) $\log_9(2x+1) = \dfrac{1}{2}$;

   (e) $\log_{1/2}(2-x) = 3$;

   (f) $\log(3x+2) = \log 7$.

4. Simplify the expressions:

   (a) $e^{3+\log x}$;

   (b) $16^{\log_2 x + 3\log_4 \sqrt{x}}$;

   (c) $\left(\dfrac{1}{3}\right)^{\log_3(x^2+4) - 2\log_3 x}$;

   (d) $25^{3\log_5 x}$;

   (e) $7^{5\log_7 x - 2\log_7 x}$;

   (f) $16^{\log_2 x}$;

5. Solve the equations :

   (a) $3^{x-5} = 4$;
   (b) $\log_{10} x + \log_{10}(x - 15) = 2$;
   (c) $\dfrac{1}{16} = 64^{4x-3}$;
   (d) $\log_3 \sqrt{2x+3} = 2$;
   (e) $4(\log x)^2 - 3\log x = 7$;
   (f) $\dfrac{1}{3}\log(x^{\log(x^3)}) - \log(x^5) + 4 = 0$.

6. Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (0.2)^{x-1}$. Identify the true and false propositions:

   (a) $f(x) < 1,\ \forall x \in \mathbb{R}$;
   (b) $f(0) = 3$;
   (c) $f(x) > 25,\ \forall x \in \mathbb{R}^-$;
   (d) $\forall x \geq 1,\ 0 < f(x) \leq 1$;
   (e) $f(x) < 0,\ \forall x \in \mathbb{R}$;
   (f) $\lim\limits_{x\to+\infty} f(x) = -\infty$.

7. Find the set os solution of the inequalities

   (a) $\left(\dfrac{1}{3}\right)^{x-1} < 9^{2-x}$;
   (b) $(0.1)^{x-x^2} \leq 0.01$;
   (c) $\log_{\frac{1}{2}}(x+5) > 0$.

8. Solve the inequalities:

   (a) $x^3\log_2(2x) + x^3\log_{\frac{1}{2}}(x+5) < 0$;
   (b) $\log_{\frac{1}{3}}(2x) < 2 - \log_{\frac{1}{3}}\left(\frac{1-x}{x}\right)$;
   (c) $3^{\frac{x^2-4}{x^2+5}} < 1$.

9. Consider the function $g(x) = 5 + \log_{\frac{1}{2}}(3x - 1)$.

   (a) Find $\mathcal{D}_g$ e $\mathcal{CD}_g$.
   (b) Solve $g(x) > 0$.
   (c) Find the zeros of g.
   (d) Study the injectivity of g.
   (e) Find the inverse of g, g^{-1}.

10. Find the domain of function f defined by $f(x) = \dfrac{5(x-2)^3}{e^{3(x-2)} - 1}$.

4.3 Trignometric functions

4.3.1 Background

The unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system. The unit circle, with radius 1 has a circumference of length 2π. In other words, for one revolution around the unit circle the length of the arc is 2π units.

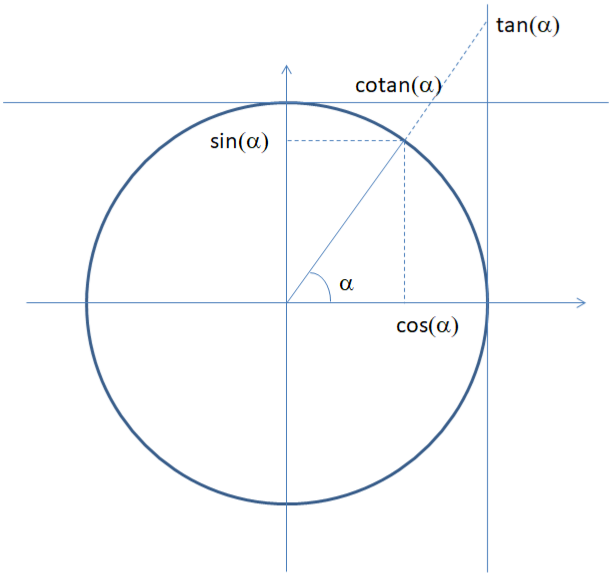


Figure 4.4: Trigonometric circle.

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos^2(\alpha) = \dfrac{1}{1+\tan^2(\alpha)}$

4.3.2 Function sine

$$\begin{aligned} f:\mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow \sin(x) \end{aligned}$$

- $\mathcal{D}_f = \mathbb{R}$ and $\mathcal{CD}_f = [-1, 1]$.
- It is a periodic function with period 2π,there is,

$$\operatorname{sen}(2\pi + x) = \operatorname{sen}(x), \quad \forall x \in \mathbb{R}$$

.

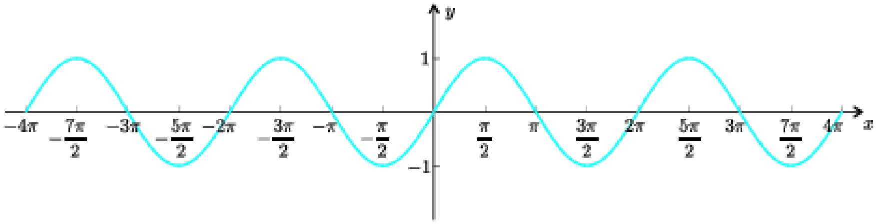


Figure 4.5: Graphical representation of $\sin(x)$.

- It is odd
$$\sin(-x) = -\sin(x), \quad \forall x \in \mathbb{R}.$$
- It is increasing in intervals
$$\left]-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right[, \ k \in \mathbb{Z},$$
e and decreasing in intervals
$$\left]\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right[, \ k \in \mathbb{Z}.$$
- Has a maximum at $x = \frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$, and minimum at $x = -\frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$. The maximun is 1 and the minimum -1.
- The zeros are of the form $x = k\pi, \ k \in \mathbb{Z}$.

4.3.3 Function cosine

$$\begin{array}{rcl} g : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \rightarrow & \cos(x) \end{array}$$

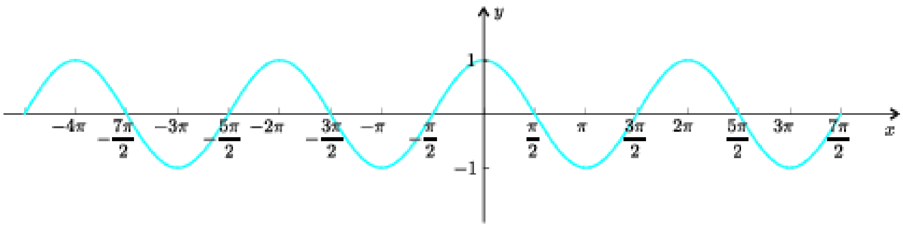


Figure 4.6: Graphical representation of $\cos(x)$.

- $\mathcal{D}_g = \mathbb{R}$ e $\mathcal{CD}_g = [-1, 1]$.
- It is periodic with period 2π, that is,
$$\cos(2\pi + x) = \cos(x), \quad \forall x \in \mathbb{R}.$$

- Is even,
$$\cos(-x) = \cos(x), \quad \forall x \in \mathbb{R}.$$
- Is increasing in the intervals
$$]\pi + 2k\pi, 2\pi + 2k\pi[\,, \;\; k \in \mathbb{Z},$$
e and decreasing at
$$]2k\pi, \pi + 2k\pi[\,, \;\; k \in \mathbb{Z}.$$
- Reaches a maximum at $x = 2k\pi, \;\; k \in \mathbb{Z}$, and the minimum at $x = \pi + 2k\pi, \;\; k \in \mathbb{Z}$.The maximun is 1 and the minimum is -1.
- The zeros are the points $x = \dfrac{\pi}{2} + k\pi, \;\; k \in \mathbb{Z}$.

4.3.4 Function tangent

$$\begin{array}{rcl} t : \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \rightarrow & \tan(x) = \frac{\sin(x)}{\cos(x)} \end{array}$$

- $\mathcal{D}_h = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}\}$ e $\mathcal{CD}_h = \mathbb{R}$.
- Is periodic with period π, that is,
$$\mathrm{tg}(\pi + x) = \mathrm{tg}(x), \quad \forall x \in \mathcal{D}_h.$$
- Is odd
$$\mathrm{tg}(-x) = -\mathrm{tg}(x), \quad \forall x \in \mathcal{D}_h.$$
- Is increasing in
$$\left]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right[, \;\; k \in \mathbb{Z}.$$
- Has no maximun or minimum.
- The zeros are $x = k\pi, \;\; k \in \mathbb{Z}$.

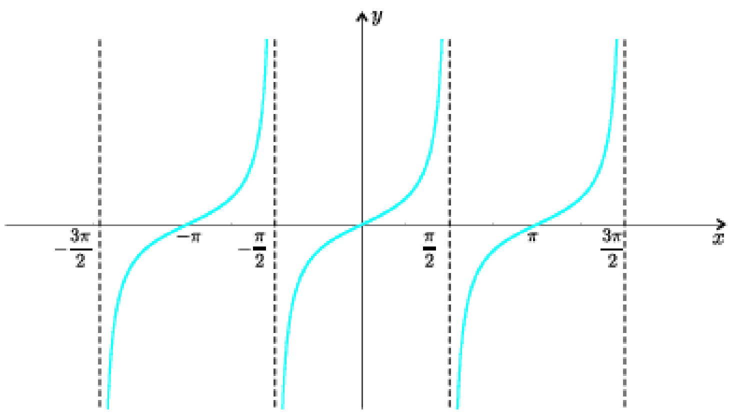


Figure 4.7: Graphical representation of tangent.

4.3.5 Function cotangent

$$\begin{aligned} t : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow \cot(x) = \tfrac{\cos(x)}{\sin(x)} \end{aligned}$$

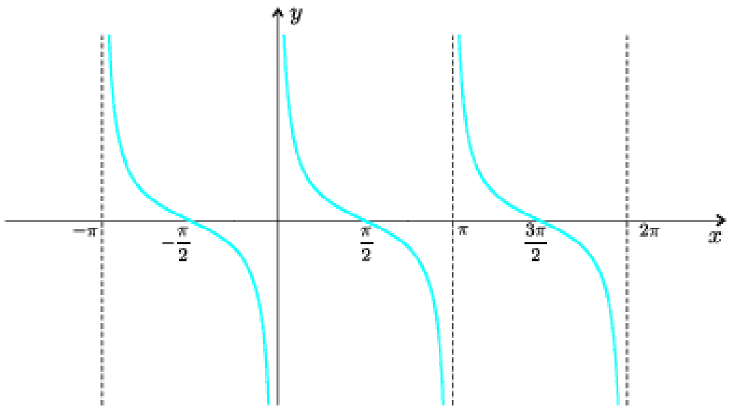


Figure 4.8: The graphical representation of cot.

- $\mathcal{D}_i = \mathbb{R} \setminus \{k\pi,\ k \in \mathbb{Z}\}$ e $\mathcal{CD}_i = \mathbb{R}$.
- Is periodic with period π, that is,

$$\operatorname{cotg}(\pi + x) = \operatorname{cotg}(x), \quad \forall x \in \mathcal{D}_i.$$

- Is odd

$$\operatorname{cotg}(-x) = -\operatorname{cotg}(x), \quad \forall x \in \mathcal{D}_i.$$

- Is increasing in intervals

$$]k\pi, \pi + k\pi[\,,\ k \in \mathbb{Z}.$$

- Has no minimum or maximum.
- The zeros are $x = \dfrac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

In Summary

$f(x)$	Domain	Range	Zeros	Odd/Even	Period	Maximum attained at	Minimum attained at
$\text{sen}(x)$	$\mathbb{R}$	$[-1,1]$	$x = k\pi$	odd	2π	$x = \frac{\pi}{2} + 2k\pi$	$x = -\frac{\pi}{2} + 2k\pi$
$\cos(x)$	$\mathbb{R}$	$[-1,1]$	$x = \frac{\pi}{2} + k\pi$	even	2π	$x = 2k\pi$	$x = \pi + 2k\pi$
$\text{tg}(x)$	$\left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi\right\}$	$\mathbb{R}$	$x = k\pi$	odd	π	—	—
$\text{cotg}(x)$	$\{x \in \mathbb{R} : x \neq k\pi\}$	$\mathbb{R}$	$x = \frac{\pi}{2} + k\pi$	odd	π	—	—

4.3.6 Inverse Trignometric Functions

Clearly the previous functions are not invertible but we can consider a restriction (sub-domain) in which they are injective and so invertible. We will choose a sub-domain where the we have full range (the range is the same of the original function).

Let $f(x) = \text{sen}\,(x)$ and consider the **main restriction of the function** the restriction of f to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let it be g than

$$\begin{aligned} g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] &\rightarrow [-1,1] \\ x &\rightarrow \text{sen}\,(x). \end{aligned}$$

Being bijective g invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = [-1,1], \;\; \mathcal{CD}_{g^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and to each $y \in [-1,1]$ we have $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which sin is y; such angle is represented by $\arcsin(y)$ and the function g^{-1} is the arc-sin function. Then for every $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1,1]$

$$y = \sin\,(x) \Leftrightarrow x = \arcsin\,(y).$$

Let $f(x) = \cos\,(x)$ and consider the **main restriction of the function** the restriction of f to the interval $[0,\pi]$. Let it be g than

$$\begin{aligned} g : [0,\pi] &\rightarrow [-1,1] \\ x &\rightarrow \cos\,(x). \end{aligned}$$

Being bijective g is invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = [-1,1], \;\; \mathcal{CD}_{g^{-1}} = [0,\pi]$$

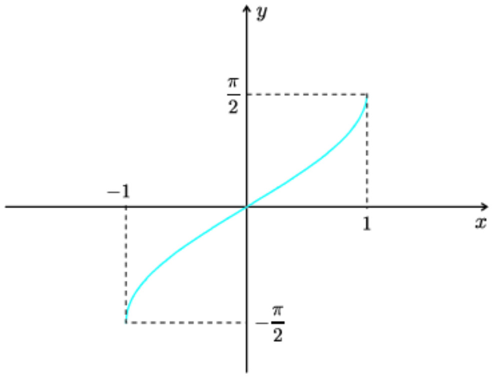


Figure 4.9: Graphical representation of arcsin.

and to each $y \in [-1, 1]$ we have $x \in [0, \pi]$ which cos is y; such angle is represented by $\arccos(y)$ and the function g^{-1} is the arc-cos function. Then for every $x \in [0, \pi]$ and $y \in [-1, 1]$

$$y = \cos(x) \Leftrightarrow x = \arccos(y).$$

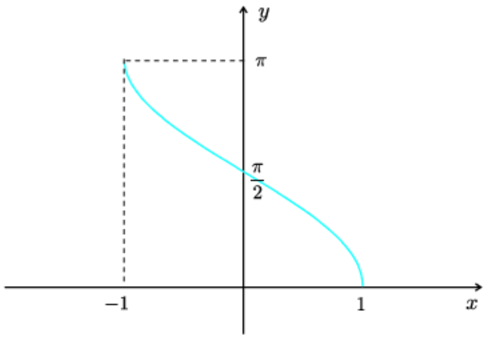


Figure 4.10: Graphical representation of arccos.

Let $f(x) = \tan(x)$ and consider the **main restriction of the function** the restriction of f to the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Let it be g than

$$g : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\quad \rightarrow \quad \mathbb{R}$$
$$x \quad \rightarrow \quad \tan(x).$$

Being bijective g invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = \mathbb{R}, \quad \mathcal{CD}_{g^{-1}} = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

and to each $y \in \mathbb{R}$ we have $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ which tan is y; such angle is represented by $\arctan(y)$ and the function g^{-1} is the arc-tan function. Then for every $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ e $y \in \mathbb{R}$

$$y = \tan(x) \Leftrightarrow x = \arctan(y).$$

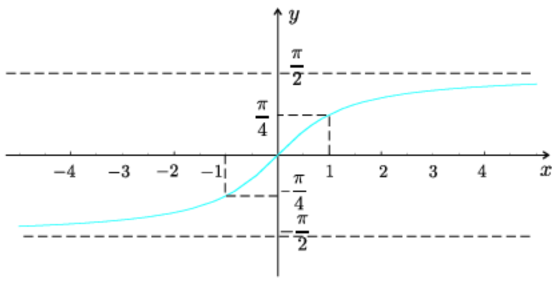


Figure 4.11: Graphical representation of arctan.

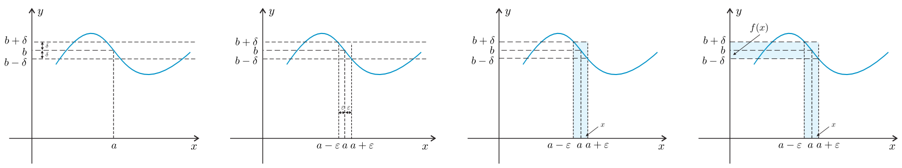
4.4 Limits and continuity

Definition 4.4.1 *(Limit of function)*
We say that

$$\lim_{x \to a} f(x) = b$$

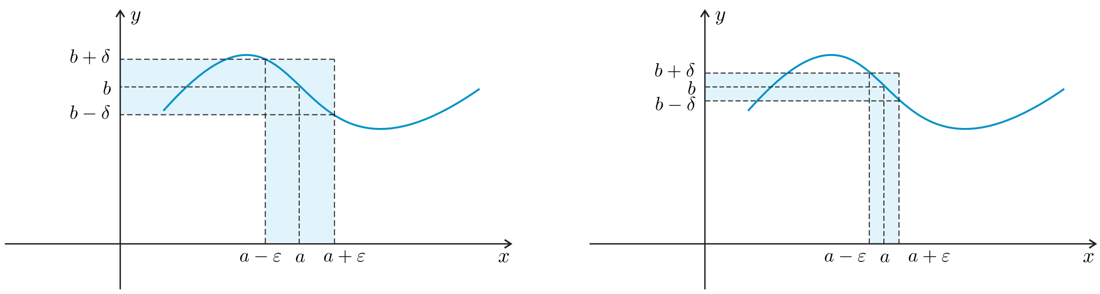
iff

$$\forall \delta > 0, \exists \epsilon > 0 : x \in D_f \wedge |x - a| < \epsilon \Rightarrow |f(x) - b| < \delta$$



(a) For every $\delta > 0$ (b) there exists a $\varepsilon > 0$ such that (c) if $|x - a| < \varepsilon$ (d) then $|f(x) - b| < \delta$

Figure 4.12



(a) (b)

Figure 4.13: The value of ε depends of δ.

Example 4.4.2 *Considering* $f(x) = \frac{x^2-1}{x-1}$ *let us prove that* $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

$$\left| \frac{x^2 - 1}{x - 1} - 2 \right| = \left| \frac{x^2 - 1 - 2(x - 1)}{x - 1} \right| = \left| \frac{x^2 - 2x + 1}{x - 1} \right| = \tag{4.2}$$

$$= \left| \frac{(x - 1)^2}{x - 1} \right| = |(x - 1)| \tag{4.3}$$

So, for every $\delta > 0$ *if* $|x - 1| < \varepsilon$ *and* $\varepsilon \leq \delta$ *then* $|f(x) - 2| < \delta$. *This concludes the proof.*

Definition 4.4.3 *(Relative limits)*

$$\lim_{x\to a^+} f(x) = b \Leftrightarrow$$

$$\forall\delta > 0, \exists\epsilon > 0 : x \in D_f \wedge x > a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

$$\lim_{x\to a^-} f(x) = b \Leftrightarrow$$

$$\forall\delta > 0, \exists\epsilon > 0 : x \in D_f \wedge x < a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

$$\lim_{\substack{x\to a\\ x\neq a}} f(x) = b \Leftrightarrow$$

$$\forall\delta > 0, \exists\epsilon > 0 : x \in D_f \wedge x \neq a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

Theorem 4.4.4 *The* $\lim_{\substack{x\to a\\ x\neq a}} f(x) = b$ *if and only if* $\lim_{x\to a^+} f(x) = b$ *and* $\lim_{x\to a^-} f(x) = b$*. If* $a \notin D_f$ *then there is no difference between* $\lim_{\substack{x\to a\\ x\neq a}} f(x)$ *and* $\lim_{x\to a} f(x)$*. If* $a \in D_f$ *then if* $\lim_{\substack{x\to a\\ x\neq a}} f(x) = b$ *then* $\lim_{x\to a} f(x)$ *exists if and only if* $b = f(a)$*. In that case also* $\lim_{x\to a} f(x) = b = f(a)$

Example 4.4.5 *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 0, & se\ x < 2 \\ 1, & se\ x \geq 2 \end{cases}$$

(see Figure 4.14).

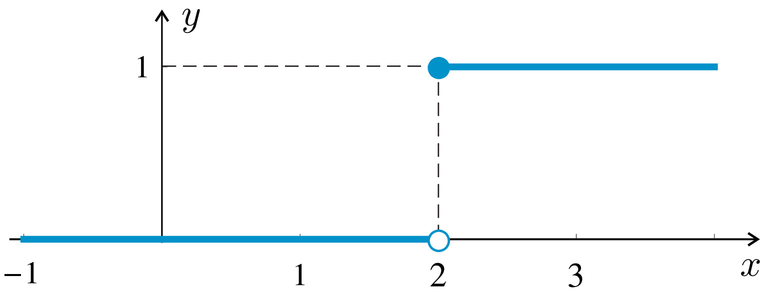


Figure 4.14

We have $\lim_{x\to 2^-} f(x) = 0$ *and* $\lim_{x\to 2^+} f(x) = 1$. *so,* $\lim_{\substack{x\to 2\\ x\neq 2}} f(x)$ *does not exists and as a consequence also* $\lim_{x\to 2} f(x)$ *does not exist.*

Example 4.4.6 *Consider the function* $f:\mathbb{R}\to\mathbb{R}$ *defined by*

$$f(x) = \begin{cases} |x-4|, & se\ x\neq 4 \\ 2, & se\ x = 4 \end{cases}$$

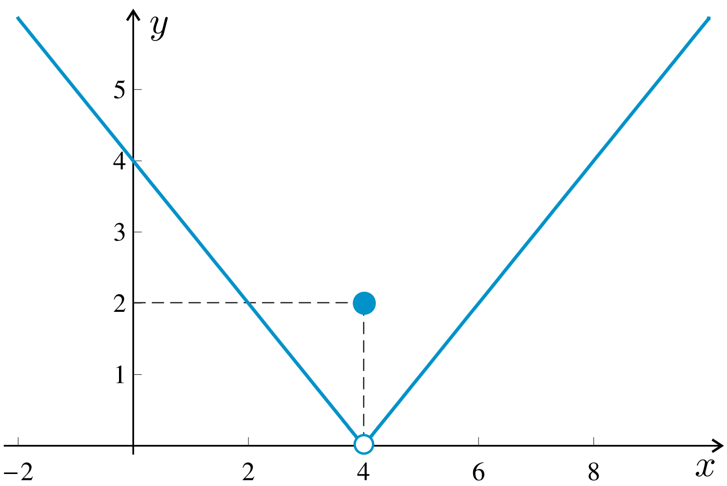


Figure 4.15

We see that $\lim_{x\to 4^-} f(x) = 0$ *and* $\lim_{x\to 4^+} f(x) = 0$. *So,* $\lim_{\substack{x\to 4\\ x\neq 4}} f(x) = 0$, *but* $\lim_{x\to 4} f(x)$ *does not exist because* $f(4) = 2 \neq 0$.

Definition 4.4.7 *We say that the limit of* f *when* $x\to +\infty$ *is* b *if*

$$\forall\,\delta > 0\ \exists\,\varepsilon > 0 : x\in D\ \wedge\ x > \frac{1}{\varepsilon} \Rightarrow |f(x) - b| < \delta$$

and we write $\lim_{x\to +\infty} f(x) = b$.

Definition 4.4.8 *We say that the limit of* f *when* $x\to -\infty$ *is* b *if*

$$\forall\,\delta > 0\ \exists\,\varepsilon > 0 : x\in D\ \wedge\ x < -\frac{1}{\varepsilon} \Rightarrow |f(x) - b| < \delta$$

and we write $\lim_{x\to -\infty} f(x) = b$.

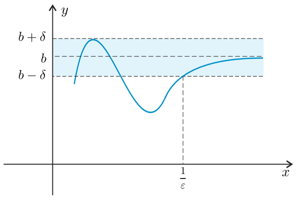


Figure 4.16: The value of εdepends on δ.

Definition 4.4.9 *We say that the limit of f when $x \to a$ is $+\infty$ if*

$$\forall\, \delta > 0\ \exists\, \varepsilon > 0 : x \in D\ \wedge\ |x-a| < \varepsilon \Rightarrow f(x) > \frac{1}{\delta}$$

and we write $\lim\limits_{x\to a} f(x) = +\infty$.

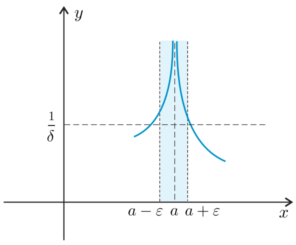


Figure 4.17: O valor de ε depende do valor de δ.

Definition 4.4.10 *We say that the limit of f when $x \to a$ is $-\infty$ if*

$$\forall\, \delta > 0\ \exists\, \varepsilon > 0 : x \in D\ \wedge\ |x-a| < \varepsilon \Rightarrow f(x) < -\frac{1}{\delta}$$

and we write $\lim\limits_{x\to a} f(x) = -\infty$.

Example 4.4.11 *a)* $\lim\limits_{x\to a^-} \dfrac{1}{x-a} = -\infty$ *e* $\lim\limits_{x\to a^+} \dfrac{1}{x-a} = +\infty$;

$\lim\limits_{x\to a} \dfrac{1}{x-a}$ *não existe.*

b) $\lim\limits_{x\to a^-} \dfrac{1}{(x-a)^2} = +\infty$ *e* $\lim\limits_{x\to a^+} \dfrac{1}{(x-a)^2} = +\infty$;

$\lim\limits_{x\to a} \dfrac{1}{(x-a)^2} = +\infty$.

c) $\lim\limits_{x\to +\infty} \dfrac{1}{x} = 0 = \lim\limits_{x\to -\infty} \dfrac{1}{x}$.

d) $\lim\limits_{x\to 0^+} (1+x)^{1/x} = \lim\limits_{y\to +\infty} \left(1+\dfrac{1}{y}\right)^y = e$.

Some special limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \tag{4.4}$$
$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \tag{4.5}$$
$$\lim_{x\to 0} \frac{e^x-1}{x} = 1 \tag{4.6}$$

$$\lim_{x\to +\infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \begin{cases} +\infty & \text{if } n > m \\ \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases} \tag{4.7}$$

Theorem 4.4.12 *The limit of f in a, when it exists is unique.*

Proof: Supose that $b, c \in \mathbb{R}$ exist such that $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} f(x) = c$ e $b \neq c$. Let $\delta = |b-c|/2$. Since $b \neq c$, then $\delta > 0$. by definition **??**,

$$\exists \varepsilon_1 > 0 : x \in D \wedge |x-a| < \varepsilon_1 \Rightarrow |f(x)-b| < \delta$$

and

$$\exists \varepsilon_2 > 0 : x \in D \wedge |x-a| < \varepsilon_2 \Rightarrow |f(x)-c| < \delta.$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $x \in D$ such that $|x-a| < \varepsilon$. Then

$$|f(x)-b| < \delta \wedge |f(x)-c| < \delta,$$

so

$$|b-c| = |b-f(x)+f(x)-c| \leq |b-f(x)| + |f(x)-c| < \delta + \delta = 2\,\delta = |b-c|,$$

which is impossible. ■

Theorem 4.4.13 *If $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ then:*

a) $\lim_{x\to a} [f(x)+g(x)] = b+c$;

b) $\lim_{x\to a} [f(x)-g(x)] = b-c$;

c) $\lim_{x\to a} [f(x)g(x)] = b\ c$;

d) *Se* $c \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c}$.

Theorem 4.4.14 *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $g : E \subset \mathbb{R} \to \mathbb{R}$ such that $g(E) \subset D$. Se $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = c$ then $\lim_{x\to a} (f \circ g)(x) = c$.*

Definition 4.4.15 *(Continuity)*
We say that f is continuous in $a \in D_f$ if $\lim\limits_{x \longrightarrow a} f(x)$ exists.
(By the definition of limit we must have $\lim\limits_{x \longrightarrow a} f(x) = f(a)$) We say that f is a continuous function if it is continuous on every point of the domain.

All the elementary functions studied before are continuous.

Exercises:

1. Study the existence of asymptotes of the following functions:

   (a) $f(x) = \dfrac{x^2+1}{x^2-1}$;

   (b) $f(x) = \begin{cases} \dfrac{4-x^2}{x^2-9}, & \text{se } x \geq 0 \\ -x - \dfrac{4}{9}, & \text{se } x < 0; \end{cases}$

   (c) $f(x) = \dfrac{4x}{4-x^2} - x + 2$;

   (d) $f(x) = x + 1 + \dfrac{1}{2x^2}$;

   (e) $f(x) = \dfrac{(x-2)^2}{x-1}$;

   (f) $f(x) = \begin{cases} x + 2 - \dfrac{1}{3-x}, & \text{se } x \geq 0 \\ x^2 + \dfrac{5}{3}, & \text{se } x < 0. \end{cases}$

2. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x}-1}{x}, & \text{se } x \neq 0 \\ k, & \text{se } x = 0 \end{cases}$$

   where $k \in \mathbb{R}$. Find k such that f is continuous in $x = 0$.

3. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{2}{\pi} \operatorname{arctg}\left(\dfrac{1}{x}\right), & \text{se } x < 0 \\ x^2 - x - a & \text{se } x \geq 0, \end{cases}$$

   where $a \in \mathbb{R}$. Study the continuity of f.

4. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{2x^2+2x-4}{x^2-4}, & \text{se } x \leq -2 \\ \dfrac{2k-3x^2}{k+1}, & \text{se } x > -2 \end{cases}$$

onde $k \in \mathbb{R} \setminus \{-1\}$.Find k such that f is continuous in $x = -2$.

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -(x^2 + 2x + 2)\, e^{-x}, & \text{se } x > -1 \\ \log\left((x^2 - 4)^2\right), & \text{se } x \leq -1. \end{cases}$$

   (a) Find the domain of f.
   (b) Study the continuity of f
   (c) Find $\lim_{x \to -2^+} f(x)$ and $\lim_{x \to -2^-} f(x)$.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \log\left(\dfrac{x^2}{|x-1|}\right), & \text{se } x \neq 0 \\ 0, & \text{se } x = 0. \end{cases}$$

   (a) Find the domain of f.
   (b) Study the continuity of f
   (c) Find the range of f.

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \log(1 - x) + x + 1, & \text{se } x < 0 \\ x^2 + 3x + 3, & \text{se } x \geq 0. \end{cases}$$

   (a) Find the domain of f.
   (b) Study the continuity of f.

8. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \log\left(\log^2(x) - 1\right)$.

   (a) Find the domain of f.
   (b) Study the continuity of f
   (c) Find the range of f

9. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\pi}{2} + \log(1 - x^2), & \text{se } x \leq 0 \\ \operatorname{arctg}\left(\dfrac{1}{x}\right) + \dfrac{1}{2}x, & \text{se } x > 0 \end{cases}$$

   (a) Find the domain of f.
   (b) Study the continuity of f
   (c) Find the range of f

4.5 Derivatives

Definition 4.5.1 *(Derivative)*
Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and a one point in the interior of D_f. We say that the derivative of f in a exists if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\overline{\mathbb{R}}$), and we designate this limit by $f'(a)$ or $\dfrac{df}{dx}(a)$.
We may also consider $x - a = h$, and write $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

If this limit is finite then we say that f is differentiable in a.If f is differentiable in every point of the domain we say that f is differentiable.

The racio $\dfrac{f(x) - f(a)}{x - a}$ is the slope of the secant. A secant line is a straight line joining two points on a function, in this case $(a, f(a))$ and $(x, f(x))$.

When Derivative of $f(x) = c$

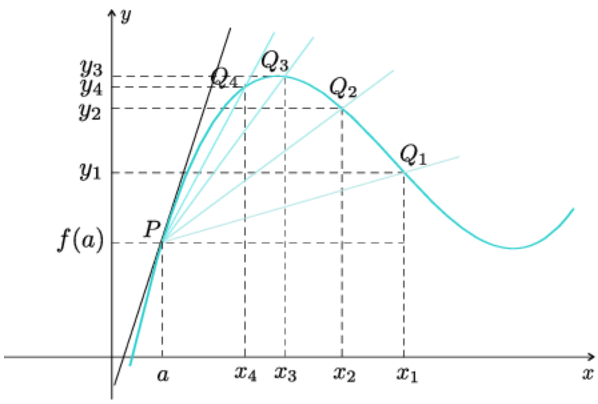


Figure 4.18: Geometric interpretation of the derivative .

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{0}{x - a} = 0$$

Derivative of $f(x) = x$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

Derivative of $f(x) = \sin(x)$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{2\,\sin\left(\frac{h}{2}\right)\,\cos\left(x + \frac{h}{2}\right)}{h} = \cos(x)$$

So

$$(\sin)'(x) = \cos(x), \quad \forall x \in \mathbb{R}.$$

Derivative of $f(x) = e^x$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x.$$

So

$$(e^x)' = e^x$$

Definition 4.5.2 *(Left and right Derivative)*
Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and a one point in the interior of D_f. The left and right derivative of f in a are given by,

$$f'_d(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

$$f'_e(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

If $f'_d(a) = f'_e(a)$ then $f'(a) = f'_d(a) = f'_e(a)$

Theorem 4.5.3 *If f is differentiable in a, then f is continuous in a.*

Proof:

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(x) - f(a) + f(a) = \lim_{x \to a} (x - a) \frac{f(x) - f(a)}{x - a} + f(a) =$$

$$= \lim_{x \to a} (x - a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} f(a) = 0.f'(a) + f(a) = f(a)$$

Theorem 4.5.4 *If f and g are differentiable in a, then $f + g$ and $f \cdot g$ are also differentiable in a, and*

$$(f + g)'(a) = f'(a) + g'(a)$$

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a).$$

If $g(a) \neq 0$, then f/g is differentiable in a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{(g(a))^2}.$$

Proof: If $f'(a)$ and $g'(a)$ are finite, then for the sum we have:

$$\begin{aligned}
(f+g)'(a) &= \lim_{x\to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)+g(x)-f(a)-g(a)}{x-a} \\
&= \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \right) \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a)+g'(a)
\end{aligned}$$

proving that $f+g$ is differentiable in a.

For the product

$$\begin{aligned}
(f\cdot g)'(a) &= \lim_{x\to a} \frac{(f\cdot g)(x)-(f\cdot g)(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)\cdot g(x)-f(a)\cdot g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)\cdot g(x)-f(a)\cdot g(x)+f(a)\cdot g(x)-f(a)\cdot g(a)}{x-a} \\
&= \lim_{x\to a} \frac{(f(x)-f(a))\cdot g(x)+f(a)\cdot (g(x)-g(a))}{x-a} \\
&= \lim_{x\to a} \left(g(x)\cdot \frac{f(x)-f(a)}{x-a} + f(a)\cdot \frac{g(x)-g(a)}{x-a} \right) \\
&= \lim_{x\to a} g(x)\cdot \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + f(a)\cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= g(a)\cdot f'(a)+f(a)\cdot g'(a)
\end{aligned}$$

we used the fact that differentiability in g at a implies continuity of g in a.

For the ratio we start by considering the special case of $f(x)=1$;

$$\left(\frac{1}{g}\right)'(a) = \lim_{x\to a} \frac{\left(\frac{1}{g}\right)(x)-\left(\frac{1}{g}\right)(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{g(x)}-\frac{1}{g(a)}}{x-a}$$

$$\begin{aligned}
&= \lim_{x\to a} \frac{\frac{g(a)-g(x)}{g(x)\cdot g(a)}}{x-a} = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}\cdot\left(-\frac{1}{g(x)\cdot g(a)}\right) \\
&= -\frac{1}{g(a)}\cdot \lim_{x\to a} \frac{1}{g(x)} \cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = -\frac{1}{g(a)}\cdot\frac{1}{g(a)}\cdot g'(a) \\
&= -\frac{g'(a)}{(g(a))^2}.
\end{aligned}$$

Now noting that $\frac{f}{g} = f\cdot\frac{1}{g}$, we have:

$$\left(\frac{f}{g}\right)'(a) = f'(a)\cdot\left(\frac{1}{g}\right)(a) + f(a)\cdot\left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{(g(a))^2}. \quad \blacksquare$$

Theorem 4.5.5 *Chain rule: If $g : E \to \mathbb{R}$ is differentiable in a and $f : D \to \mathbb{R}$ is differentiable in $b = g(a)$, then $f \circ g$ is differentiable in a and*

$$(f \circ g)'(a) = f'(b)\cdot g'(a) = f'(g(a))\cdot g'(a).$$

Theorem 4.5.6 *Basic derivative rules*

f	f'
c	0
cg	cg'
x	1
x^p	px^{p-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
e^x	e^x
$\log(x)$	$\frac{1}{x}$

4.6 Exercises

1. For $f(x) = \sin(x^2+2)$ find $f'(x)$.

2. For $f(x) = \tan(e^x)$ find $f'(x)$.

3. For $f(x) = \log(\sqrt{x+2}+x)$ find $f'(x)$.

4. For $f(x) = x\log(x^2+1)$ find $f'(x)$.

5. For $f(x) = \sqrt[3]{x+2}$ find $f'(x)$.

6. For $f(x) = x^2\sqrt[4]{2x+6}$ find $f'(x)$.

7. For $f(x) = \frac{x+1}{e^x+x}$ find $f'(x)$.

7. For $f(x) = e^{\frac{1}{x}+2x}$ find $f'(x)$.

8. Study the continuity and differentiability of f, defined by

$$f(x) = \begin{cases} (x-1)^3, & \text{se } x > 0 \\ \frac{1}{4}x^2 - 1, & \text{se } x \leq 0. \end{cases}$$

9. Study the continuity and differentiability of f, defined by

$$f(x) = \begin{cases} -\pi - 1 - x, & \text{se } x \leq -\pi \\ \cos(x), & \text{se } -\pi < x < 0 \\ \dfrac{1}{1+x^2}, & \text{se } x \geq 0. \end{cases}$$

10. Study the continuity and differentiability of f, defined by

$$f(x) = 2\, e^{x^2 - 4x}.$$

11. Study the continuity and differentiability of f, defined by $g(x) = \dfrac{x^2 e^{4x}}{4}$.

12. Study the continuity and differentiability of f, defined by

$$f(x) = \begin{cases} e^{x^2+1}, & \text{se } x \leq 0 \\ x\cos(x), & \text{se } x > 0. \end{cases}$$