Mathematics

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Functions

Definition

(Function)

Let A e B be two sets. A function f is a rule that assigns to each element x in A exactly one element, $y = f(x)$, in B.

The variable x is the independent variable and y is the dependent variable.

Definition

(Domain and Range)

Given a real function f of real variable, the domain of f is the set of values in $\mathbb{R}$ such that $f(x)$ can be algebraically calculated. The range is the set of values $y = f(x)$ for every which x in the Domain of f.

Definition

(Properties)

A function f from A to B is

- Injective if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
- Surjective if $\forall y \in B, \exists x \in A : y = f(x)$.
- Bijective if it is injective and surjective.
- Even if $f(x) = f(-x)$.
- Odd if $f(x) = -f(-x)$.
- Increasing if $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$.
- Strictly increasing if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.
- Decreasing if $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$.
- Strictly decreasing if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

Definition

(Composition of functions)

Composition of function is a sequence of nested functions, where the input of one function is the output of the previous function. For the composition of two functions we say f after g and write $f \circ g$ and the expression is given by $f \circ g(x) = f(g(x))$.

The domain of $f \circ g$ is given by

$$D_{f \circ g} = \{x \in \mathbb{R} : x \in D_g \wedge y = g(x) \in D_f\}$$

Definition

(Roots, maximum and minimum)

We say that x_0 is a root or a zero of f if $f(x) = 0$.
$(f(x_1))$ is a relative or local minimum of f if

Definition

(Inverse function)

We say that f and g are inverse functions if $f \circ g = g \circ f = I$ where I is the identity function $I(x) = x$.

Definition

(Algebric operations on functions)

- $(f+g)(x) = f(x) + g(x)$ and
$$D_{f+g} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(fg)(x) = f(x).g(x)$ and
$$D_{fg} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(f-g)(x) = f(x) - g(x)$ and
$$D_{f-g} = \{x \in \mathbb{R} : x \in D_g \wedge x \in D_f\}$$
- $(f/g)(x) = f(x)/g(x)$ and
$$D_{f/g} = \{x \in \mathbb{R} : x \in D_g \wedge g(x) \neq 0 \wedge x \in D_f\}$$

Definition
(Stepwise function)
We say that f is a stepwise function if

$$f = \begin{cases} g_1(x) & \text{if } x \in A_1 \\ g_2(x) & \text{if } x \in A_2 \\ \dots & \\ g_k(x) & \text{if } x \in A_k \end{cases}$$

where $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$ and

$$D_f = \{x \in \mathbb{R} : (x \in A_1 \wedge x \in D_{g_1}) \vee \cdots \vee (x \in A_k \wedge x \in D_{g_k})\}$$

Exercises

1. Find the domain of the following functions

a) $f = \begin{cases} \sqrt{x-1} & \text{if } x \geq 0 \\ \dfrac{x-1}{x+2} & \text{if } x < 0 \end{cases}$

b) $g = \begin{cases} x^2 + 2 & \text{if } x \geq 4 \\ \dfrac{1}{x^2 - 4} & \text{if } x < 4 \end{cases}$

c) $h = \begin{cases} \dfrac{1}{2x^2 - 8x + 6} & \text{if } x \leq 1 \\ \dfrac{1}{x^2} & \text{if } x > 1 \end{cases}$

2. For $f(x) = \frac{1}{x^2}$ and $r(x) = 2x - 1$ write $f \circ r$ and $r \circ f$.

3. For $g(x) = x^3 + 3$ and $h(x) = x + 2$ write $g \circ h$ and $h \circ g$.
4. Find the inverse function of $f(x) = 3x - 7$.
5. For $g(x) = x + 1$ and $s(x) = x^3$ write $g \circ s$. Define the inverse of g, the inverse of s and the inverse of $g \circ s$ and relate $(g \circ s)^{-1}$ with g^{-1} and s^{-1} .
6. Check if the function $f(x) = (x-1)^3+2$ have inverse and in case of a positive answer find the expression of f^{-1}.
7. Check if the function $h(x) = x^2 - 6$ have inverse and in case of a positive answer find the expression of h^{-1}.

Exponential and logarithmic functions

The exponential function is given by the expression

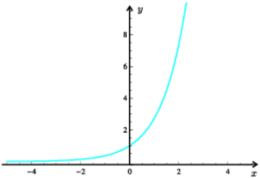
$$f(x) = a^x$$

with $a > 0$. The domain is $\mathbb{R}$, the range is $\mathbb{R}^+$.

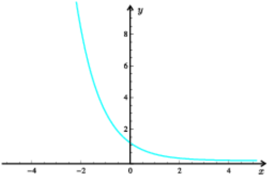
- $a^x \cdot a^y = a^{x+y}, \forall a \in \mathbb{R}^+, \ \forall x, y \in \mathbb{R}$;
- $\dfrac{a^x}{a^y} = a^{x-y}, \forall a \in \mathbb{R}^+, \ \forall x, y \in \mathbb{R}$;
- $(a^x)^y = a^{x \cdot y}, \forall a \in \mathbb{R}^+, \ \forall x, y \in \mathbb{R}$;
- $a^x \cdot b^x = (a \cdot b)^x, \forall a, b \in \mathbb{R}^+, \ \forall x \in \mathbb{R}$;
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x, \forall a, b \in \mathbb{R}^+, \ \forall x \in \mathbb{R}$.

The function has no zeros. It is strictly increasing for $a > 1$ and strictly decreasing for $0 < a < 1$.

Let us remind some properties of exponentials.



(a) The function a^x, $a > 1$.



(b) The function a^x, $0 < a < 1$.

Figure 1: Graphical representation of exponential function.

Among exponential functions it is relevant, for its practical applications, the function of base $a = e$ where e is the number of Neper. In general we refer to e^x simply as the exponential function.

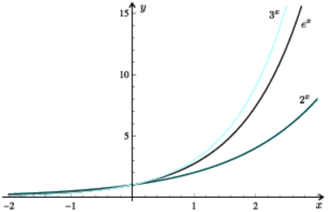


Figure 2: Function a^x, with $a = 2$, $a = e$ e $a = 3$.

The inverse of the exponential function is the logarithmic function. In fact we have

$$,x = a^y \Leftrightarrow \log_a x = y$$

and

$$\log_a a^y = y \text{ and } a^{\log_a x} = x. \tag{1}$$

From the definition of logarithm we have the following properties

$$\log_a a = 1 \text{ (because } a^1 = a)$$

e

$$\log_a 1 = 0 \text{ (because } a^0 = 1).$$

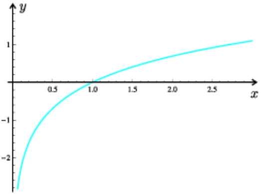
We have also the following properties, in which we consider $a, b \in \mathbb{R}^+ \setminus \{1\}$, $x, y \in \mathbb{R}^+$, $z \in \mathbb{R}$ e $n \in \mathbb{N}$:

- $\log_a(x.y) = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $log_a\left(x^z\right) = z \cdot \log_a x$
- $log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
- $\log_b x = \frac{\log_a x}{\log_a b}$

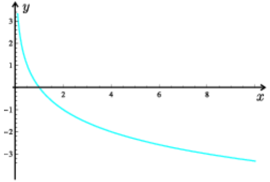
Note: In the case of $a = e$ (Neper number) we will adopt the notation $\log_e x = \log x$.

Properties

- Domain is $\mathbb{R}^+$ and range is $\mathbb{R}$;
- The function has only one root at $x = 1$.
- The function is surjective ans injective, so is bijective;



(a) The function $\log_a x$, $a > 1$.



(b) The function $\log_a x$, $0 < a < 1$.

Figure 3: Graphical representation of $log_a(x)$.

Exercises

1. Solve the equations:
   a) $2^x = 128$;
   b) $10^x = 100$;
   c) $15^x = 225$;
   d) $4^{2x+1} = \dfrac{1}{64}$;
   e) $3^{2-x} = 81$;
   f) $5^{3x+2} = \dfrac{1}{125}$.
2. Simplify the expressions:
   a) $\log_3 9 + \log_3 36 - \log_3 4$;

b) $\dfrac{\log_5 \frac{1}{8}}{\log_5 2}$;

c) $\log_{10}(x+3) - 4\log_{10} x$;

d) $4\log x - 6\log(x+2)$;

e) $\log_b y^3 + \log_b y^2 - \log_b y^4$;

f) $\dfrac{1}{2}\log_{1/3} x^2 + 5\log_{1/3} x$;

g) $\log_{1/5} 16 + \log_{1/5} 20 - \log_{1/5} 4^3$;

h) $\dfrac{1}{3}(3\log_2 x - \log_2 \frac{1}{y^3} + 6\log_2 z)$.

3. Solve the equations

a) $\log_b 256 = 4$;

b) $\log_5 125 = x$;

c) $\log_b 8 = \dfrac{3}{2}$;

d) $\log_9(2x+1) = \dfrac{1}{2}$;

e) $\log_{1/2}(2-x) = 3$;

f) $\log(3x+2) = \log 7$.

4. Simplify the expressions:

a) $e^{3+\log x}$;

b) $16^{\log_2 x + 3\log_4 \sqrt{x}}$;

c) $\left(\dfrac{1}{3}\right)^{\log_3(x^2+4) - 2\log_3 x}$;

d) $25^{3\log_5 x}$;

e) $7^{5\log_7 x - 2\log_7 x}$;

f) $16^{\log_2 x}$;

5. Solve the equations :

a) $3^{x-5} = 4$;

b) $\log_{10} x + \log_{10}(x - 15) = 2$;

c) $\frac{1}{16} = 64^{4x-3}$;

d) $\log_3 \sqrt{2x + 3} = 2$;

e) $4(\log x)^2 - 3\log x = 7$;

f) $\frac{1}{3}\log(x^{\log(x^3)}) - \log(x^5) + 4 = 0$.

6. Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (0.2)^{x-1}$. Identify the true and false propositions:

   a) $f(x) < 1, \forall x \in \mathbb{R}$;

   b) $f(0) = 3$;

   c) $f(x) > 25, \forall x \in \mathbb{R}^-$;

   d) $\forall x \geq 1, 0 < f(x) \leq 1$;

   e) $f(x) < 0, \forall x \in \mathbb{R}$;

   f) $\lim_{x \to +\infty} f(x) = -\infty$.

7. Find the set os solution of the inequalities

   a) $\left(\frac{1}{3}\right)^{x-1} < 9^{2-x}$;

b) $(0.1)^{x-x^2} \leq 0.01$;

c) $\log_{\frac{1}{2}}(x+5) > 0$.

8. Solve the inequalities:

   a) $x^3 \log_2(2x) + x^3 \log_{\frac{1}{2}}(x+5) < 0$;

   b) $\log_{\frac{1}{3}}(2x) < 2 - \log_{\frac{1}{3}}\left(\frac{1-x}{x}\right)$;

   c) $3^{\frac{x^2-4}{x^2+5}} < 1$.

9. Consider the function $g(x) = 5 + \log_{\frac{1}{2}}(3x-1)$.

   a) Find $\mathcal{D}_g$ e $\mathcal{CD}_g$.

   b) Solve $g(x) > 0$.

   c) Find the zeros of g.

d) Study the injectivity of g.

e) Find the inverse of g, g^{-1}.

10. Find the domain of function f defined by $f(x) = \dfrac{5(x-2)^3}{e^{3(x-2)}-1}$.

Trignometric functions

Background

The unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system. The unit circle, with radius 1 has a circumference of length 2π. In other words, for one revolution around the unit circle the length of the arc is 2π units.

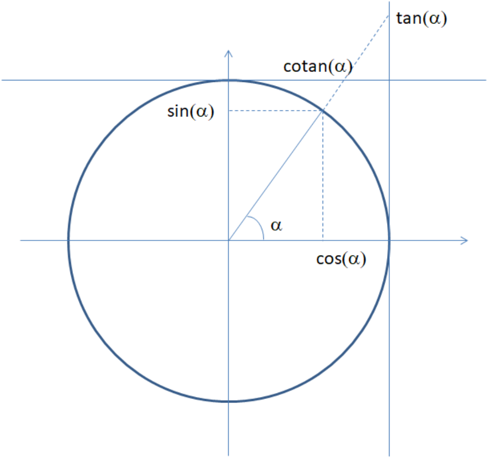


Figure 4: Trigonometric circle.

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos^2(\alpha) = \dfrac{1}{1+\tan^2(\alpha)}$

Function sine

$$\begin{aligned} f:\mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow \sin(x) \end{aligned}$$

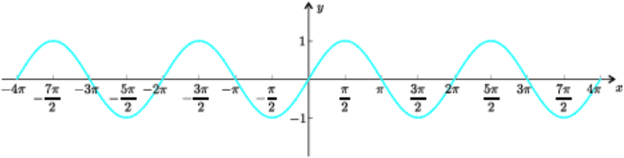


Figure 5: Graphical representation of $\sin(x)$.

- $\mathcal{D}_f = \mathbb{R}$ and $\mathcal{CD}_f = [-1, 1]$.
- It is a periodic function with period 2π,there is,

$$\text{sen}\,(2\pi + x) = \text{sen}\,(x), \quad \forall x \in \mathbb{R}$$

.

- It is odd

$$\sin\,(-x) = -\sin\,(x), \quad \forall x \in \mathbb{R}.$$

- It is increasing in intervals

$$\left]-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi\right[,\ k\in\mathbb{Z},$$

e and decreasing in intervals

$$\left]\frac{\pi}{2}+2k\pi,\frac{3\pi}{2}+2k\pi\right[,\ k\in\mathbb{Z}.$$

- Has a maximum at $x=\dfrac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$, and minimum at $x=-\dfrac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$. The maximun is 1 and the minimum -1.
- The zeros are of the form $x=k\pi,\ k\in\mathbb{Z}$.

Function cosine

$$\begin{aligned} g : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow \cos(x) \end{aligned}$$

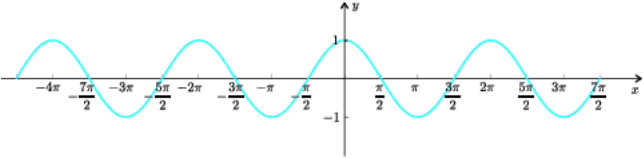


Figure 6: Graphical representation of $\cos(x)$.

- $\mathcal{D}_g = \mathbb{R}$ e $\mathcal{CD}_g = [-1, 1]$.
- It is periodic with period 2π, that is,

$$\cos(2\pi + x) = \cos(x), \quad \forall x \in \mathbb{R}.$$

- Is even,

$$\cos(-x) = \cos(x), \quad \forall x \in \mathbb{R}.$$

- Is increasing in the intervals

$$]\pi + 2k\pi, 2\pi + 2k\pi[\,, \; k \in \mathbb{Z},$$

  e and decreasing at

$$]2k\pi, \pi + 2k\pi[\,, \; k \in \mathbb{Z}.$$

- Reaches a maximum at $x = 2k\pi,\ k \in \mathbb{Z}$, and the minimum at $x = \pi + 2k\pi,\ k \in \mathbb{Z}$.The maximun is 1 and the minimum is -1.
- The zeros are the points $x = \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

Function tangent

$$
\begin{aligned}
t:\mathbb{R} &\rightarrow \mathbb{R} \\
x &\rightarrow \tan(x) = \frac{\sin(x)}{\cos(x)}
\end{aligned}
$$

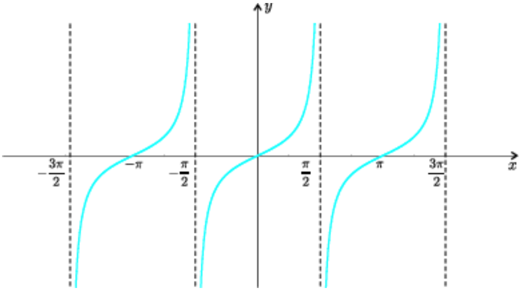


Figure 7: Graphical representation of tangent.

- $\mathcal{D}_h = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$ e $\mathcal{CD}_h = \mathbb{R}$.
- Is periodic with period π, that is,

$$\operatorname{tg}(\pi + x) = \operatorname{tg}(x), \quad \forall x \in \mathcal{D}_h.$$

- Is odd

$$\operatorname{tg}(-x) = -\operatorname{tg}(x), \quad \forall x \in \mathcal{D}_h.$$

- Is increasing in

$$\left]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right[, \ k \in \mathbb{Z}.$$

- Has no maximun or minimum.
- The zeros are $x = k\pi,\ k \in \mathbb{Z}$.

Function cotangent

$$\begin{aligned} t : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow \cot(x) = \frac{\cos(x)}{\sin(x)} \end{aligned}$$

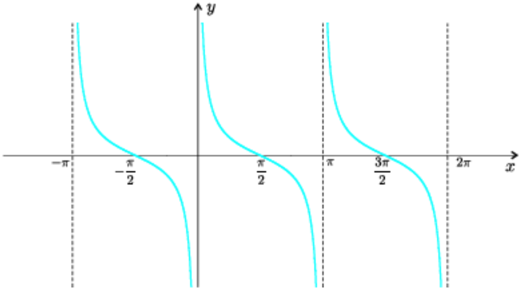


Figure 8: The graphical representation of $\cot$.

- $\mathcal{D}_i = \mathbb{R} \setminus \{k\pi,\, k \in \mathbb{Z}\}$ e $\mathcal{CD}_i = \mathbb{R}$.
- Is periodic with period π, that is,

$$\operatorname{cotg}(\pi + x) = \operatorname{cotg}(x), \quad \forall x \in \mathcal{D}_i.$$

- Is odd

$$\operatorname{cotg}(-x) = -\operatorname{cotg}(x), \quad \forall x \in \mathcal{D}_i.$$

- Is increasing in intervals

$$]k\pi, \pi + k\pi[\,,\ k \in \mathbb{Z}.$$

- Has no minimum or maximum.
- The zeros are $x = \dfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

$f(x)$	Domain	Range	Zeros	Odd/Even
$\operatorname{sen}(x)$	$\mathbb{R}$	$[-1, 1]$	$x = k\pi$	odd
$\cos(x)$	$\mathbb{R}$	$[-1, 1]$	$x = \frac{\pi}{2} + k\pi$	even
$\operatorname{tg}(x)$	$\left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi\right\}$	$\mathbb{R}$	$x = k\pi$	odd
$\operatorname{cotg}(x)$	$\{x \in \mathbb{R} : x \neq k\pi\}$	$\mathbb{R}$	$x = \frac{\pi}{2} + k\pi$	odd

Inverse Trignometric Functions

Clearly the previous functions are not invertible but we can consider a restriction (sub-domain) in which they are injective and so invertible. We will choose a sub-domain where the we have full range (the range is the same of the original function).

Let $f(x) = \operatorname{sen}(x)$ and consider the **main restriction of the function** the restriction of f to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let it be g then

$$\begin{aligned} g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] &\to [-1, 1] \\ x &\to \operatorname{sen}(x). \end{aligned}$$

Being bijective g invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = [-1, 1], \quad \mathcal{CD}_{g^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and to each $y \in [-1, 1]$ we have $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which $\sin$ is y; such angle is represented by $\arcsin(y)$ and the function g^{-1} is the arc-sin function. Then for every $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$

$$y = \sin(x) \Leftrightarrow x = \arcsin(y).$$

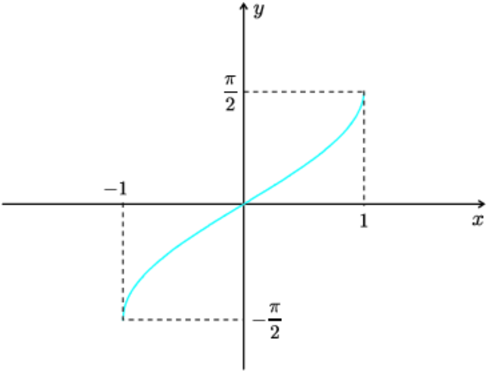


Figure 9: Graphical representation of arcsin.

Let $f(x) = \cos(x)$ and consider the **main restriction of**

the function the restriction of f to the interval $[0, \pi]$. Let it be g then

$$\begin{aligned} g : [0, \pi] &\rightarrow [-1, 1] \\ x &\rightarrow \cos(x). \end{aligned}$$

Being bijective g is invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = [-1, 1], \;\; \mathcal{CD}_{g^{-1}} = [0, \pi]$$

and to each $y \in [-1, 1]$ we have $x \in [0, \pi]$ which $\cos$ is y; such angle is represented by $\arccos(y)$ and the function g^{-1} is the arc-cos function. Then for every $x \in [0, \pi]$ and $y \in [-1, 1]$

$$y = \cos(x) \Leftrightarrow x = \arccos(y).$$

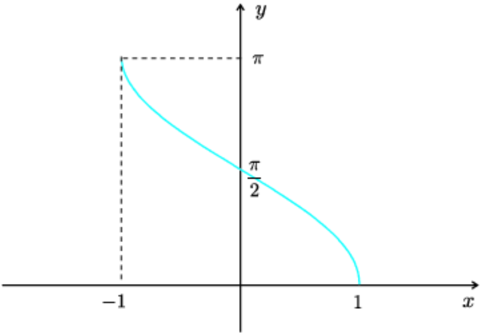


Figure 10: Graphical representation of arccos.

Let $f(x) = \tan(x)$ and consider the **main restriction of the function** the restriction of f to the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Let it be g than

$$g : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\;\to\; \mathbb{R}$$
$$x \;\to\; \tan(x).$$

Being bijective g invertible and let g^{-1} be the inverse. For g^{-1}, we have

$$\mathcal{D}_{g^{-1}} = \mathbb{R}, \quad \mathcal{CD}_{g^{-1}} = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

and to each $y \in \mathbb{R}$ we have $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ which $\tan$ is y; such angle is represented by $\arctan(y)$ and the function

g^{-1} is the arc-tan function. Then for every $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ e $y \in \mathbb{R}$

$$y = \tan(x) \Leftrightarrow x = \arctan(y).$$

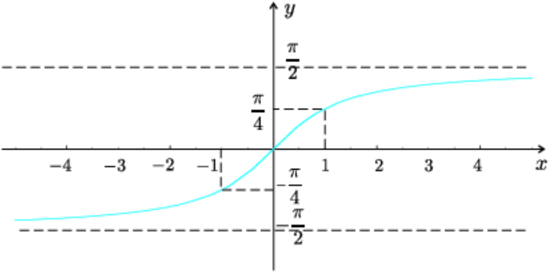


Figure 11: Graphical representation of arctan.

Limits and continuity

Definition

(Limit of function)

We say that

$$\lim_{x\to a} f(x) = b$$

iff

$$\forall\delta > 0, \exists\epsilon > 0 : x \in D_f \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

Definition
(Relative limits)

$$\lim_{x \to a^+} f(x) = b \Leftrightarrow$$

$$\forall \delta > 0, \exists \epsilon > 0 : x \in D_f \wedge x > a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

$$\lim_{x \to a^-} f(x) = b \Leftrightarrow$$

$$\forall \delta > 0, \exists \epsilon > 0 : x \in D_f \wedge x < a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

$$\lim_{\substack{x \to a \\ x \neq a}} f(x) = b \Leftrightarrow$$

$$\forall \delta > 0, \exists \epsilon > 0 : x \in D_f \wedge x \neq a \wedge |x-a| < \epsilon \Rightarrow |f(x)-b| < \delta$$

Definition
(Continuity)
We say that f is continuous in $a \in D_f$ if $\lim_{x \longrightarrow a} f(x)$ exists.
(By the definition of limit we must have $\lim_{x \longrightarrow a} f(x) = f(a)$) We say that f is a continuous function if it is continuous on every point of the domain.

All the elementary functions studied before are continuous.

Some special limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \tag{2}$$

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0 \tag{3}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \tag{4}$$

$$\lim_{x \to +\infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \begin{cases} +\infty & \text{if } n > m \\ \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases} \tag{5}$$