Research Article

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Fredholm discrete Wiener–Hopf operators on reflexive rearrangement-invariant Banach sequence spaces are one-sided invertible

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Abstract: Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space and let a be a periodic distribution generating the bounded Laurent operator $L(a)$ on $X(\mathbb{Z})$. We prove that if the discrete Wiener–Hopf operator $T(a)$ is Fredholm on the subspace $X(\mathbb{Z}_+) = \{f = \{f_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}) : f_k = 0 \text{ for } k < 0\}$ and has index κ, then $T(a)$ is left-invertible if $\kappa \le 0$, right-invertible if $\kappa \ge 0$, and invertible if $\kappa = 0$. The proof is based on ideas by Hartman–Wintner (1954), Coburn (1966), Simonenko (1968), and Duduchava (1975), who proved the analogous result for $\ell^2(\mathbb{Z})$ and $\ell^p(\mathbb{Z})$ with $1 < p < \infty$.

Keywords: Discrete Wiener–Hopf operator, Laurent operator, rearrangement-invariant Banach sequence spaces, Fredholmness, one-sided invertibility.

MSC 2020: ???

Dedicated to Professor Roland Duduchava on the occasion of his 80th birthday.

1 Introduction

For a Banach space $\mathcal{X}$, let $\mathcal{B}(\mathcal{X})$ and $\mathcal{K}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$ and its closed two-sided ideal consisting of all compact linear operators on $\mathcal{X}$, respectively. An operator $A \in \mathcal{B}(\mathcal{X})$ is said to be Fredholm on $\mathcal{X}$ if $\alpha(A) := \dim \operatorname{Ker} A < \infty$ and $\beta(A) := \dim \mathcal{X}/\operatorname{Im} A < \infty$, where $\operatorname{Ker} A := \{x \in \mathcal{X} : Ax = 0\}$ and $\operatorname{Im} A := A(\mathcal{X})$ are the kernel and the range of A, respectively. The set of all Fredholm operators on $\mathcal{X}$ will be denoted by $\Phi(\mathcal{X})$. For each $A \in \Phi(\mathcal{X})$, the index of A is defined by

$$\operatorname{Ind} A := \alpha(A) - \beta(A).$$

For a unital Banach algebra $\mathcal{A}$, let $\mathcal{GA}$ denote the group of all invertible elements of $\mathcal{A}$. The spectrum and the essential spectrum of an operator $A \in \mathcal{B}(\mathcal{X})$ are defined by

$$\operatorname{Spec} A := \big\{\lambda \in \mathbb{C} \ : \ A - \lambda I \notin \mathcal{GB}(\mathcal{X})\big\}, \quad \operatorname{Spec}_{\mathrm{e}} A := \big\{\lambda \in \mathbb{C} \ : \ A - \lambda I \notin \Phi(\mathcal{X})\big\},$$

respectively. It is clear that $\operatorname{Spec}_{\mathrm{e}} A \subset \operatorname{Spec} A$.

Let $\ell^0(\mathbb{Z})$ denote the linear space of all sequences $f = \{f_k\}_{k\in\mathbb{Z}}$ with $f_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. Let $X(\mathbb{Z}) \subset \ell^0(\mathbb{Z})$ be a separable Banach sequence space (see [3, Ch. 1] and Section 2 below). The class of Banach sequence spaces includes classical Lebesgue sequence spaces $\ell^p(\mathbb{Z})$ with $1 \le p \le \infty$, Orlicz sequence spaces $\ell^\Phi(\mathbb{Z})$, and Lorentz sequence spaces $\ell^{p,q}(\mathbb{Z})$ with $1 < p < \infty$, $1 \le q \le \infty$, among others, as well as their weighted analogues.

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Let $\mathcal{P}'$ be the space of periodic distributions (see, e.g., [2, Ch. 3 and 5]) and let $S_0(\mathbb{Z})$ denote the set of all finitely supported sequences. For $a \in \mathcal{P}'$ and $\varphi \in S_0(\mathbb{Z})$, we define the convolution $a * \varphi$ as the sequence

$$(a * \varphi)_j = \sum_{k \in \mathbb{Z}} \widehat{a}_{j-k} \varphi_k, \quad j \in \mathbb{Z},$$

where $\{\widehat{a}_j\}_{j \in \mathbb{Z}}$ is the sequence of Fourier coefficients of the distribution a. By $M_{X(\mathbb{Z})}$ we denote the collection of all distributions $a \in \mathcal{P}'$ for which $a * \varphi \in X(\mathbb{Z})$ whenever $\varphi \in S_0(\mathbb{Z})$ and

$$\|a\|_{M_{X(\mathbb{Z})}} := \sup \left\{ \frac{\|a * \varphi\|_{X(\mathbb{Z})}}{\|\varphi\|_{X(\mathbb{Z})}} \; : \; \varphi \in S_0(\mathbb{Z}), \; \varphi \neq 0 \right\} < \infty.$$

Since $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z})$ whenever $X(\mathbb{Z})$ is separable (see [13, Lemma 3]), for $a \in M_{X(\mathbb{Z})}$, the operator from $S_0(\mathbb{Z})$ to $X(\mathbb{Z})$ defined by $\varphi \mapsto a * \varphi$ extends to a bounded operator

$$L(a) : X(\mathbb{Z}) \to X(\mathbb{Z}), \quad \varphi \mapsto a * \varphi,$$

which is referred to as the Laurent operator with symbol a.

It is known that $M_{\ell^1(\mathbb{Z})}$ coincides with the Wiener algebra W and $M_{\ell^2(\mathbb{Z})}$ is $L^\infty(-\pi, \pi)$, the space of all 2π-periodic essentially bounded functions. In all other cases, a reasonable description of $M_{X(\mathbb{Z})}$ is unknown. If $X(\mathbb{Z})$ is reflexive and reflection-invariant, then $M_{X(\mathbb{Z})}$ is continuously embedded into $L^\infty(-\pi, \pi)$ and is a Banach algebra with respect to pointwise operations and the norm $\|a\|_{M_{X(\mathbb{Z})}} = \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}$ (see [15, Theorems 5.5–5.6]). If $X(\mathbb{Z})$ is translation-invariant, then the Young convolution inequality (see [13, Theorem 6]) implies that $W \subset M_{X(\mathbb{Z})}$. Under some technical conditions (which we do not require in the main result of this paper), $M_{X(\mathbb{Z})}$ contains all 2π-periodic functions of bounded variation (see [15, Theorem 1.2(c)]).

Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and let P denote the discrete Riesz projection on $X(\mathbb{Z})$ defined for $\varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \in X(\mathbb{Z})$ by

$$(P\varphi)_j := \begin{cases} \varphi_j & \text{if } j \in \mathbb{Z}_+, \\ 0 & \text{if } j \in \mathbb{Z} \setminus \mathbb{Z}_+. \end{cases}$$

Consider the following subspaces of $X(\mathbb{Z})$:

$$S_0(\mathbb{Z}_+) := PS_0(\mathbb{Z}) = \{\varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \in S_0(\mathbb{Z}) : \varphi_j = 0 \text{ for } j \in \mathbb{Z} \setminus \mathbb{Z}_+\},$$
$$X(\mathbb{Z}_+) := PX(\mathbb{Z}) = \{\varphi = \{\varphi_j\}_{j \in \mathbb{Z}} \in X(\mathbb{Z}) : \varphi_j = 0 \text{ for } j \in \mathbb{Z} \setminus \mathbb{Z}_+\}.$$

The subspace $S_0(\mathbb{Z}_+)$ is dense in $X(\mathbb{Z}_+)$ if the space $X(\mathbb{Z})$ is separable in view of [13, Lemma 3]. Therefore, for every $a \in M_{X(\mathbb{Z})}$, the operator from $S_0(\mathbb{Z}_+)$ to $X(\mathbb{Z}_+)$ defined by $\varphi \mapsto P(a * \varphi)$, extends to a bounded operator

$$T(a) : X(\mathbb{Z}_+) \to X(\mathbb{Z}_+), \quad \varphi \mapsto P(a * \varphi),$$

which is referred to as the discrete Wiener–Hopf (or Toeplitz) operator with symbol a. Clearly, we have

$$\|T(a)\|_{\mathcal{B}(X(\mathbb{Z}_+))} \leq \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}.$$

Moreover, if $X(\mathbb{Z})$ is reflexive and rearrangement-invariant (see [3, Ch. 2] and Section 7 below), then for every $a \in M_{X(\mathbb{Z})}$, one has

$$\|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))} = \inf_{K \in \mathcal{K}(X(\mathbb{Z}))} \|L(a) + K\|_{\mathcal{B}(X(\mathbb{Z}))} = \|T(a)\|_{\mathcal{B}(X(\mathbb{Z}_+))} = \inf_{K \in \mathcal{K}(X(\mathbb{Z}_+))} \|T(a) + K\|_{\mathcal{B}(X(\mathbb{Z}_+))}$$

(see [14, Theorem 2.6]).

The Fredholm theory of discrete Wiener–Hopf operators on $\ell^p(\mathbb{Z}_+)$ with $1 \leq p \leq \infty$ is a classical topic of operator theory. For various classes of symbols, it is well documented in the monographs by Gohberg and Fel'dman [10], Prössdorf [18], Böttcher and Silbermann [6, 7]. Recently, the authors have begun to extend this theory to the setting of more general Banach sequence spaces (see [13–15]). The aim of this paper is to make one more step in this direction.

Our main result is the following theorem, which has its roots in the works by Hartman and Wintner [12], Coburn [8], Simonenko [21], and especially Duduchava [9], who was the first to consider discrete Wiener–Hopf operators with discontinuous symbols on $\ell^p(\mathbb{Z}_+)$ for $p \neq 2$. It is our pleasure to dedicate this paper to Professor Roland Duduchava on the occasion of his 80th birthday.

Theorem 1.1. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space and $a \in M_{X(\mathbb{Z})}$. Suppose the discrete Wiener–Hopf operator $T(a) : X(\mathbb{Z}_+) \to X(\mathbb{Z}_+)$ is Fredholm and has index κ. Then $T(a)$ is left-invertible if $\kappa \leq 0$, right-invertible if $\kappa \geq 0$, and invertible if $\kappa = 0$. Moreover,*

$$\operatorname{Spec} T(a) = \operatorname{Spec}_{\mathrm{e}} T(a) \cup \{\lambda \in \mathbb{C} \setminus \operatorname{Spec}_{\mathrm{e}} T(a) : \operatorname{Ind} T(a - \lambda) \neq 0\}. \tag{1.1}$$

The scheme of the proof is as follows:

$$T(a) \in \Phi(X(\mathbb{Z}_+)) \overset{(i)}{\Longrightarrow} L(a) \in \mathcal{GB}(X(\mathbb{Z})) \overset{(ii)}{\Longrightarrow} a \in \mathcal{G}M_{X(\mathbb{Z})} \overset{(iii)}{\Longrightarrow} \alpha(T(a)) = 0 \text{ or } \alpha(T(\overline{a})) = 0 \tag{1.2}$$

and

$$\begin{cases} T(a) \in \Phi(X(\mathbb{Z}_+)), \\ \alpha(T(a)) = 0 \text{ or } \alpha(T(\overline{a})) = 0 \end{cases} \overset{(iv)}{\Longrightarrow} \begin{cases} T(a) \text{ is left-invertible} & \text{if } \operatorname{Ind} T(a) \leq 0, \\ T(a) \text{ is right-invertible} & \text{if } \operatorname{Ind} T(a) \geq 0, \\ T(a) \text{ is invertible} & \text{if } \operatorname{Ind} T(a) = 0. \end{cases} \tag{1.3}$$

Hartman and Wintner [12, Part I] proved that the spectrum of the Toeplitz operator $\mathfrak{T}(a)$ with a real-valued 2π-periodic symbol $a \in L^\infty(-\pi, \pi)$ on the Hardy space H^2 is equal to the segment $[m, M]$, where $m := \operatorname{ess\,inf} a(t)$ and $M := \operatorname{ess\,sup} a(t)$. Simonenko's result [21, Lemma 2] can be restated as follows: if $a \in L^\infty(-\pi, \pi)$ and $\mathfrak{T}(a)$ is Fredholm on a Hardy space H^p with $1 < p < \infty$, then $a \in \mathcal{G}L^\infty(-\pi, \pi)$. Since the Toeplitz operator $\mathfrak{T}(a)$ on H^2 and the discrete Wiener–Hopf operator $T(a)$ on $\ell^2(\mathbb{Z}_+)$ are unitarily equivalent and $M_{\ell^2(\mathbb{Z})} = L^\infty(-\pi, \pi)$, one has $T(a) \in \Phi(\ell^2(\mathbb{Z}_+)) \Longrightarrow a \in \mathcal{G}L^\infty(-\pi, \pi)$, which gives the implications (i) and (ii) for $\ell^2(\mathbb{Z})$. Böttcher and Silbermann [7, Propositions 2.28(b) and 2.30(b)] proved the implications (i) and (ii) for $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ and coined this result as the "Hartman–Wintner theorem" (although the attribution "Hartman–Wintner–Simonenko theorem" used later by Böttcher and Karlovich for Toeplitz operators in [4, Theorem 6.20], is probably more correct).

Coburn observed in the proof of [8, Theorem 4.1] that if $a \in L^\infty(-\pi, \pi) \setminus \{0\}$, then for the Toeplitz operator $\mathfrak{T}(a); H^2 \to H^2$, one has $\alpha(\mathfrak{T}(a)) = 0$ of $\alpha(\mathfrak{T}(\overline{a})) = 0$. Simonenko [21, Theorem 5] proved this fact independently for H^p with $1 < p < \infty$ (in the equivalent language of the Riemann boundary value problem). Inspired by these results, Duduchava [9] proved that if $a \in M_{\ell^p(\mathbb{Z})} \setminus \{0\}$ with $1 < p < \infty$, then $\alpha(T(a)) = 0$ or $\alpha(T(\overline{a})) = 0$. Note that Duduchava's result is stronger than the implication (iii) for $\ell^p(\mathbb{Z})$. Its proof given in [7, Proposition 2.38(b)] essentially uses the fact that for $1 \leq p < \infty$, either $\ell^p(\mathbb{Z})$ or its dual $(\ell^p(\mathbb{Z}))^* = \ell^{p'}(\mathbb{Z})$, where $1/p + 1/p' = 1$, is contained in $\ell^2(\mathbb{Z})$. Note that this is not the case for more general Banach sequence spaces (for instance, weighted ℓ^p spaces). Böttcher and Seybold [5, Theorem 7.4] were able to prove the implication (iii) for the case of weighted ℓ^p spaces with symmetric Muckenhoupt weights.

Our paper is organised as follows. In Section 2, we recall the definition of a Banach sequence space and its associate space $X'(\mathbb{Z})$, as well as the fact that if $X(\mathbb{Z})$ is separable, then $X'(\mathbb{Z}) = (X(\mathbb{Z}))^*$. In Section 3, we prove that if $X(\mathbb{Z})$ is reflexive, then $(L(a))^* = L(\overline{a})$ on $X'(\mathbb{Z}) = (X(\mathbb{Z}))^*$ and $(T(a))^* = T(\overline{a})$ on $X'(\mathbb{Z}_+) = (X(\mathbb{Z}_+))^*$ with respect to the pairing $(f, g) = \sum_{k \in \mathbb{Z}} f_k \overline{g_k}$. Section 4 is a warmup before the proof of the implication (i). There we prove that if $X(\mathbb{Z})$ is reflexive and translation-invariant, then $L(a) \in \Phi(X(\mathbb{Z}))$ if and only if $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$. Section 5 is dedicated to the proof of the implication (i) in the setting of reflexive translation-invariant Banach sequence spaces. In Section 6, we prove the implication (ii) and its converse for reflexive reflection-invariant Banach sequence spaces contained in the set $S'(\mathbb{Z})$ of all sequences of slow growth. In Section 7, we prove that if $X(\mathbb{Z})$ is rearrangement-invariant and $w = \{w_k\}_{k \in \mathbb{Z}}$ is a symmetric Muckenhoupt weight in $A_p^{\mathrm{sym}}(\mathbb{Z})$ for some $p \in (1, \infty)$, then the weighted Banach sequence space $X(\mathbb{Z}, w) = \{f = \{f_k\}_{k \in \mathbb{Z}} \in \ell^0(\mathbb{Z}) : fw = \{f_k w_k\}_{k \in \mathbb{Z}} \in X(\mathbb{Z})\}$ is contained in $S'(\mathbb{Z})$. This allows us to reformulate the implication (ii) and its converse for $X(\mathbb{Z}, w)$. In Section 8, we extend [5, Lemmas 7.2–7.3] and give some conditions guaranteeing that a formally defined convolution $x * y$ of two sequences $x = \{x_k\}_{k\in\mathbb{Z}}$ and $y = \{y_k\}_{k\in\mathbb{Z}}$ is well defined and associative. These facts are used in Section 9, where we extend [5, Theorem 7.4] and prove the implication (iii) in the setting of a weighted Banach sequence space $X(\mathbb{Z}, w)$ built upon a reflexive rearrangement-invariant Banach sequence space $X(\mathbb{Z})$ and a symmetric weight w "controlled" by another weight v. All above gives the proof of the implications (i)–(iii) for reflexive rearrangement-invariant Banach sequence spaces. In Section 10, we provide the standard proof of implication (iv) for the sake of completeness.

2 Banach sequence spaces

Let $\ell^0_+(\mathbb{Z})$ be the cone of nonnegative sequences in $\ell^0(\mathbb{Z})$. According to [3, Ch. 1, Definition 1.1], a Banach function norm $\varrho : \ell^0_+(\mathbb{Z}) \to [0,\infty]$ is a mapping that satisfies the following axioms for all $f, g \in \ell^0_+(\mathbb{Z})$, for all sequences $\{f^{(n)}\}_{n\in\mathbb{N}}$ in $\ell^0_+(\mathbb{Z})$, for all finite subsets $E \subset \mathbb{Z}$, and all constants $\alpha \ge 0$:

$$
\begin{aligned}
&(\mathrm{A1}) &&\varrho(f) = 0 \Leftrightarrow f = 0,\ \varrho(\alpha f) = \alpha\varrho(f),\ \varrho(f+g) \le \varrho(f) + \varrho(g),\\
&(\mathrm{A2}) &&0 \le g \le f \quad \Rightarrow \quad \varrho(g) \le \varrho(f) \quad \text{(the lattice property)},\\
&(\mathrm{A3}) &&0 \le f^{(n)} \uparrow f \quad \Rightarrow \quad \varrho(f^{(n)}) \uparrow \varrho(f) \quad \text{(the Fatou property)},\\
&(\mathrm{A4}) &&\varrho(\mathbb{1}_E) < \infty,\\
&(\mathrm{A5}) &&\sum_{k\in E} f_k \le C_E\,\varrho(f),
\end{aligned}
$$

where $\mathbb{1}_E$ is the characteristic (indicator) function of E, and the constant $C_E \in (0,\infty)$ may depend on ϱ and E, but is independent of f. The set $X(\mathbb{Z})$ of all sequences $f \in \ell^0(\mathbb{Z})$ for which $\varrho(|f|) < \infty$ is called a Banach sequence space. For each $f \in X(\mathbb{Z})$, the norm of f is defined as

$$\|f\|_{X(\mathbb{Z})} := \varrho(|f|).$$

The set $X(\mathbb{Z})$ equipped with the natural linear space operations and this norm becomes a Banach space (see [3, Ch. 1, Theorems 1.4 and 1.6]). If ϱ is a Banach function norm, its associate norm ϱ' is defined on $\ell^0_+(\mathbb{Z})$ as

$$\varrho'(g) := \sup\left\{\sum_{k\in\mathbb{Z}} f_k g_k \ :\ f = \{f_k\}_{k\in\mathbb{Z}} \in \ell^0_+(\mathbb{Z}),\ \varrho(f) \le 1\right\}, \quad g \in \ell^0_+(\mathbb{Z}).$$

It is a Banach function norm itself [3, Ch. 1, Theorem 2.2]. The Banach sequence space $X'(\mathbb{Z})$ determined by the Banach function norm ϱ' is called the associate space (Köthe dual) of $X(\mathbb{Z})$. It follows from the Hölder inequality for Banach sequence spaces (see [3, Ch. 1, Theorem 2.4]) that $X'(\mathbb{Z})$ can be viewed as a subspace of the Banach dual space $(X(\mathbb{Z}))^*$.

If $X(\mathbb{Z})$ is separable, then $(X(\mathbb{Z}))^*$ and $X'(\mathbb{Z})$ are isometrically isomorphic (see [3, Ch. 1, Corollaries 4.3 and 5.6]). More precisely, for every $F \in (X(\mathbb{Z}))^*$ there is a unique $y = \{y_n\}_{n\in\mathbb{Z}} \in X'(\mathbb{Z})$ such that for all $x = \{x_n\}_{n\in\mathbb{Z}} \in X(\mathbb{Z})$,

$$F(x) = (x, y) := \sum_{n\in\mathbb{Z}} x_n \overline{y_n} \tag{2.1}$$

and $\|F\|_{(X(\mathbb{Z}))^*} = \|y\|_{X'(\mathbb{Z})}$.

3 Adjoint operators of Laurent and discrete Wiener–Hopf operators

In most of our results, we will assume that $X(\mathbb{Z})$ is reflexive, which is equivalent to separability of both $X(\mathbb{Z})$ and $X'(\mathbb{Z})$ (see [3, Ch. 1, Corollaries 4.4 and 5.6]). In that case, $(X(\mathbb{Z}))^*$ is isometrically isomorphic to $X'(\mathbb{Z})$ and $(X(\mathbb{Z}_+))^*$ is isometrically isomorphic to $X'(\mathbb{Z}_+)$ with respect to the pairing defined in (2.1) (see [14, Lemma 2.1]). Moreover, since both $X(\mathbb{Z})$ and $X'(\mathbb{Z})$ are separable, the set $S_0(\mathbb{Z})$ is dense in both $X(\mathbb{Z})$ and $X'(\mathbb{Z})$ and the set $S_0(\mathbb{Z}_+)$ is dense in both $X(\mathbb{Z}_+)$ and $X'(\mathbb{Z}_+)$ (see [13, Lemma 3]).

Recall that the complex conjugate $\overline{a}$ of a 2π-periodic distribution is defined as $\overline{a}(u) := \overline{a(\overline{u})}$ for $u \in \mathcal{P}$, where $\mathcal{P}$ is the set of all infinitely differentiable 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$.

The following standard fact will be repeatedly used in what follows.

Lemma 3.1. *Let $X(\mathbb{Z})$ be a reflexive Banach sequence space. If $a \in M_{X(\mathbb{Z})}$, then $\overline{a} \in M_{X'(\mathbb{Z})}$ and*

$$(L(a))^* = L(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z})), \quad (T(a))^* = T(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}_+)).$$

Proof. If $a \in M_{X(\mathbb{Z})}$, then $\overline{a} \in M_{X'(\mathbb{Z})}$ (see [15, Lemma 5.2]). Hence $L(a) \in \mathcal{B}(X(\mathbb{Z}))$, $L(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}))$ and $T(a) \in \mathcal{B}(X(\mathbb{Z}_+))$, $T(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}_+))$. It is easy to check that

$$(L(a)\varphi, \psi) = (\varphi, L(\overline{a})\psi), \quad \varphi, \psi \in S_0(\mathbb{Z}) \tag{3.1}$$

(see the proof of [15, Lemma 5.2]). Let $f \in X(\mathbb{Z})$ and $g \in X'(\mathbb{Z})$. In view of the density of $S_0(\mathbb{Z})$ in $X(\mathbb{Z})$ and in $X'(\mathbb{Z})$, there exist sequences $\varphi^{(n)}, \psi^{(n)} \in S_0(\mathbb{Z})$, $n \in \mathbb{N}$, such that

$$\|f - \varphi^{(n)}\|_{X(\mathbb{Z})} \to 0, \quad \|g - \psi^{(n)}\|_{X'(\mathbb{Z})} \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

It follows from (3.1) and Hölder's inequality for Banach sequence spaces (see [3, Ch. 1, Theorem 2.4]) that

$$\begin{aligned}
|(L(a)f, g) - (f, L(\overline{a})g)| \le{}& \big|(L(a)f, g) - \big(L(a)\varphi^{(n)}, g\big)\big| + \big|\big(L(a)\varphi^{(n)}, g\big) - \big(L(a)\varphi^{(n)}, \psi^{(n)}\big)\big| \\
&+ \big|\big(L(a)\varphi^{(n)}, \psi^{(n)}\big) - \big(\varphi^{(n)}, L(\overline{a})\psi^{(n)}\big)\big| \\
&+ \big|\big(\varphi^{(n)}, L(\overline{a})\psi^{(n)}\big) - \big(f, L(\overline{a})\psi^{(n)}\big)\big| + \big|\big(f, L(\overline{a})\psi^{(n)}\big) - (f, L(\overline{a})g)\big| \\
\le{}& \big\|L(a)\big(f - \varphi^{(n)}\big)\big\|_{X(\mathbb{Z})}\|g\|_{X'(\mathbb{Z})} + \big\|L(a)\varphi^{(n)}\big\|_{X(\mathbb{Z})}\big\|g - \psi^{(n)}\big\|_{X'(\mathbb{Z})} \\
&+ \big\|\varphi^{(n)} - f\big\|_{X(\mathbb{Z})}\big\|L(\overline{a})\psi^{(n)}\big\|_{X'(\mathbb{Z})} + \|f\|_{X(\mathbb{Z})}\big\|L(\overline{a})\big(\psi^{(n)} - g\big)\big\|_{X'(\mathbb{Z})} \\
\le{}& \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}\|g\|_{X'(\mathbb{Z})}\big\|f - \varphi^{(n)}\big\|_{X(\mathbb{Z})} \\
&+ \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}\left(\sup_{n\in\mathbb{N}}\big\|\varphi^{(n)}\big\|_{X(\mathbb{Z})}\right)\big\|g - \psi^{(n)}\big\|_{X'(\mathbb{Z})} \\
&+ \|L(\overline{a})\|_{\mathcal{B}(X'(\mathbb{Z}))}\left(\sup_{n\in\mathbb{N}}\big\|\psi^{(n)}\big\|_{X'(\mathbb{Z})}\right)\big\|\varphi^{(n)} - f\big\|_{X(\mathbb{Z})} \\
&+ \|L(\overline{a})\|_{\mathcal{B}(X'(\mathbb{Z}))}\|f\|_{X(\mathbb{Z})}\big\|\psi^{(n)} - g\big\|_{X'(\mathbb{Z})}.
\end{aligned}$$

The above inequality and (3.2) imply that

$$(L(a)f, g) = (f, L(\overline{a})g), \quad f \in X(\mathbb{Z}), \quad g \in X'(\mathbb{Z}).$$

This means that $(L(a))^* = L(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}))$.

The proof of the equality $(T(a))^* = T(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}))$ is analogous. Similarly to (3.1) established in the proof of [15, Lemma 5.2], one can prove that

$$(T(a)\varphi, \psi) = (P(a * \varphi), \psi) = (\varphi, P(\overline{a} * \psi)) = (\varphi, T(\overline{a})\psi), \quad \varphi, \psi \in S_0(\mathbb{Z}_+). \tag{3.3}$$

Since $S_0(\mathbb{Z}_+)$ is dense both in $X(\mathbb{Z}_+)$ and in $X'(\mathbb{Z}_+)$, $T(a) \in \mathcal{B}(X(\mathbb{Z}_+))$, and $T(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}_+))$, arguing as above, we can prove that (3.3) implies that

$$(T(a)f, g) = (f, T(\overline{a})g), \quad f \in X(\mathbb{Z}_+), \quad g \in X'(\mathbb{Z}_+),$$

which means that $(T(a))^* = T(\overline{a}) \in \mathcal{B}(X'(\mathbb{Z}_+))$. ☐

4 Fredholmness of Laurent operators is equivalent to their invertibility

For $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}} \in \ell^0(\mathbb{Z})$, define the translation (shift) operator

$$(U\varphi)_j = \varphi_{j-1}, \quad j \in \mathbb{Z}.$$

A Banach sequence space $X(\mathbb{Z})$ is said to be translation-invariant if $U\varphi \in X(\mathbb{Z})$ and $\|U\varphi\|_{X(\mathbb{Z})} = \|\varphi\|_{X(\mathbb{Z})}$ for all $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}} \in X(\mathbb{Z})$. If $X(\mathbb{Z})$ is translation-invariant, then it is clear that U^{-1} is a bounded operator on $X(\mathbb{Z})$ given by

$$(U^{-1}\varphi)_j = \varphi_{j+1}, \quad j \in \mathbb{Z}.$$

As usual, $U^0 := I$ and $U^{-n} := (U^{-1})^n$ for $n \in \mathbb{N}$. It is easy to see that $\|U^m\|_{\mathcal{B}(X(\mathbb{Z}))} = 1$ for all $m \in \mathbb{Z}$.

The following result is analogous to [7, Proposition 2.29(d)].

Theorem 4.1. *Let $X(\mathbb{Z})$ be a reflexive translation-invariant Banach sequence space and $a \in M_{X(\mathbb{Z})}$. Then $L(a) \in \Phi(X(\mathbb{Z}))$ if and only if $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$.*

Proof. The sufficiency part is trivial. Let us prove the necessity portion.

Suppose $L(a)$ is Fredholm. Then, by [17, Ch. I, Theorem 3.1], there exists $R \in \mathcal{B}(X(\mathbb{Z}))$ such that

$$RL(a) - I = K_1, \quad L(a)R - I = K_2, \tag{4.1}$$

where $K_1, K_2 \in \mathcal{K}(X(\mathbb{Z}))$. It is clear that $R \neq 0$. It follows from [14, Lemma 2.3(a)] that

$$U^{-n}L(a)U^n = L(a), \quad n \in \mathbb{N}. \tag{4.2}$$

It follows from the first equality in (4.1) and equality (4.2) that for all $f \in X(\mathbb{Z})$ and all $n \in \mathbb{N}$,

$$f = (U^{-n}RU^n)(U^{-n}L(a)U^n)f - (U^{-n}K_1U^n)f = (U^{-n}RU^n)L(a)f - (U^{-n}K_1U^n)f.$$

Since the operators $U^{\pm n}$ are isometries on $X(\mathbb{Z})$, the above equality implies that for all $f \in X(\mathbb{Z})$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|f\|_{X(\mathbb{Z})} &\leq \|U^{-n}RU^n\|_{\mathcal{B}(X(\mathbb{Z}))}\|L(a)f\|_{X(\mathbb{Z})} + \|(U^{-n}K_1U^n)f\|_{X(\mathbb{Z})} \\ &= \|R\|_{\mathcal{B}(X(\mathbb{Z}))}\|L(a)f\|_{X(\mathbb{Z})} + \|(U^{-n}K_1U^n)f\|_{X(\mathbb{Z})}.\end{aligned}$$

It follows from [14, Lemma 2.2(a)] that

$$\lim_{n\to\infty} \|(U^{-n}K_1U^n)f\|_{X(\mathbb{Z})} = 0.$$

Hence, for all $f \in X(\mathbb{Z})$,

$$\|f\|_{X(\mathbb{Z})} \leq \|R\|_{\mathcal{B}(X(\mathbb{Z}))}\|L(a)f\|_{X(\mathbb{Z})}.$$

Since $R \neq 0$, the above inequality implies that $\operatorname{Ker} L(a) = \{0\}$.

It follows from the second equality in (4.1) and Lemma 3.1 that

$$R^*(L(a))^* - I^* = R^*L(\overline{a}) - I = K_2^*.$$

Note that $K_2^* \in \mathcal{K}(X'(\mathbb{Z}))$ in view of [20, Theorem 4.19]. Repeating the above argument with $L(\overline{a})$ acting on $X'(\mathbb{Z})$ in place of $L(a)$ acting on $X(\mathbb{Z})$, we conclude that for all $g \in X'(\mathbb{Z})$,

$$\|g\|_{X'(\mathbb{Z})} \leq \|R^*\|_{\mathcal{B}(X'(\mathbb{Z}))}\|L(\overline{a})g\|_{X'(\mathbb{Z})}.$$

Hence, $\operatorname{Ker} L(\overline{a}) = \{0\}$. Therefore, by [20, Corollary (b) to Theorem 4.12], $\operatorname{Im} L(a)$ is dense in $X(\mathbb{Z})$. Since $L(a)$ is Fredholm, $\operatorname{Im} L(a)$ is closed in $X(\mathbb{Z})$ (see, e.g., [1, Lemma 4.38]). Thus $\operatorname{Im} L(a) = X(\mathbb{Z})$. So, $L(a)$ is bijective. In view of the Banach isomorphism theorem, $L(a)$ is invertible on $X(\mathbb{Z})$. □

5 An analogue of the Hartman–Wintner–Simonenko theorem

Now we are in a position to prove the implication (i) in (1.2).

Theorem 5.1. *Let $X(\mathbb{Z})$ be a reflexive translation-invariant Banach sequence space. If $a \in M_{X(\mathbb{Z})}$ and $T(a) \in \Phi(X(\mathbb{Z}_+))$, then $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$.*

Proof. The proof is analogous to that of [7, Proposition 2.30(b)]. Since the space $X(\mathbb{Z})$ is reflexive, the set $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z})$ and in its associate space $X'(\mathbb{Z})$ (see [15, Lemma 2.1(b)]).

Let $T(a) \in \Phi(X(\mathbb{Z}_+))$. Then $\dim \operatorname{Ker} T(a) < \infty$. It follows from [17, Ch. I, Lemma 2.1] that there is an operator $K \in \mathcal{K}(X(\mathbb{Z}_+))$ and a constant $\delta > 0$ such that for all $\varphi \in X(\mathbb{Z}_+)$,

$$\|T(a)\varphi\|_{X(\mathbb{Z}_+)} + \|K\varphi\|_{X(\mathbb{Z}_+)} \geq \delta\|\varphi\|_{X(\mathbb{Z}_+)}. \tag{5.1}$$

Let $Q := I - P$. Since for all $\psi \in X(\mathbb{Z})$, one has $\varphi = P\psi \in X(\mathbb{Z}_+)$ and

$$\|\psi\|_{X(\mathbb{Z})} \leq \|P\psi\|_{X(\mathbb{Z})} + \|Q\psi\|_{X(\mathbb{Z})},$$

inequality (5.1) implies that

$$\|PL(a)P\psi\|_{X(\mathbb{Z})} + \|PKP\psi\|_{X(\mathbb{Z})} + \delta\|Q\psi\|_{X(\mathbb{Z})} \geq \delta\|\psi\|_{X(\mathbb{Z})}.$$

Since $X(\mathbb{Z})$ is translation-invariant, for every $\psi \in X(\mathbb{Z})$ and every $n \in \mathbb{N}$, one has $U^n\psi \in X(\mathbb{Z})$. Therefore, it follows from the above inequality that

$$\|PL(a)PU^n\psi\|_{X(\mathbb{Z})} + \|PKPU^n\psi\|_{X(\mathbb{Z})} + \delta\|QU^n\psi\|_{X(\mathbb{Z})} \geq \delta\|U^n\psi\|_{X(\mathbb{Z})}.$$

Note that U^{-n} are isometries on $X(\mathbb{Z})$ for all $n \in \mathbb{Z}$, so the above inequality yields that for all $n \in \mathbb{N}$ and all $\psi \in X(\mathbb{Z})$,

$$\|U^{-n}PL(a)PU^n\psi\|_{X(\mathbb{Z})} + \|U^{-n}PKPU^n\psi\|_{X(\mathbb{Z})} + \delta\|U^{-n}QU^n\psi\|_{X(\mathbb{Z})} \geq \delta\|\psi\|_{X(\mathbb{Z})}. \tag{5.2}$$

Since $K \in \mathcal{K}(X(\mathbb{Z}_+))$, we have $PKP \in \mathcal{K}(X(\mathbb{Z}))$. In this case it follows from [14, Lemma 2.2(a)] that

$$\lim_{n\to\infty} \|U^{-n}PKPU^n\psi\|_{X(\mathbb{Z})} = 0. \tag{5.3}$$

By [14, Lemma 2.3(a)], we have

$$U^{-n}PL(a)PU^n = (U^{-n}PU^n)L(a)(U^{-n}PU^n). \tag{5.4}$$

For every set $E \subset \mathbb{R}$, we denote by $\mathbb{1}_E$ the characteristic (indicator) function of the set $E \cap \mathbb{Z}$ considered as an element of $\ell^0_+(\mathbb{Z})$. Let $f \in S_0(\mathbb{Z})$. It is easy to see that $U^{-n}PU^nf = \mathbb{1}_{[-n,+\infty)}f$ for all $n \in \mathbb{N}$. Since f has finite support, this equality implies that

$$\lim_{n\to\infty} \|U^{-n}PU^nf - f\|_{X(\mathbb{Z})} = \lim_{n\to\infty} \|\mathbb{1}_{(-\infty,-n-1]}f\|_{X(\mathbb{Z})} = 0. \tag{5.5}$$

It is evident that for all $n \in \mathbb{N}$,

$$\|U^{-n}PU^n - I\|_{\mathcal{B}(X(\mathbb{Z}))} \leq 2. \tag{5.6}$$

It follows from (5.5)–(5.6) and [19, Lemma 1.4.1(b)] that $U^{-n}PU^n \to I$ strongly as $n \to \infty$. This observation, (5.4) and [19, Lemma 1.4.4] imply that $U^{-n}QU^n = I - U^{-n}PU^n \to 0$ and $U^{-n}PL(a)PU^n \to L(a)$ strongly as $n \to \infty$. Thus, for all $\psi \in X(\mathbb{Z})$,

$$\lim_{n\to\infty} \|U^{-n}QU^n\psi\|_{X(\mathbb{Z})} = 0, \quad \lim_{n\to\infty} \|U^{-n}PL(a)PU^n\psi\|_{X(\mathbb{Z})} = \|L(a)\psi\|_{X(\mathbb{Z})}. \tag{5.7}$$

Passing to the limit in inequality (5.2) as $n \to \infty$ and taking into account equalities (5.3) and (5.7), we conclude that

$$\|L(a)\psi\|_{X(\mathbb{Z})} \geq \delta\|\psi\|_{X(\mathbb{Z})}, \quad \psi \in X(\mathbb{Z}).$$

This means that $\operatorname{Ker} L(a) = \{0\}$.

It follows from Lemma 3.1 and [1, Theorem 4.42] that the operator $(T(a))^* = T(\overline{a})$ is Fredholm on the space $X'(\mathbb{Z}_+) = (X(\mathbb{Z}_+))^*$. Repeating the above argument for $T(\overline{a})$ acting on $X'(\mathbb{Z}_+)$ in place of $T(a)$ acting on $X(\mathbb{Z}_+)$, we conclude that $\operatorname{Ker} L(\overline{a}) = \{0\}$. Thus $L(a)$ is invertible on $X(\mathbb{Z})$ (see the end of the proof of Theorem 4.1). □

6 Invertibility of Laurent operators is equivalent to invertibility of corresponding periodic multipliers

We say that a Banach sequence space $X(\mathbb{Z})$ is reflection-invariant if $\|\varphi\|_{X(\mathbb{Z})} = \|\widetilde{\varphi}\|_{X(\mathbb{Z})}$ for every $\varphi \in X(\mathbb{Z})$, where $\widetilde{\varphi}$ denotes the reflection of a sequence $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}}$ defined by $\widetilde{\varphi}_k := \varphi_{-k}$ for $k \in \mathbb{Z}$.

A sequence $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}}$ is said to be of slow growth if there are some positive constants c and r such that $|\varphi_k| \le c|k|^r$ for all $k \in \mathbb{Z}\setminus\{0\}$. The set of all sequences of slow growth is denoted by $S'(\mathbb{Z})$.

The following result is analogous to [7, Proposition 2.28(b)].

Theorem 6.1. *Let $X(\mathbb{Z})$ be a reflexive reflection-invariant Banach sequence space such that $X(\mathbb{Z}) \subset S'(\mathbb{Z})$ and let $a \in M_{X(\mathbb{Z})}$. Then $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$ if and only if $a \in \mathcal{G}M_{X(\mathbb{Z})}$. If $a \in \mathcal{G}M_{X(\mathbb{Z})}$, then $(L(a))^{-1} = L(a^{-1})$.*

Proof. *Sufficiency.* Recall that if $X(\mathbb{Z})$ is reflexive and reflection-invariant, then $M_{X(\mathbb{Z})}$ is a Banach algebra under pointwise multiplication (see [15, Theorem 5.6]). If $a \in \mathcal{G}M_{X(\mathbb{Z})}$ and $b \in M_{X(\mathbb{Z})}$ is the inverse of a, then

$$L(a)L(b) = L(ab) = L(\mathbb{1}) = I, \quad L(b)L(a) = L(ba) = L(\mathbb{1}) = I,$$

that is, $L(b)$ is the inverse of $L(a)$.

Necessity. Suppose $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$ and B is the inverse of $L(a)$ in $\mathcal{B}(X(\mathbb{Z}))$. The invertibility of $L(a)$ implies that the equation $L(a)\varphi = e_0$ has a solution $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}} \in X(\mathbb{Z})$. Then

$$(a*\varphi)_j = \sum_{k\in\mathbb{Z}} \widehat{a}_{j-k}\varphi_k = (e_0)_j = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Since $X(\mathbb{Z}) \subset S'(\mathbb{Z})$, by [2, Ch. 5, Theorem 1.2], there exists a distribution $b \in \mathcal{P}'$ such that

$$\widehat{b}_n = \varphi_n, \quad n \in \mathbb{Z}. \tag{6.2}$$

For this $b \in \mathcal{P}'$ and for $\psi = \{\psi_k\}_{k\in\mathbb{Z}} \in S_0(\mathbb{Z})$, consider

$$(L(b)\psi)_i := (b*\psi)_i = \sum_{j\in\mathbb{Z}} \widehat{b}_{i-j}\psi_j, \quad i \in \mathbb{Z}.$$

It follows from (6.1)–(6.2) that for all $j \in \mathbb{Z}$,

$$\begin{aligned}(L(a)L(b)\psi)_j &= \sum_{k\in\mathbb{Z}} \widehat{a}_{j-k}(L(b)\psi)_k = \sum_{k\in\mathbb{Z}} \widehat{a}_{j-k}\Big(\sum_{m\in\mathbb{Z}} \widehat{b}_{k-m}\psi_m\Big) \\ &= \sum_{k\in\mathbb{Z}} \widehat{a}_{j-k}\Big(\sum_{m\in\mathbb{Z}} \varphi_{k-m}\psi_m\Big) = \sum_{m\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} \widehat{a}_{j-k}\varphi_{k-m}\Big)\psi_m \\ &= \sum_{m\in\mathbb{Z}}\Big(\sum_{i\in\mathbb{Z}} \widehat{a}_{j-m-i}\varphi_i\Big)\psi_m = \sum_{m\in\mathbb{Z}} (e_0)_{j-m}\psi_m = \psi_j.\end{aligned} \tag{6.3}$$

Note that the change of order of summation is possible because $\psi \in S_0(\mathbb{Z})$, and so one of the sums is finite. Analogously,

$$\begin{aligned}(L(b)L(a)\psi)_j &= \sum_{k\in\mathbb{Z}} \widehat{b}_{j-k}(L(a)\psi)_k = \sum_{k\in\mathbb{Z}} \widehat{b}_{j-k}\Big(\sum_{m\in\mathbb{Z}} \widehat{a}_{k-m}\psi_m\Big) \\ &= \sum_{k\in\mathbb{Z}} \varphi_{j-k}\Big(\sum_{m\in\mathbb{Z}} \widehat{a}_{k-m}\psi_m\Big) = \sum_{m\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} \varphi_{j-k}\widehat{a}_{k-m}\Big)\psi_m \\ &= \sum_{m\in\mathbb{Z}}\Big(\sum_{i\in\mathbb{Z}} \widehat{a}_{j-m-i}\varphi_i\Big)\psi_m = \sum_{m\in\mathbb{Z}} (e_0)_{j-m}\psi_m = \psi_j.\end{aligned} \tag{6.4}$$

Taking into account that $B = (L(a))^{-1}$, we conclude from (6.3)–(6.4) that for all $\psi = \{\psi_i\}_{i\in\mathbb{Z}} \in S_0(\mathbb{Z})$,

$$(B\psi)_i = (L(b)\psi)_i = (b * \psi)_i, \quad i \in \mathbb{Z}. \tag{6.5}$$

Since the operator B is bounded, it follows from (6.5) that

$$\begin{aligned}\|b\|_{M_{X(\mathbb{Z})}} &= \sup\left\{\frac{\|b*\psi\|_{X(\mathbb{Z})}}{\|\psi\|_{X(\mathbb{Z})}} \ :\ \psi \in S_0(\mathbb{Z})\setminus\{0\}\right\}\\ &= \sup\left\{\frac{\|B\psi\|_{X(\mathbb{Z})}}{\|\psi\|_{X(\mathbb{Z})}} \ :\ \psi \in S_0(\mathbb{Z})\setminus\{0\}\right\} \le \|B\|_{\mathcal{B}(X(\mathbb{Z}))} < \infty.\end{aligned}$$

Therefore, $b \in M_{X(\mathbb{Z})}$.

It remains to show that $b \in M_{X(\mathbb{Z})}$ is the inverse of $a \in M_{X(\mathbb{Z})}$. Since $M_{X(\mathbb{Z})} \hookrightarrow L^\infty(-\pi,\pi)$ (see [15, Theorem 5.5]), we conclude that $a, b \in L^\infty(-\pi,\pi) \hookrightarrow L^2(-\pi,\pi)$. Therefore, $ab \in L^2(-\pi,\pi)$ and

$$(\widehat{ab})_n = \sum_{k\in\mathbb{Z}} \widehat{a}_{n-k}\widehat{b}_k, \quad n \in \mathbb{Z}. \tag{6.6}$$

Combining (6.1)–(6.2) and (6.6), we see that $(\widehat{ab})_n = \widehat{\mathbb{1}}_n$ for all $n \in \mathbb{Z}$. By the uniqueness theorem for Fourier series (see, e.g., [16, Ch. 1, Theorem 2.7]), $ab = \mathbb{1}$ a.e. on $(-\pi,\pi)$. Thus, b is the inverse of a in $M_{X(\mathbb{Z})}$. ☐

7 When is a weighted rearrangement-invariant Banach sequence space contained in $S'(\mathbb{Z})$?

The distribution function of a sequence $f = \{f_k\}_{k\in\mathbb{Z}} \in \ell^0(\mathbb{Z})$ is defined by

$$d_f(\lambda) := m\{k \in \mathbb{Z} : |f_k| > \lambda\}, \quad \lambda \ge 0,$$

where $m(S)$ denotes the measure (cardinality) of a set $S \subset \mathbb{Z}$. One says that sequences $f = \{f_k\}_{k\in\mathbb{Z}}$, $g = \{g_k\}_{k\in\mathbb{Z}} \in \ell^0(\mathbb{Z})$ are equimeasurable if $d_f = d_g$. A Banach function norm $\varrho : \ell^0_+(\mathbb{Z}) \to [0,\infty]$ is said to be rearrangement-invariant if $\varrho(f) = \varrho(g)$ for every pair of equimeasurable sequences $f = \{f_k\}_{k\in\mathbb{Z}}$, $g = \{g_k\}_{k\in\mathbb{Z}} \in \ell^0_+(\mathbb{Z})$. In that case, the Banach sequence space $X(\mathbb{Z})$ generated by ϱ is said to be a rearrangement-invariant Banach sequence space (cf. [3, Ch. 2, Definition 4.1]). It follows from [3, Ch. 2, Proposition 4.2] that if a Banach sequence space $X(\mathbb{Z})$ is rearrangement-invariant, then its associate space $X'(\mathbb{Z})$ is also a rearrangement-invariant Banach sequence space. It is easy to see that, for every $f = \{f_k\}_{k\in\mathbb{Z}} \in \ell^0_+(\mathbb{Z})$, the sequences f and Uf are equimeasurable. Hence, every rearrangement-invariant Banach sequence space is also translation-invariant.

A weight on $\mathbb{Z}$ is a sequence $w = \{w_k\}_{k\in\mathbb{Z}}$ of positive numbers. A weight $w = \{w_k\}_{k\in\mathbb{Z}}$ is said to be symmetric if $w_{-k} = w_k$ for all $k \in \mathbb{Z}$.

For a Banach sequence space $X(\mathbb{Z})$ and a weight $w = \{w_k\}_{k\in\mathbb{Z}}$, the weighted Banach sequence space $X(\mathbb{Z}, w)$ consists of all sequences $f = \{f_k\}_{k\in\mathbb{Z}}$ such that $fw := \{f_k w_k\}_{k\in\mathbb{Z}}$ belongs to $X(\mathbb{Z})$. It is easy to see that $X(\mathbb{Z}, w)$ is itself a Banach sequence space with respect to the norm

$$\|f\|_{X(\mathbb{Z},w)} := \|fw\|_{X(\mathbb{Z})}.$$

Lemma 7.1. *If $X(\mathbb{Z})$ is a rearrangement-invariant Banach sequence space and w is a weight such that there exist constants $C, r \in (0,\infty)$ such that $1/w_n \le C|n|^r$ for all $n \in \mathbb{Z}\setminus\{0\}$, then $X(\mathbb{Z}, w) \subset S'(\mathbb{Z})$.*

Proof. Let $x = \{x_n\}_{n\in\mathbb{Z}} \in X(\mathbb{Z}, w)$. Then it follows from [3, Ch. 2, Corollary 6.8] that $xw \in X(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$. Hence $|x_n w_n| \le \|xw\|_{\ell^\infty(\mathbb{Z})}$ for all $n \in \mathbb{Z}$. Therefore,

$$|x_n| \le \frac{\|xw\|_{\ell^\infty(\mathbb{Z})}}{w_n} \le C\|xw\|_{\ell^\infty(\mathbb{Z})}|n|^r, \quad n \in \mathbb{Z}\setminus\{0\}.$$

This means that x is of slow growth. ☐

Let $l, n \in \mathbb{Z}$ satisfy $l \leq n$. We call a set of the form $J = \{l, \dots, n\}$ an interval of $\mathbb{Z}$. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. A weight $w = \{w_k\}_{k\in\mathbb{Z}}$ is said to belong to the Muckenhoupt class $A_p(\mathbb{Z})$ if

$$[w]_{A_p(\mathbb{Z})} := \sup_{J\subset\mathbb{Z}} \frac{1}{m(J)} \left(\sum_{k\in J} w_k^p\right)^{1/p} \left(\sum_{k\in J} w_k^{-p'}\right)^{1/p'} < \infty,$$

where the supremum is taken over all intervals $J \subset \mathbb{Z}$. The collection of all symmetric weights in the Muckenhoupt class $A_p(\mathbb{Z})$ will be denoted by $A_p^{\mathrm{sym}}(\mathbb{Z})$.

Lemma 7.2. *If $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$ for some $p \in (1, \infty)$, then there exists $C \in (0, \infty)$ such that $1/w_n \leq C(|n| + 1)$ for all $n \in \mathbb{Z}$.*

Proof. It is clear that $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$ if and only if $w^{-1} \in A_{p'}^{\mathrm{sym}}(\mathbb{Z})$, where $1/p + 1/p' = 1$. By [5, Lemma 7.1], there exists $C_{p',w^{-1}} \in (0, \infty)$ such that

$$\frac{w_n}{w_{n+k}} = \frac{w_{n+k}^{-1}}{w_n^{-1}} \leq C_{p',w^{-1}}(|k| + 1), \quad k, n \in \mathbb{Z}.$$

Then

$$\frac{1}{w_k} \leq \frac{C_{p',w^{-1}}}{w_0}(|k| + 1), \quad k \in \mathbb{Z}.$$

Thus, the desired inequality holds with $C := C_{p',w^{-1}}/w_0$. □

The above two lemmas immediately imply the following.

Corollary 7.3. *If $X(\mathbb{Z})$ is a rearrangement-invariant Banach sequence space and $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$ for some $p \in (1, \infty)$, then $X(\mathbb{Z}, w) \subset S'(\mathbb{Z})$.*

Recall that if $X(\mathbb{Z})$ is a reflexive rearrangement-invariant Banach sequence space and w is a symmetric weight, then $X(\mathbb{Z}, w)$ is a reflexive reflection-invariant Banach sequence space (see [15, Lemma 6.1]). This observation, Corollary 7.3, and Theorem 6.1 yield the following.

Theorem 7.4. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space, $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$ for some $p \in (1, \infty)$, and $a \in M_{X(\mathbb{Z},w)}$. Then $L(a) \in \mathcal{GB}(X(\mathbb{Z}, w))$ if and only if $a \in \mathcal{G}M_{X(\mathbb{Z},w)}$. If $a \in \mathcal{G}M_{X(\mathbb{Z},w)}$, then $(L(a))^{-1} = L(a^{-1})$.*

Let us also formulate explicitly one corollary of the above theorem, which seems to be new.

Corollary 7.5. *Let $1 < p < \infty$, $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$, and $a \in M_{\ell^p(\mathbb{Z},w)}$. Then $L(a) \in \mathcal{GB}(\ell^p(\mathbb{Z}, w))$ if and only if $a \in \mathcal{G}M_{\ell^p(\mathbb{Z},w)}$. If $a \in \mathcal{G}M_{\ell^p(\mathbb{Z},w)}$, then $(L(a))^{-1} = L(a^{-1})$.*

8 Two lemmas on convolutions

The convolution of two sequences $f = \{f_k\}_{k\in\mathbb{Z}}$ and $g = \{g_k\}_{k\in\mathbb{Z}}$ is formally defined by

$$(f * g)_n = \sum_{k\in\mathbb{Z}} f_k g_{n-k}, \quad n \in \mathbb{Z}.$$

The aim of this section is to provide some conditions ensuring that this convolution is well defined and associative. These conditions will play an important role in the proof of the implication (*iii*) in (1.2), which will be given in the next section.

A weight $w : \mathbb{Z} \to (0, \infty)$ is said to be controlled by a weight $v : \mathbb{Z} \to (0, \infty)$ if

$$\frac{w_{n+k}}{w_n} \leq v_k$$

for all $k, n \in \mathbb{Z}$.

It follows from [5, Lemma 7.1] that if $w \in A_p^{\mathrm{sym}}(\mathbb{Z})$ for some $p \in (1,\infty)$, then there is a constant $C_{p,w} > 0$ depending only on p and w such that the weight w is controlled by the weight v defined by

$$v_k := C_{p,w}(|k|+1), \quad k \in \mathbb{Z}.$$

Lemma 8.1. *Let $X(\mathbb{Z})$ be a rearrangement-invariant Banach sequence space and let $w : \mathbb{Z} \to (0,\infty)$ be a symmetric weight controlled by a weight $v : \mathbb{Z} \to (0,\infty)$. If $y = \{y_k\}_{k\in\mathbb{Z}} \in X'(\mathbb{Z}, w^{-1})$ and $x = \{x_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}, w)$, then for every $n \in \mathbb{Z}$,*

$$|(y * x)_n| \le v_{-n} \|y\|_{X'(\mathbb{Z},w^{-1})} \|x\|_{X(\mathbb{Z},w)}. \tag{8.1}$$

Proof. The proof is similar to the proof of [5, Lemma 7.2]. By Hölder's inequality for Banach sequence spaces (see [3, Ch. 1, Theorem 2.4]),

$$\begin{aligned} |(y * x)_n| &= \left| \sum_{k\in\mathbb{Z}} y_k x_{n-k} \right| \le \|\{y_k x_{n-k}\}_{k\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} \\ &\le \left\| \{y_k w_k^{-1}\}_{k\in\mathbb{Z}} \right\|_{X'(\mathbb{Z})} \|\{x_{n-k} w_k\}_{k\in\mathbb{Z}}\|_{X(\mathbb{Z})} . \end{aligned} \tag{8.2}$$

Since the counting measure on $\mathbb{Z}$ is reflection-invariant and translation-invariant, we have for every $\lambda \ge 0$,

$$m\{k \in \mathbb{Z} : |x_{n-k} w_k| > \lambda\} = m\{k \in \mathbb{Z} : |x_{k-n} w_{-k}| > \lambda\} = m\{l \in \mathbb{Z} : |x_l w_{n-l}| > \lambda\},$$

that is, the sequences $\{x_{n-k} w_k\}_{k\in\mathbb{Z}}$ and $\{x_l w_{n-l}\}_{l\in\mathbb{Z}}$ are equimeasurable. Since $X(\mathbb{Z})$ is rearrangement-invariant, the above observation implies that

$$\|\{x_{n-k} w_k\}_{k\in\mathbb{Z}}\|_{X(\mathbb{Z})} = \|\{x_l w_{n-l}\}_{k\in\mathbb{Z}}\|_{X(\mathbb{Z})} . \tag{8.3}$$

Taking into account that w is symmetric and is controlled by a weight v, we have

$$\frac{w_{n-l}}{w_l} = \frac{w_{l-n}}{w_l} \le v_{-n}, \quad l, n \in \mathbb{Z}.$$

Hence, for all $l, n \in \mathbb{Z}$,

$$|x_l| w_{n-l} \le v_{-n} |x_l| w_l,$$

which implies that

$$\|\{x_l w_{n-l}\}_{l\in\mathbb{Z}}\|_{X(\mathbb{Z})} \le v_{-n} \|\{x_l w_l\}_{l\in\mathbb{Z}}\|_{X(\mathbb{Z})} . \tag{8.4}$$

Combining (8.2)–(8.4), we arrive at

$$|(y * x)_n| \le v_{-n} \left\| \{y_k w_k^{-1}\}_{k\in\mathbb{Z}} \right\|_{X'(\mathbb{Z})} \|\{x_k w_k\}_{k\in\mathbb{Z}}\|_{X(\mathbb{Z})} ,$$

which immediately yields (8.1). □

The following statement extends [5, Lemma 7.3].

Lemma 8.2. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space, let $w : \mathbb{Z} \to (0,\infty)$ be a symmetric weight, and let $a \in M_{X(\mathbb{Z},w)}$. Suppose there is a weight $v : \mathbb{Z} \to (0,\infty)$ such that w is controlled by v. If $y = \{y_k\}_{k\in\mathbb{Z}} \in X'(\mathbb{Z}, w^{-1})$ and $x = \{x_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}, w)$, then $(y * a) * x$ and $y * (a * x)$ are well defined sequences and*

$$[(y * a) * x]_n = [y * (a * x)]_n \quad \textit{for all} \quad n \in \mathbb{Z}. \tag{8.5}$$

Proof. It follows from [15, Theorem 5.6 and Lemma 6.1] that $M_{X(\mathbb{Z},w)}$ is a Banach algebra under pointwise multiplication and the norm

$$\|a\|_{M_{X(\mathbb{Z},w)}} = \|L(a)\|_{\mathcal{B}(X(\mathbb{Z},w))}.$$

Moreover, [15, Corollary 5.4] implies that $M_{X(\mathbb{Z},w)} = M_{X'(\mathbb{Z},w^{-1})}$ and

$$\|a\|_{M_{X(\mathbb{Z},w)}} = \|a\|_{M_{X'(\mathbb{Z},w^{-1})}}.$$

Then

$$\|y*a\|_{X'(\mathbb{Z},w^{-1})} \le \|a\|_{M_{X'(\mathbb{Z},w^{-1})}}\|y\|_{X'(\mathbb{Z},w^{-1})} = \|a\|_{M_{X(\mathbb{Z},w)}}\|y\|_{X'(\mathbb{Z},w^{-1})} < \infty$$

and

$$\|a*x\|_{X(\mathbb{Z},w)} \le \|a\|_{M_{X(\mathbb{Z},w)}}\|x\|_{X(\mathbb{Z},w)} < \infty.$$

We infer from Lemma 8.1 and the above inequalities that for all $n \in \mathbb{Z}$,

$$|[(y*a)*x]_n| \le v_{-n}\|y*a\|_{X'(\mathbb{Z},w^{-1})}\|x\|_{X(\mathbb{Z},w)} \le v_{-n}\|a\|_{M_{X(\mathbb{Z},w)}}\|y\|_{X'(\mathbb{Z},w^{-1})}\|x\|_{X(\mathbb{Z},w)} < \infty \tag{8.6}$$

and

$$|[y*(a*x)]_n| \le v_{-n}\|y\|_{X'(\mathbb{Z},w^{-1})}\|a*x\|_{X(\mathbb{Z},w)} \le v_{-n}\|y\|_{X'(\mathbb{Z},w^{-1})}\|a\|_{M_{X(\mathbb{Z},w)}}\|x\|_{X(\mathbb{Z},w)} < \infty. \tag{8.7}$$

So, both sequences $(y*a)*x$ and $y*(a*x)$ are well-defined.

If $x, y \in S_0(\mathbb{Z})$, then for all $n \in \mathbb{Z}$,

$$\begin{aligned}
[(y*a)*x]_n &= \sum_{k\in\mathbb{Z}}(y*a)_k x_{n-k} = \sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} y_j \widehat{a}_{k-j}\Big)x_{n-k} = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_j\widehat{a}_{k-j}x_{n-k} \\
&\overset{(*)}{=} \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} y_j\widehat{a}_{k-j}x_{n-k} = \sum_{j\in\mathbb{Z}} y_j\Big(\sum_{k\in\mathbb{Z}}\widehat{a}_{k-j}x_{n-k}\Big) \overset{(**)}{=} \sum_{j\in\mathbb{Z}} y_j\Big(\sum_{l\in\mathbb{Z}}\widehat{a}_l x_{n-j-l}\Big) \\
&= \sum_{j\in\mathbb{Z}} y_j(a*x)_{n-j} = [y*(a*x)]_n.
\end{aligned} \tag{8.8}$$

The change of order of summation in (*) is justified because both sums are finite in view of the hypothesis that x and y are finitely supported. Also note that we have made the change of variables $l = k - j$ in (**).

Taking into account that $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z}, w)$ and in $X'(\mathbb{Z}, w^{-1})$ in view of [15, Lemmas 2.1 and 6.1], for $x \in X(\mathbb{Z}, w)$ and $y \in X'(\mathbb{Z}, w^{-1})$, there exist sequences $\{x^{(m)}\}_{m\in\mathbb{N}}$ and $\{y^{(m)}\}_{m\in\mathbb{N}}$ in $S_0(\mathbb{Z})$ such that

$$\lim_{m\to\infty}\big\|x - x^{(m)}\big\|_{X(\mathbb{Z},w)} = 0, \quad \lim_{m\to\infty}\big\|y - y^{(m)}\big\|_{X'(\mathbb{Z},w^{-1})} = 0. \tag{8.9}$$

For every $m \in \mathbb{N}$, we have

$$(y*a)*x = ((y-y^{(m)})*a)*x + (y^{(m)}*a)*(x-x^{(m)}) + (y^{(m)}*a)*x^{(m)} \tag{8.10}$$

and

$$y*(a*x) = (y-y^{(m)})*(a*x) + y^{(m)}*(a*(x-x^{(m)})) + y^{(m)}*(a*x^{(m)}). \tag{8.11}$$

It follows from (8.6)–(8.8) and (8.10)–(8.11) that for every $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
|[(y*a)*x]_n - [y*(a*x)]_n| \le{}& \Big|[((y-y^{(m)})*a)*x]_n\Big| + \Big|[(y^{(m)}*a)*(x-x^{(m)})]_n\Big| \\
&+ \Big|[(y^{(m)}*a)*x^{(m)}]_n - [y^{(m)}*(a*x^{(m)})]_n\Big| \\
&+ \Big|[(y-y^{(m)})*(a*x)]_n\Big| + \Big|[y^{(m)}*(a*(x-x^{(m)}))]_n\Big| \\
\le{}& 2v_{-n}\big\|y-y^{(m)}\big\|_{X'(\mathbb{Z},w^{-1})}\|a\|_{M_{X(\mathbb{Z},w)}}\|x\|_{X(\mathbb{Z},w)} \\
&+ 2v_{-n}\big\|y^{(m)}\big\|_{X'(\mathbb{Z},w^{-1})}\|a\|_{M_{X(\mathbb{Z},w)}}\big\|x-x^{(m)}\big\|_{X(\mathbb{Z},w)}.
\end{aligned}$$

Fix $n \in \mathbb{Z}$. Passing to the limit as $m \to \infty$ in the above inequality and taking into account (8.9), we arrive at (8.5). □

9 An analogue of the Coburn–Duduchava theorem

The following result is an extension of the version of the Coburn–Duduchava theorem proved by Böttcher and Seybold [5, Theorem 7.4]. It provides the proof of the implication *(iii)* in (1.2).

Theorem 9.1. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space, let $w : \mathbb{Z} \to (0, \infty)$ be a symmetric weight, and let $a \in M_{X(\mathbb{Z},w)}$ be invertible in the Banach algebra $M_{X(\mathbb{Z},w)}$. Suppose there is a weight $v : \mathbb{Z} \to (0, \infty)$ such that w is controlled by v. Then the discrete Wiener–Hopf operator $T(a)$ has a trivial kernel on $X(\mathbb{Z}_+, w)$ or the discrete Wiener–Hopf operator $T(\overline{a})$ has a trivial kernel on $X'(\mathbb{Z}_+, w^{-1})$.*

Proof. The proof is analogous to that of [5, Theorem 7.4]. Assume the contrary, that is, assume that there are $x_+ = \{(x_+)_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}_+, w) \setminus \{0\}$ and $y_+ = \{(y_+)_k\}_{k\in\mathbb{Z}} \in X'(\mathbb{Z}_+, w^{-1}) \setminus \{0\}$ such that

$$T(a)x_+ = 0, \quad T(\overline{a})y_+ = 0.$$

It follows that

$$x_- := L(a)x_+ = a * x_+$$

satisfies $x_- \in X(\mathbb{Z}, w)$ and $(x_-)_n = 0$ for $n \geq 0$. Analogously,

$$y_- := L(\overline{a})y_+ = Va * y_+$$

satisfies $y_- \in X'(\mathbb{Z}, w^{-1})$ and $(y_-)_n = 0$ for $n \geq 0$, where V is defined by

$$(Vf)_n := \overline{f}_{-n}, \quad n \in \mathbb{Z}.$$

Let $f = \{f_n\}_{n\in\mathbb{Z}}$, $g = \{g_n\}_{n\in\mathbb{Z}}$, and $h = f * g$. Then for $n \in \mathbb{Z}$,

$$(Vh)_n = \overline{h}_{-n} = \overline{\sum_{k\in\mathbb{Z}} f_k g_{-n-k}} = \sum_{k\in\mathbb{Z}} \overline{f}_k \overline{g}_{-n-k} = \sum_{k\in\mathbb{Z}} \overline{f}_{-k} \overline{g}_{-n+k} = \sum_{k\in\mathbb{Z}} (Vf)_k (Vg)_{n-k} = (Vf * Vg)_n,$$

that is,

$$V(f * g) = Vf * Vg.$$

Taking into account the above equality and Lemma 8.2, we get

$$\begin{aligned} Vy_- * x_+ &= V(Va * y_+) * x_+ = (V^2 a * Vy_+) * x_+ = (a * Vy_+) * x_+ \\ &= (Vy_+ * a) * x_+ = Vy_+ * (a * x_+) = Vy_+ * x_-. \end{aligned} \tag{9.1}$$

If $n \leq 0$, then we write

$$(Vy_- * x_+)_n = \sum_{k=-\infty}^{0} (\overline{y}_-)_{-k} (x_+)_{n-k} + \sum_{k=1}^{\infty} (\overline{y}_-)_{-k} (x_+)_{n-k}.$$

Note that the first term is equal to zero because $(\overline{y}_-)_{-k} = 0$ for $k \leq 0$, the second term is also equal to zero because $n - k > 0$ and $(x_+)_{n-k} = 0$ for $k \geq 1$. Thus,

$$(Vy_- * x_+)_n = 0 \quad \text{for} \quad n \leq 0. \tag{9.2}$$

Analogously, if $n \geq 0$, then write

$$(Vy_+ * x_-)_n = \sum_{k=-\infty}^{0} (\overline{y}_+)_{-k} (x_-)_{n-k} + \sum_{k=1}^{\infty} (\overline{y}_+)_{-k} (x_-)_{n-k}$$

The first term is equal to zero because $n - k \geq 0$ and $(x_-)_{n-k} = 0$ for $k \leq 0$. The second term is equal to zero because $(\overline{y}_+)_{-k} = 0$ for $k > 0$. Thus,

$$(Vy_+ * x_-)_n = 0 \quad \text{for} \quad n \geq 0. \tag{9.3}$$

It follows from (9.1)–(9.3) that

$$(Vy_- * x_+)_n = 0 = (Vy_+ * x_-)_n \quad \text{for} \quad n \in \mathbb{Z}. \tag{9.4}$$

Thus,

$$Vy_+ * x_- = 0.$$

Since $(\overline{y}_+)_{-k} = 0$ for $k > 0$ and $(x_-)_{n-k} = 0$ for $n - k \geq 0$, the nonzero terms in the expression

$$(Vy_+ * x_-)_n = \sum_{k \in \mathbb{Z}} (\overline{y}_+)_{-k} (x_-)_{n-k}$$

may only appear if $n + 1 \leq k \leq 0$. So, it follows from (9.4) that if $n \leq -1$, then

$$0 = (Vy_+ * x_-)_n = \sum_{k=n+1}^{0} (\overline{y}_+)_{-k} (x_-)_{n-k} = \sum_{k=0}^{-n-1} (\overline{y}_+)_k (x_+)_{n+k}. \tag{9.5}$$

Since $y_+ \in X'(\mathbb{Z}_+, w^{-1}) \setminus \{0\}$, there exists $m \in \mathbb{Z}_+$ such that $(\overline{y}_+)_k = 0$ for $k < m$ and $(\overline{y}_+)_m \neq 0$.

For $j \in \mathbb{N}$, take $n = -m - j$ in (9.5). Then

$$0 = \sum_{k=m}^{m+j-1} (\overline{y}_+)_k (x_-)_{k-m-j}, \quad j \in \mathbb{N},$$

that is,

$$\begin{aligned}
&(\overline{y}_+)_m (x_-)_{-1} = 0, \\
&(\overline{y}_+)_m (x_-)_{-2} + (\overline{y}_+)_{m+1} (x_-)_{-1} = 0, \\
&(\overline{y}_+)_m (x_-)_{-3} + (\overline{y}_+)_{m+1} (x_-)_{-2} + (\overline{y}_+)_{m+2} (x_-)_{-1} = 0, \\
&\dots
\end{aligned}$$

Since $(\overline{y}_+)_m \neq 0$, the first of the above equalities implies that $(x_-)_{-1} = 0$, then the second of the above equalities yields $(x_-)_{-2} = 0$, and so on. Hence,

$$0 = (x_-)_{-1} = (x_-)_{-2} = (x_-)_{-3} = \dots$$

Thus $x_- = 0$, which implies that $L(a)x_+ = 0$.

Since $a \in M_{X(\mathbb{Z},w)}$ is invertible in the Banach algebra $M_{X(\mathbb{Z},w)}$, the operator $L(a)$ is invertible with the inverse $L(a^{-1})$. Therefore,

$$x_+ = L(a^{-1}) L(a) x_+ = L(a^{-1}) x = 0.$$

This contradicts our assumption. □

10 Proof of the main result

Let us prove Theorem 1.1. Recall that every rearrangement-invariant space is translation-invariant. So, we can apply Theorem 5.1. If $T(a) \in \Phi(X(\mathbb{Z}_+))$, then $L(a) \in \mathcal{GB}(X(\mathbb{Z}))$ by Theorem 5.1. In that case, $a \in \mathcal{G}M_{X(\mathbb{Z})}$ in view of Theorem 7.4 (with $w = 1$). Then, by Theorem 9.1 (again, with $w = 1$),

$$\alpha(T(a)) = \dim \operatorname{Ker} T(a) = 0 \quad \text{or} \quad \alpha(T(\overline{a})) = \dim \operatorname{Ker} T(\overline{a}) = 0. \tag{10.1}$$

It remains to prove the implication (iv) in (1.3). Since $T(a)$ is Fredholm, the subspaces $\operatorname{Ker} T(a)$ and $\operatorname{Im} T(a)$ have direct complements in $X(\mathbb{Z}_+)$ (see, e.g., [1, Lemma 4.39]).

Suppose

$$\kappa = \operatorname{Ind} T(a) = \alpha(T(a)) - \beta(T(a)) \leq 0. \tag{10.2}$$

If $\alpha(T(a)) > 0$, then $\alpha(T(\overline{a})) = 0$ (see (10.1)). Therefore, Lemma 3.1 and [1, Theorem 4.42] imply that

$$\beta(T(a)) = \alpha((T(a))^*) = \alpha(T(\overline{a})) = 0,$$

which contradicts (10.2). Hence, $\operatorname{Ker} T(a) = \{0\}$. Since the kernel of $T(a)$ is trivial and the image of $T(a)$ is complemented, the operator $T(a)$ is left-invertible on $X(\mathbb{Z}_+)$ in view of [11, Ch. 2, Theorem 5.1].

The case $\kappa \geq 0$ is considered in a similar fashion. Suppose

$$\kappa = \operatorname{Ind} T(a) = \alpha(T(a)) - \beta(T(a)) \geq 0. \tag{10.3}$$

If $\alpha(T(\overline{a})) > 0$, then $\alpha(T(a)) = 0$ (see (10.1))). Therefore, Lemma 3.1 and [1, Theorem 4.42] yield that

$$\beta(T(a)) = \alpha((T(a))^*) = \alpha(T(\overline{a})) > 0,$$

so $\alpha(T(a)) - \beta(T(a)) < 0$, which contradicts (10.3). Thus, $\beta(T(a)) = 0$. This implies that $\operatorname{Im} T(a) = X(\mathbb{Z}_+)$. Since $T(a)$ is surjective and the kernel of $T(a)$ is complemented, it follows from [11, Ch. 2, Theorem 5.2] that $T(a)$ is right-invertible on $X(\mathbb{Z}_+)$.

Finally, if $\kappa = 0$, then it follows from the above that $T(a)$ is simultaneously left-invertible and right-invertible. Thus $T(a)$ is invertible on $X(\mathbb{Z}_+)$ if and only if it is Fredholm and its index is equal to zero. Formula (1.1) is just a reformulation of this statement. □

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