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## ЗМІСТ

# Invertibility in Banach algebras of functional operators with non-Carleman shifts

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**Abstract**

We prove the inverse closedness of the Banach algebra $\mathfrak{A}_p$ of functional operators with non-Carleman shifts, which have only two fixed points, in the Banach algebra of all bounded linear operators on $L^p$.
We suppose that $1 \le p \le \infty$ and the generators of the algebra $\mathfrak{A}_p$ have essentially bounded data. An invertibility criterion for functional operators in $\mathfrak{A}_p$ is obtained in terms of the invertibility of a family of discrete operators on $l^p$. An effective invertibility criterion is established for binomial difference operators with $l^\infty$ coefficients on the spaces $l^p$. Using the reduction to binomial difference operators, we give effective criteria of invertibility for binomial functional operators on the spaces $L^p$.

## 1. Introduction

Let $\alpha$ be an orientation-preserving homeomorphism of $[0,1]$ onto itself, which has only two fixed points 0 and 1. So, $\alpha(0)=0$ and $\alpha(1)=1$, but $\alpha(t) \ne t$ for $t \in \mathbb{I} := (0,1)$. The function $\alpha$ is referred to as a *shift*. Since the shift $\alpha$ does not satisfy the generalized Carleman condition (see, e.g., [11, 14]), $\alpha$ is called a *non-Carleman shift.*

Denote by $\beta := \alpha_{-1}$ the inverse function to $\alpha$. Since $\alpha$ and $\beta$ strictly monotonically increase on $[0,1]$, their derivatives exist and are positive almost everywhere on $\mathbb{I}$. If $\log \alpha' \in L^\infty := L^\infty(\mathbb{I})$, then the shift operator $U_\alpha$ defined by

$$(U_\alpha \varphi)(t) := (\alpha'(t))^{1/p} \varphi[\alpha(t)], \quad t \in \mathbb{I},$$

is an isometry on the Lebesgue space $L^p := L^p(\mathbb{I})$ for every $p \in [1,\infty]$. Its inverse is given by $U_\alpha^{-1} = U_\beta$. Put $\alpha_0(t) = t$ and $\alpha_n(t) = \alpha[\alpha_{n-1}(t)]$ for $n \in \mathbb{Z}$ and $t \in [0,1]$. Then $U_\alpha^n = U_{\alpha_n}$ for $n \in \mathbb{Z}$.

Fix an arbitrary point $x \in \mathbb{I}$. Let $\gamma$ be a half-open segment with endpoints $x$ and $\alpha(x)$ such that $x \in \gamma$ but $\alpha(x) \notin \gamma$. Notice that either $x < \alpha(x)$ and then 1 is the attracting point of $\alpha$, or $\alpha(x) < x$ and then 0 is the attracting point of $\alpha$ (see, e.g., [11, Chapter 1, Section 3]). The shift $\alpha$ generates the cyclic group, which is algebraically isomorphic to the group $\mathbb{Z}$ of all integer numbers. In view of this important property, we can consider the following orbital decomposition

$$\mathbb{I} = \bigcup_{n \in \mathbb{Z}} \alpha_n(\gamma), \quad \alpha_i(\gamma) \cap \alpha_j(\gamma) = \emptyset \quad (i \ne j). \tag{1}$$

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For a Banach space $X$, let $\mathfrak{B}(X)$ be the Banach algebra of all bounded linear operators on $X$. Denote by $\mathfrak{A}_p$ the smallest Banach subalgebra of $\mathfrak{B}(L^p), 1 \leq p \leq \infty$, containing the operators $U_\alpha, U_\alpha^{-1}$ and all the operators of multiplication by functions in $L^\infty$. Thus, in what follows, always $p \in [1, \infty]$ and the generators of $\mathfrak{A}_p$ have $L^\infty$ data. Following [1, 11], operators $A \in \mathfrak{A}_p$ are called *functional operators*.

Functional operators and their discrete analogues play an important role in the theory of functional differential operators (see [1], [13] and the references therein), theory of singular integral operators, convolution type operators and pseudodifferential operators with shifts (see, e.g., [2], [11], [14], [19]), theory of dynamical systems [5], etc.

The paper is devoted to several facts about the invertibility of operators $A \in \mathfrak{A}_p$. Using the Bochner-Phillips theorem, in Section 2 we prove the inverse closedness of $\mathfrak{A}_p$ in $\mathfrak{B}(L^p)$. Similar results for Wiener algebras of functional operators on Lebesgue spaces over locally compact commutative groups, which are based on an idea of [18], were established in the most general form in [12, Chapter 2] (see also [3, Section 26] for the inverse closedness of Wiener subalgebras in $C^*$-algebras of abstract functional operators with discrete commutative groups of shifts). Note also that in the case of piecewise continuous coefficients, the inverse closedness of Banach algebras of functional operators with discrete subexponential groups of shifts on Lebesgue spaces over piecewise smooth contours was obtained by other methods in [7], [8].

In Section 3, making use of a decomposition of the space $L^p$ into the direct integral of the spaces $l^p$ and generalizing the approach of [16, Section 26], we get an invertibility criterion for operators $A \in \mathfrak{A}_p$ in terms of the invertibility of a family of discrete operators on $l^p$. In the case of piecewise continuous coefficients analogous results for $A \in \mathfrak{A}_p$ $(1 < p < \infty)$ and for functional operators with discrete subexponential groups of shifts on the spaces $L^p$ $(1 \leq p \leq \infty)$ were obtained in [9], [15] and in [7], [8], respectively. The latter results were extended to $C^*$-algebras of functional operators with discrete amenable groups of shifts in [6], [7] (see also [1, Chapter 3] and [3, Section 21]).

In Section 4 we get an effective invertibility criterion for binomial difference operators with $l^\infty$ coefficients on the spaces $l^p$, $1 \leq p \leq \infty$ (cf. [3, Theorem 17.3] for a related general $C^*$-algebra result).

Finally, in Section 5 we obtain two effective invertibility criteria for binomial functional operators in $\mathfrak{A}_p$ on the basis of Sections 3 and 4. Those criteria are qualitatively different from that for binomial functional operators with data in $C[0, 1]$ (see [11, Chapter 2, Section 4]).

## 2. Inverse closedness of $\mathfrak{A}_p$ in $\mathfrak{B}(L^p)$

### 2.1. The Bochner-Phillips theorem

For a unital Banach algebra $\mathcal{B}$, let $\mathcal{GB}$ denote the group of all invertible elements in $\mathcal{B}$. Let $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}$ with the same identity element. The algebra $\mathcal{A}$ is said to be *inverse closed* in $\mathcal{B}$, if for every $a \in \mathcal{A}$ such that $a \in \mathcal{GB}$, we have $a \in \mathcal{GA}$.

Let $G$ be a discrete commutative group, $K = K(G)$ its character group (all characters are continuous, and $K$ is a compact group), and $\mathcal{L}$ a unital Banach algebra. Let $C(K, \mathcal{L})$ be the Banach algebra of all continuous functions on $K$ with values in $\mathcal{L}$, and let $W(K, \mathcal{L})$ denote the subalgebra of $C(K, \mathcal{L})$ which consists of the functions

$$T : K \to \mathcal{L}, \quad \chi \mapsto \sum_{\lambda \in G} \chi(\lambda) b_\lambda,$$

where $b_\lambda \in \mathcal{L}$, $\chi(\lambda)$ is the value of the character $\chi$ at the element $\lambda \in G$, and

$$\|T\|_W := \sum_{\lambda \in G} \|b_\lambda\|_{\mathcal{L}} < \infty.$$

**Theorem 1.** (see [4, Theorem 4] and [12, Theorem 1.4.12]). *The algebra $W(K, \mathcal{L})$ is inverse closed in $C(K, \mathcal{L})$, that is, if $T(\chi)$ has an inverse in $\mathcal{L}$ for every $\chi \in K$, then $T$ has an inverse in $W(K, \mathcal{L})$.*

## 2.2. Operators which commute with operators of multiplication

The following statement is known, but for the convenience of readers, we give its proof.

**Proposition 2.** *Suppose $1 \le p < \infty$ and $\Delta$ is a finite segment of the real line. Every operator $D \in \mathfrak{B}(L^p(\Delta))$ which commutes with all the operators $\varphi I$, where $\varphi \in C(\Delta)$, has the form $D = dI$, where $d \in L^\infty(\Delta)$.*

***Proof.*** By assumption, $D(gf) = gDf$ for any $g \in C(\Delta)$ and any $f \in L^p(\Delta)$. In particular, taking $f = 1$, we get $Dg = ag$ where $a := D(1) \in L^p(\Delta)$. Thus $D$ is the operator of multiplication by the function $a \in L^p(\Delta)$ at least on the subset of continuous functions. Since $D \in \mathfrak{B}(L^p(\Delta))$, one can show (assuming the contrary) that $a \in L^\infty(\Delta)$. Hence for every $g \in C(\Delta)$,

$$\|Dg\|_{L^p(\Delta)} = \|ag\|_{L^p(\Delta)} \le \|a\|_{L^\infty(\Delta)} \|g\|_{L^p(\Delta)}. \tag{2}$$

Since the set $C(\Delta)$ is dense in $L^p(\Delta)$, the inequality (2) allows us to extend $D$ by continuity to the whole space $L^p(\Delta)$ as the operator of multiplication by the function $a \in L^\infty(\Delta)$. ■

## 2.3. Inverse closedness of the Wiener algebra of functional operators

Let $\mathbb{J}$ be a measurable $\alpha$-invariant (that is, $\alpha(\mathbb{J}) = \mathbb{J}$) subset of $\mathbb{I}$. We denote by $\mathcal{W}_p(\mathbb{J})$ $(1 \le p \le \infty)$ the set of all operators $A \in \mathfrak{B}(L^p(\mathbb{J}))$ which can be represented in the form

$$A = \sum_{n \in \mathbb{Z}} a_n U_\alpha^n \tag{3}$$

where $\log \alpha' \in L^\infty(\mathbb{I})$, $a_n \in L^\infty(\mathbb{J})$ and

$$\|A\|_{\mathcal{W}_p(\mathbb{J})} := \sum_{n \in \mathbb{Z}} \|a_n\|_{L^\infty(\mathbb{J})} < \infty. \tag{4}$$

It is easy to see that $\mathcal{W}_p(\mathbb{J})$ is a Banach algebra with the norm $\|\cdot\|_{\mathcal{W}_p(\mathbb{J})}$. This algebra is called the *Wiener algebra* of functional operators. If $\mathbb{J} = \mathbb{I}$, we will write $\mathcal{W}_p$ instead of $\mathcal{W}_p(\mathbb{I})$.

**Theorem 3.** *The Wiener algebra $\mathcal{W}_p$ is inverse closed in $\mathfrak{B}(L^p)$.*

***Proof.*** First let $1 \le p < \infty$. For $G = \mathbb{Z}$, its character group $K(G)$ coincides with the unit circle $\mathbb{T}$, and $z(n) = z^n$ for $z \in \mathbb{T}$, $n \in \mathbb{Z}$. We apply Theorem 1 to $\mathcal{L} = \mathfrak{B}(L^p)$, $G = \mathbb{Z}$, and $K(G) = \mathbb{T}$.

For an invertible operator $A \in \mathcal{W}_p$ given by (3), consider the function

$$a : \mathbb{T} \to \mathcal{L}, \quad a(z) := \sum_{n \in \mathbb{Z}} z^n a_n U_\alpha^n. \tag{5}$$

It follows from (3)–(4) that $a \in W(\mathbb{T}, \mathcal{L})$. For each $z \in \mathbb{T}$, let $\varphi_z$ denote a function in $\mathcal{G}L^\infty$ which satisfies

$$\varphi_z[\alpha_n(t)] = z^n \varphi_z(t), \quad t \in \gamma, \quad n \in \mathbb{Z}. \tag{6}$$

Put $\Phi_z := \varphi_z I$. In view of (6),

$$\Phi_z^{-1} U_\alpha \Phi_z = z U_\alpha, \quad z \in \mathbb{T}. \tag{7}$$

For every $z \in \mathbb{T}$, we infer from (5) and (7) that

$$a(z) = \sum_{n\in\mathbb{Z}} z^n a_n U_\alpha^n = \Phi_z^{-1} \left( \sum_{n\in\mathbb{Z}} a_n U_\alpha^n \right) \Phi_z = \Phi_z^{-1} A \Phi_z. \tag{8}$$

Since $A$ is invertible in $\mathcal{L}$, $a(z)$ is invertible in $\mathcal{L}$ for every $z \in \mathbb{T}$, in view of (8). Then, due to Theorem 1, $a$ is invertible in $W(\mathbb{T}, \mathcal{L})$. Thus, its inverse has the form

$$a^{-1}(\eta) = \sum_{n\in\mathbb{Z}} \eta^n \mathcal{D}_n, \quad \eta \in \mathbb{T}, \tag{9}$$

where $\mathcal{D}_n \in \mathcal{L}$ for every $n \in \mathbb{N}$, and $\sum_{n\in\mathbb{Z}} \|\mathcal{D}_n\|_{\mathcal{L}} < \infty$. Since $A = a(1)$, from (9) we get

$$A^{-1} = a^{-1}(1) = \sum_{n\in\mathbb{Z}} \mathcal{D}_n. \tag{10}$$

Let us show that there exist functions $d_n \in L^\infty$ such that $\mathcal{D}_n = d_n U_\alpha^n$ for all $n \in \mathbb{Z}$. It follows from (8) that

$$\Phi_\zeta^{-1} a(z) \Phi_\zeta = \sum_{n\in\mathbb{Z}} \Phi_\zeta^{-1} \Big( z^n a_n U_\alpha^n \Big) \Phi_\zeta = \sum_{n\in\mathbb{Z}} \zeta^n z^n a_n U_\alpha^n = a(\zeta z), \quad z, \zeta \in \mathbb{T}.$$

Therefore,

$$\Phi_\zeta^{-1} a^{-1}(z) \Phi_\zeta = a^{-1}(\zeta z), \quad z, \zeta \in \mathbb{T}. \tag{11}$$

Letting $\eta = z$ and $\eta = \zeta z$ in (9), we obtain from (11) that

$$\sum_{n\in\mathbb{Z}} z^n \Phi_\zeta^{-1} \mathcal{D}_n \Phi_\zeta = \sum_{n\in\mathbb{Z}} z^n \zeta^n \mathcal{D}_n, \quad z, \zeta \in \mathbb{T}. \tag{12}$$

Considering (12) as a function of $z$ and comparing the coefficients of $z^n$, we get $\Phi_\zeta^{-1} \mathcal{D}_n \Phi_\zeta = \zeta^n \mathcal{D}_n$ for every $n \in \mathbb{Z}$, and thus

$$\Phi_\zeta^{-1} \mathcal{D}_n \Phi_\zeta U_\alpha^{-n} = \zeta^n \mathcal{D}_n U_\alpha^{-n}, \quad n \in \mathbb{Z}. \tag{13}$$

It follows from (7) that

$$\Phi_\zeta U_\alpha^{-n} = \zeta^n U_\alpha^{-n} \Phi_\zeta, \quad \zeta \in \mathbb{T}, \quad n \in \mathbb{Z}. \tag{14}$$

Combining (13)–(14), we obtain

$$\Phi_\zeta^{-1} \mathcal{D}_n U_\alpha^{-n} \Phi_\zeta = \mathcal{D}_n U_\alpha^{-n}, \quad n \in \mathbb{Z}. \tag{15}$$

Taking into account (1), consider the space $L^p = L^p(\mathbb{I})$ as the direct sum of its subspaces $L^p(\alpha_i(\gamma))$, $i \in \mathbb{Z}$. The operator $D_n := \mathcal{D}_n U_\alpha^{-n} \in \mathcal{L}$ can be represented in this direct sum of subspaces by the operator matrix $\{\Pi_i D_n \Pi_j\}_{i,j=-\infty}^{+\infty}$, where $\Pi_k := (\chi_\gamma \circ \alpha_k)I$ $(k \in \mathbb{Z})$ and $\chi_\gamma$ is the characteristic function of $\gamma$. Since (15) is valid for every function $\varphi_\zeta \in \mathcal{G}L^\infty$ satisfying (6), we choose $\varphi_\zeta(t) = \zeta^i$ for $t \in \alpha_i(\gamma)$, $i \in \mathbb{Z}$; and from (15) we get

$$\Pi_i \zeta^{-i} D_n \zeta^j \Pi_j = \Pi_i D_n \Pi_j, \quad i, j, n \in \mathbb{Z}.$$

Thus, $\zeta^{j-i}\Pi_i D_n \Pi_j = \Pi_i D_n \Pi_j$ whenever $i, j, n \in \mathbb{Z}$. Choosing $\zeta \neq 1$, we get $\Pi_i D_n \Pi_j = 0$ for $i \neq j$. Hence,

$$D_n = \operatorname{diag}\{\Pi_i D_n \Pi_i\}_{i=-\infty}^{+\infty}, \quad n \in \mathbb{Z}. \tag{16}$$

We are left with proving that each operator $\Pi_i D_n \Pi_i$ $(i, n \in \mathbb{Z})$ is an operator of multiplication by a function in $L^\infty(\alpha_i(\gamma))$. Without loss of generality assume $i = 0$. It follows from (15) that $\Phi_\zeta^{-1} D_n \Phi_\zeta = D_n$, whence, in view of (6) and (16), we get

$$\varphi^{-1}\chi_\gamma D_n \chi_\gamma \varphi I = \chi_\gamma D_n \chi_\gamma I \tag{17}$$

for every $\varphi \in \mathcal{G}L^\infty(\gamma)$. Thus, by (17) and Proposition 2, there exists a function $d_{n,0} \in L^\infty(\gamma)$ such that

$$\Pi_0 D_n \Pi_0 = \chi_\gamma D_n \chi_\gamma I = d_{n,0} I \in \mathfrak{B}(L^p(\gamma)), \quad n \in \mathbb{Z}.$$

Hence, for every $i, n \in \mathbb{Z}$, there are functions $d_{n,i} \in L^\infty(\alpha_i(\gamma))$ for which $\Pi_i D_n \Pi_i = d_{n,i} I \in \mathfrak{B}(L^p(\alpha_i(\gamma)))$. It follows from (16) that $D_n = d_n I$, where $d_n(t) = d_{n,i}(t)$ for $t \in \alpha_i(\gamma)$, $i \in \mathbb{Z}$. Finally,

$$\mathcal{D}_n = D_n U_\alpha^n = d_n U_\alpha^n, \quad \text{where} \quad d_n \in L^\infty \quad (n \in \mathbb{Z}). \tag{18}$$

Combining (10) and (18), we complete the proof for $p \in [1, \infty)$.

Let $p = \infty$. Obviously, for every operator $A \in \mathcal{W}_\infty$ of the form (3), there exists the operator

$$B = \sum_{n \in \mathbb{Z}} (\overline{a}_n \circ \alpha_{-n}) U_\alpha^{-n} \in \mathcal{W}_1$$

such that $A = B^*$. But according to [10, Chapter III, Theorem 5.30], the invertibility of $A$ on $L^\infty$ is equivalent to the invertibility of $B$ on $L^1$. Thus, if $A \in \mathcal{W}_\infty$ is invertible, then $B \in \mathcal{W}_1$ is invertible and, by the part already proved, $B^{-1} \in \mathcal{W}_1$. Then $A^{-1} = (B^{-1})^* \in \mathcal{W}_\infty$. ■

**Corollary 4.** *If $\mathbb{J} \subset \mathbb{I}$ is an $\alpha$-invariant subset of positive measure, then the Wiener algebra $\mathcal{W}_p(\mathbb{J})$ is inverse closed in $\mathfrak{B}(L^p(\mathbb{J}))$.*

***Proof.*** Setting $\widetilde{a}_0 := \chi_\mathbb{J} a_0 + \chi_{\mathbb{I}\setminus\mathbb{J}}$ and $\widetilde{a}_n := \chi_\mathbb{J} a_n$ $(n \neq 0)$, we get the operator

$$\widetilde{A} := \sum_{n \in \mathbb{Z}} \widetilde{a}_n U_\alpha^n \in \mathcal{W}_p.$$

Since $\mathbb{J} \subset \mathbb{I}$ is $\alpha$-invariant, $\widetilde{A} = \operatorname{diag}\{A, I\}$ in $\mathfrak{B}(L^p) = \mathfrak{B}(L^p(\mathbb{J}) \dot{+} L^p(\mathbb{I}\setminus\mathbb{J}))$. Hence, $\widetilde{A}$ is invertible on $L^p$ whenever $A$ is invertible on $L^p(\mathbb{J})$. Applying Theorem 3 to $\widetilde{A}$, we get $A^{-1} = (\widetilde{A})^{-1}|_{L^p(\mathbb{J})} \in \mathcal{W}_p(\mathbb{J})$. ■

## 2.4. Inverse closedness of $\mathfrak{A}_p$ in $\mathfrak{B}(L^p)$

Since the Wiener algebra $\mathcal{W}_p$ is dense in the algebra $\mathfrak{A}_p$, from Theorem 3 we immediately get the following.

**Corollary 5.** *The algebra $\mathfrak{A}_p$ is inverse closed in $\mathfrak{B}(L^p)$.*

## 3. Relations with discrete operators

### 3.1. Direct integral of spaces $l^p$

Let $1 \le p \le \infty$. By analogy with [16, Chapter V, Section 26.5], we say that a vector-valued function

$$f : \gamma \to l^p, \quad t \mapsto \{f_n(t)\}_{n\in\mathbb{Z}} \tag{19}$$

is *measurable* if for an arbitrary vector $\eta \in l^q$ $(p^{-1} + q^{-1} = 1)$, the complex-valued function

$$f_\eta : \gamma \to \mathbb{C}, \quad t \mapsto (f(t), \eta) := \sum_{n\in\mathbb{Z}} f_n(t)\overline{\eta}_n$$

is measurable on $\gamma$ with respect to the Lebesgue measure. Since

$$\|f(t)\|_{l^p} = \Big(\sum_{k\in\mathbb{Z}} |(f(t), e_k)|^p\Big)^{1/p}, \quad 1 \le p < \infty, \qquad \|f(t)\|_{l^\infty} = \sup_{k\in\mathbb{Z}} |(f(t), e_k)|,$$

where $e_k \in l^q$, $(e_k)_k = 1$ and $(e_k)_n = 0$ $(n \ne k)$, it follows from [16, Chapter I, Section 6.10, VI and VII] that for a measurable vector-valued function (19), the non-negative function $t \mapsto \|f(t)\|_{l^p}$ is measurable on $\gamma$ as well.

Further we consider the Banach space $L^p(\gamma, l^p)$ of all measurable vector-valued functions $f : \gamma \to l^p$ with the norm

$$\|f\|_{L^p(\gamma,l^p)} := \Big(\int_\gamma \|f(t)\|^p_{l^p}\, dt\Big)^{1/p}, \quad 1 \le p < \infty, \qquad \|f\|_{L^\infty(\gamma,l^\infty)} := \operatorname*{ess\,sup}_{t\in\gamma} \|f(t)\|_{l^\infty}.$$

Thus, a function $f \in L^p(\gamma, l^p)$ is defined on $\gamma$ with the possible exclusion of a set of measure zero. For $1 \le p < \infty$, the space $L^p(\gamma, l^p)$ is called the *direct integral of spaces $l^p$*.

Let $\mathcal{L}(l_p)$ be the algebra of all linear (but in general unbounded) operators acting on $l_p$. Following [16, Chapter V, Section 26.5], an operator-valued function $\mathcal{A} : \gamma \to \mathcal{L}(l^p)$ is said to be *measurable* if $\mathcal{A}(t) \in \mathcal{L}(l^p)$ is defined for all $t \in \gamma$ with the possible exclusion of a set of measure zero, and for an arbitrary measurable vector-valued function $\xi : \gamma \to l^p$, the vector-valued function $\mathcal{A}\xi : \gamma \to l^p$, $t \mapsto \mathcal{A}(t)\xi(t)$ is defined for almost all $t \in \gamma$ and is measurable on $\gamma$.

If $\mathcal{A} : \gamma \to \mathcal{L}(l^p)$ is a measurable operator-valued function, then according to [16, Chapter V, Section 26.5, II], we conclude that for $p \in [1, \infty)$ the non-negative function $t \mapsto \|\mathcal{A}(\cdot)\|_{\mathfrak{B}(l^p)}$ is measurable on $\gamma$ too. If $p = \infty$, then again the function

$$\|\mathcal{A}(\cdot)\|_{\mathfrak{B}(l^\infty)} = \sup_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |(\mathcal{A}(\cdot)e_n, e_i)|$$

is measurable on $\gamma$ as the supremum of the sequence of the non-negative functions $\sum_{n\in\mathbb{Z}} |\mathfrak{a}_{in}(\cdot)|$ which are measurable on $\gamma$ together with $\mathfrak{a}_{in}(\cdot) := (\mathcal{A}(\cdot)e_n, e_i)$.

**Proposition 6.** *A measurable operator-valued function $\mathcal{A} : \gamma \to \mathcal{L}(l^p)$, $t \mapsto \mathcal{A}(t)$ defines a bounded linear operator*

$$M_\mathcal{A} : L^p(\gamma, l^p) \to L^p(\gamma, l^p), \quad (M_\mathcal{A} f)(t) = \mathcal{A}(t)f(t), \quad t \in \gamma,$$

*if and only if the function* $t \mapsto \|\mathcal{A}(t)\|_{\mathfrak{B}(l^p)}$ *belongs to* $L^\infty(\gamma)$*. In that case,*

$$\|M_{\mathcal{A}}\|_{\mathfrak{B}(L^p(\gamma,l^p))} = \Big\| \|\mathcal{A}(\cdot)\|_{\mathfrak{B}(l^p)} \Big\|_{L^\infty(\gamma)}.$$

***Proof.*** Sufficiency for all $p \in [1,\infty]$ and necessity for $p \in [1,\infty)$ are proved by analogy with [16, Chapter V, Section 26.5, III]. Let us prove necessity for $p = \infty$.

Suppose $M_{\mathcal{A}} = \mathcal{A}(\cdot)I \in \mathfrak{B}(L^\infty(\gamma, l^\infty))$. Then

$$\|\mathcal{A}(t)\|_{\mathfrak{B}(l^\infty)} = \sup_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\mathfrak{a}_{in}(t)| \quad \text{for almost all} \quad t \in \gamma.$$

Assume that the function $t \mapsto \|\mathcal{A}(t)\|_{\mathfrak{B}(l^\infty)}$ does not belong to $L^\infty(\gamma)$. Then for every $m \in \mathbb{N}$ there is a measurable subset $\gamma_m \subset \gamma$ such that $\operatorname{mes}\gamma_m \neq 0$ and

$$\sup_{i\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\mathfrak{a}_{in}(t)| > m+1 \quad \text{for almost all} \quad t \in \gamma_m.$$

Then there are $i_m \in \mathbb{Z}$ and measurable subsets $\Delta_m \subset \gamma_m$ such that $\operatorname{mes}\Delta_m \neq 0$ and

$$\|\mathcal{A}(t)\|_{\mathfrak{B}(l^\infty)} \geq \sum_{n\in\mathbb{Z}} |\mathfrak{a}_{i_m n}(t)| \geq m \quad \text{for almost all} \quad t \in \Delta_m. \tag{20}$$

For $t \in \gamma$, $m \in \mathbb{N}$, and $n \in \mathbb{Z}$, set

$$f_n^{(m)}(t) := \begin{cases} \overline{\mathfrak{a}_{i_m n}(t)}/|\mathfrak{a}_{i_m n}(t)| & \text{if} \quad \mathfrak{a}_{i_m n}(t) \neq 0, \\ 1 & \text{if} \quad \mathfrak{a}_{i_m n}(t) = 0. \end{cases} \tag{21}$$

As every function $f_n^{(m)}$ is measurable on $\gamma$ together with $\mathfrak{a}_{i_m n}(\cdot) = (\mathcal{A}(\cdot)e_n, e_{i_m})$, the vector functions

$$f^{(m)} : \gamma \to l^\infty, \quad t \mapsto \{f_n^{(m)}(t)\}_{n\in\mathbb{Z}}$$

belong to $L^\infty(\gamma, l^\infty)$ and have norm 1. Hence, from (20)–(21) we get for $m \in \mathbb{N}$,

$$\|\mathcal{A}\|_{\mathfrak{B}(L^\infty(\gamma,l^\infty))} \geq \|\mathcal{A}(\cdot)f^{(m)}(\cdot)\|_{L^\infty(\gamma,l^\infty)} \geq \operatorname*{ess\,inf}_{t\in\Delta_m} \|\mathcal{A}(t)f^{(m)}(t)\|_{l^\infty}$$

$$\geq \operatorname*{ess\,inf}_{t\in\Delta_m} \left( \sup_{i\in\mathbb{Z}} \Big| \sum_{n\in\mathbb{Z}} \mathfrak{a}_{in}(t) f_n^{(m)}(t) \Big| \right) \geq \operatorname*{ess\,inf}_{t\in\Delta_m} \Big| \sum_{n\in\mathbb{Z}} \mathfrak{a}_{i_m n}(t) f_n^{(m)}(t) \Big| = \operatorname*{ess\,inf}_{t\in\Delta_m} \sum_{n\in\mathbb{Z}} |\mathfrak{a}_{i_m n}(t)| \geq m,$$

which contradicts the boundedness of the operator $M_{\mathcal{A}}$ on $L^\infty(\gamma, l^\infty)$. Thus, the function $t \mapsto \|\mathcal{A}(t)\|_{\mathfrak{B}(l^\infty)}$ belongs to $L^\infty(\gamma)$. ∎

## 3.2. Invertibility of functional operators in terms of discrete operators

Below we apply the results of Section 3.1 to study the invertibility of functional operators.

Introduce the isometric isomorphism

$$\sigma : L^p(\mathbb{I}) \to L^p(\gamma, l^p), \quad f \mapsto \psi \quad \text{where} \quad \psi : \gamma \to l^p, \quad t \mapsto \{(U_\alpha^n f)(t)\}_{n\in\mathbb{Z}}. \tag{22}$$

**Lemma 7.** *If* $A \in \mathfrak{A}_p$*, then the operator* $\widehat{A} := \sigma A \sigma^{-1} \in \mathfrak{B}(L^p(\gamma, l^p))$ *is given by* $(\widehat{A}\psi)(t) = \mathcal{A}(t)\psi(t)$ *for almost all* $t \in \gamma$*, where* $\mathcal{A}$ *is an operator-valued function in* $L^\infty(\gamma, \mathfrak{B}(l^p))$ *which has the form*

$$\mathcal{A}(t) = (a_{j-i}[\alpha_i(t)])_{i,j\in\mathbb{Z}} \quad \textit{for almost all} \quad t \in \gamma. \tag{23}$$

***Proof.*** First suppose $A \in \mathcal{W}_p$, that is, $A$ has the form (3) and satisfies (4) with $\mathbb{J} = \mathbb{I}$. For $\psi \in L^p(\gamma, l^p)$ and almost all $t \in \gamma$, we get

$$(\widehat{A}\psi)(t) = (\sigma A \sigma^{-1}\psi)(t) = \Big\{\sum_{n\in\mathbb{Z}} a_n[\alpha_i(t)]\Big(U_\alpha^{i+n}\sigma^{-1}\psi\Big)(t)\Big\}_{i\in\mathbb{Z}} = \Big\{\sum_{n\in\mathbb{Z}} a_n[\alpha_i(t)]\psi_{i+n}(t)\Big\}_{i\in\mathbb{Z}}$$

$$= \Big\{\sum_{j\in\mathbb{Z}} a_{j-i}[\alpha_i(t)]\psi_j(t)\Big\}_{i\in\mathbb{Z}} = \Big(a_{j-i}[\alpha_i(t)]\Big)_{i,j\in\mathbb{Z}} \{\psi_j(t)\}_{j\in\mathbb{Z}} = \mathcal{A}(t)\psi(t),$$

that is, the operator $\widehat{A} \in \mathfrak{B}(L^p(\gamma, l^p))$ is the operator of multiplication by the matrix function (23) where $a_n$ are the coefficients of the operator $A$.

Clearly, the operators $\mathcal{A}(t)$ belong to $\mathcal{L}(l^p)$ for almost all $t \in \gamma$. Since all the entries $a_{j-i} \circ \alpha_i$ of the matrix function $\mathcal{A}$ belong to $L^\infty(\gamma)$, we have for almost all $t \in \gamma$ and all $i, j \in \mathbb{Z}$,

$$|a_{j-i}[\alpha_i(t)]| \le \|a_{j-i} \circ \alpha_i\|_{L^\infty(\gamma)} \le \|a_{j-i}\|_{L^\infty}. \tag{24}$$

It follows from (24) and (4) that for almost all $t \in \gamma$,

$$\|\mathcal{A}(t)\|_{\mathfrak{B}(l^p)} \le \sum_{n\in\mathbb{Z}} \|a_n\|_{L^\infty} = \|A\|_{\mathcal{W}_p} < \infty. \tag{25}$$

Hence, $\mathcal{A}(t) \in \mathfrak{B}(l^p)$ for those $t$.

Let us show that the operator-valued function $\mathcal{A} : \gamma \mapsto \mathfrak{B}(l^p)$ is measurable. For every measurable vector-valued function $\xi : \gamma \to l^p$ and every $\eta \in l^q$, it follows from (24) and Hölder's inequality that

$$\Big|\sum_{n\in\mathbb{Z}} (\mathcal{A}(t)\xi(t))_n \overline{\eta}_n\Big| \le \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \|a_{j-i}\|_{L^\infty} |\xi_j(t)||\eta_i| \le \sum_{n\in\mathbb{Z}} \|a_n\|_{L^\infty} \sum_{j\in\mathbb{Z}} |\xi_j(t)|\,|\eta_{j-n}|$$

$$\le \|A\|_{\mathcal{W}_p} \|\xi(t)\|_{l^p} \|\eta\|_{l^q} < \infty \quad \text{(a.e. on } \gamma). \tag{26}$$

Since the functions $t \mapsto a_{j-i}[\alpha_i(t)]$ and $t \mapsto \xi_j(t)\overline{\eta}_i$ are measurable on $\gamma$, it follows from (26) and [16, Chapter I, Section 6.10, VII] that the function

$$\mathcal{A}_{\xi,\eta} : \gamma \to \mathbb{C}, \quad t \mapsto \sum_{n\in\mathbb{Z}} (\mathcal{A}(t)\xi(t))_n \overline{\eta}_n = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} a_{j-i}[\alpha_i(t)]\xi_j(t)\overline{\eta}_i,$$

is well defined for almost all $t \in \gamma$ and is measurable on $\gamma$. This means, by definition, that the operator-valued function $\mathcal{A}$ is measurable on $\gamma$. As was shown before, $\mathcal{A}$ defines a bounded linear operator

$$\widehat{A} : L^p(\gamma, l^p) \to L^p(\gamma, l^p), \quad (\widehat{A}\psi)(t) = \mathcal{A}(t)\psi(t), \quad t \in \gamma.$$

Then due to necessity in Proposition 6, the function $t \mapsto \|\mathcal{A}(t)\|_{\mathfrak{B}(l^p)}$ belongs to $L^\infty(\gamma)$ and

$$\Big\| \|\mathcal{A}(\cdot)\|_{\mathfrak{B}(l^p)} \Big\|_{L^\infty(\gamma)} = \|\widehat{A}\|_{\mathfrak{B}(L^p(\gamma, l^p))}.$$

Thus, the assertion is proved for the algebra $\mathcal{W}_p$ which is dense in $\mathfrak{A}_p$. To get the assertion for the whole algebra $\mathfrak{A}_p$ it only remains to make use of the extension by continuity.

Indeed, if $A \in \mathfrak{A}_p \setminus \mathcal{W}_p$, then there is a sequence $\{A_n\} \subset \mathcal{W}_p$ such that $\lim_{n\to\infty} \|A - A_n\|_{\mathfrak{B}(L^p)} = 0$. In that case the operator-valued function $\mathcal{A} : \gamma \to \mathfrak{B}(l^p)$ is defined as the limit of the sequence of the operator-valued functions $\mathcal{A}_n \in L^\infty(\gamma, \mathfrak{B}(l^p))$ in $L^\infty(\gamma, \mathfrak{B}(l^p))$. Clearly, $\mathcal{A}$ also belongs to $L^\infty(\gamma, \mathfrak{B}(l^p))$, $\mathcal{A}$ has the form (23), and $\mathcal{A}$ is independent of a choice of the sequence $\{\mathcal{A}_n\} \subset \mathcal{W}_p$. The functions $t \mapsto a_{ij}(t) := (\mathcal{A}(t)e_j, e_i)$ are equal to $\lim_{n\to\infty} a_{ij}^{(n)}(\cdot)$ in $L^\infty(\gamma)$, where $a_{ij}^{(n)}(t) = (\mathcal{A}_n(t)e_j, e_i)$ are the entries of the band-dominated (see, e.g., [17]) operators $\mathcal{A}_n(t) \in \mathfrak{B}(l^p)$. ∎

Let $\delta_{i,j}$ stand for the Kronecker delta. From (23) it follows directly that the operator-valued functions $\mathcal{A} \in L^\infty(\gamma, \mathfrak{B}(l^p))$, associated with operators $A \in \mathfrak{A}_p$ by Lemma 7, satisfy the following important relation evoked by (1).

**Lemma 8.** *If $A \in \mathfrak{A}_p$, then for almost all $t \in \gamma$ and all $n \in \mathbb{Z}$,*

$$\mathcal{A}[\alpha_n(t)] = \mathcal{V}^n \mathcal{A}(t) \mathcal{V}^{-n} \quad where \quad \mathcal{V} := (\delta_{i,j-1})_{i,j\in\mathbb{Z}}.$$

Now we are in a position to formulate the main result of this section.

**Theorem 9.** *An operator $A \in \mathfrak{A}_p$ is invertible on $L^p$ if and only if for almost all $t \in \gamma$ the operators $\mathcal{A}(t)$, given by Lemma 7, are invertible on $l^p$ and the operator-valued function $\mathcal{A}^{-1} : \gamma \to \mathfrak{B}(L^p)$, $t \mapsto (\mathcal{A}(t))^{-1}$ belongs to $L^\infty(\gamma, \mathfrak{B}(l^p))$, that is, $\mathcal{A}^{-1}$ is measurable and*

$$\left\| \| (\mathcal{A}(\cdot))^{-1} \|_{\mathfrak{B}(l^p)} \right\|_{L^\infty(\gamma)} < \infty. \tag{27}$$

***Proof.*** *Necessity.* If an operator $A \in \mathfrak{A}_p$ is invertible on $L^p$, then in view of Corollary 5, there exists $B := A^{-1} \in \mathfrak{A}_p$. Then the operator $\widehat{B} := \sigma B \sigma^{-1} \in \mathfrak{B}(L^p(\gamma, l^p))$ is inverse to the operator $\widehat{A} = \sigma A \sigma^{-1} \in \mathfrak{B}(L^p(\gamma, l^p))$. Since, by Lemma 7, $\widehat{A} = \mathcal{A}(\cdot)I$ and $\widehat{B} = \mathcal{B}(\cdot)I$, where $\mathcal{A}, \mathcal{B} \in L^\infty(\gamma, \mathfrak{B}(l^p))$, the equality $\widehat{B} = (\widehat{A})^{-1}$ implies that $\mathcal{B} = \mathcal{A}^{-1}$. This means that for almost all $t \in \gamma$ the operators $\mathcal{A}(t)$ are invertible on $l^p$ and $\mathcal{A}^{-1} \in L^\infty(\gamma, \mathfrak{B}(l^p))$. Necessity is proved.

*Sufficiency.* If for almost all $t \in \gamma$ the operators $\mathcal{A}(t)$ are invertible on the space $l^p$ and (27) is fulfilled, then by sufficiency of Proposition 6, the measurable essentially bounded operator-valued function $\mathcal{B} : \gamma \to \mathfrak{B}(l^p)$, $t \mapsto (\mathcal{A}(t))^{-1}$ generates the bounded linear operator $\widehat{B} = \mathcal{B}(\cdot)I \in \mathfrak{B}(L^p(\gamma, l^p))$. Since $\mathcal{B} = \mathcal{A}^{-1}$, the operator $\widehat{B}$ is the inverse operator for $\widehat{A} = \mathcal{A}(\cdot)I \in \mathfrak{B}(L^p(\gamma, l^p))$. Therefore, the operator $B = \sigma^{-1}\widehat{B}\sigma$ is the inverse operator for $A = \sigma^{-1}\widehat{A}\sigma$. ∎

# 4. Invertibility of binomial difference operators

## 4.1. Quantities characterizing invertibility

In this section we will find criteria for the invertibility of the difference operator

$$D := aI - bV$$

on the spaces $l^p$, $1 \le p \le \infty$, where $a, b \in l^\infty$ and the isometric shift operator $V$ is given by $(Vf)_n = f_{n+1}$ for $n \in \mathbb{Z}$. Clearly, $V$ is invertible on $l^p$, and one can check straightforwardly the following.

**Proposition 10.** *For $c \in l^\infty$, the spectral radius of operators $cV$ and $cV^{-1}$ on $l^p, 1 \le p \le \infty$, is calculated by*

$$r(c) := \lim_{n\to\infty} \left( \sup_{k\in\mathbb{Z}} |c_{k+1}c_{k+2}\cdots c_{k+n}| \right)^{1/n}.$$

**Proposition 11.** *Suppose there exist $C > 0$ and $m \in \mathbb{Z}$ such that $0 < C \le |c_n| < +\infty$ for $n \in \mathbb{Z} \setminus \{m\}$.*

(a) *For*

$$\rho_k^-(c) := \limsup_{n\to+\infty} |c_{k-1}c_{k-2}\dots c_{k-n}|^{-1/n} \quad (k \le m),$$

$$\rho_k^+(c) := \limsup_{n\to+\infty} |c_{k+1}c_{k+2}\dots c_{k+n}|^{1/n} \quad (k \ge m),$$

*we get*

$$0 \le \rho_k^-(c) \le \rho_s^-(c) \le C^{-1} \quad (k < s \le m), \qquad C \le \rho_k^+(c) \le \rho_s^+(c) \le +\infty \quad (m \le k < s). \tag{28}$$

*Moreover, there exist limits*

$$\rho_-(c) := \lim_{k\to-\infty} \rho_k^-(c) \in [0, C^{-1}], \quad \rho_+(c) := \lim_{k\to+\infty} \rho_k^+(c) \in [C, +\infty].$$

(b) *If $\rho_-(c) > 1$, then the function $\varphi : \mathbb{Z} \to \mathbb{C}$ whose values for $n < m$ have the form*

$$\varphi_n = dc_n^{-1}c_{n+1}^{-1}\dots c_{m-1}^{-1}, \quad d \in \mathbb{C}\setminus\{0\},$$

*does not belong to $l^p$.*

(c) *If $\rho_+(c) > 1$, then the function $\varphi : \mathbb{Z} \to \mathbb{C}$ whose values for $n > m$ have the form*

$$\varphi_n = dc_{m+1}c_{m+2}\dots c_{n-1}, \quad d \in \mathbb{C}\setminus\{0\}, \tag{29}$$

*does not belong to $l^p$.*

***Proof.*** (a) Let us consider the "positive" case. Suppose $m \le k < s$, then

$$|c_{k+1}c_{k+2}\dots c_{k+n}|^{1/n} = |c_{s+1}c_{s+2}\dots c_{s+n}|^{1/n} \cdot \left(\frac{|c_{k+1}c_{k+2}\dots c_s|}{|c_{k+n+1}c_{k+n+2}\dots c_{s+n}|},\right)^{1/n} \quad n \in \mathbb{N}. \tag{30}$$

Further, we have

$$\limsup_{n\to+\infty} \left(\frac{|c_{k+1}c_{k+2}\dots c_s|}{|c_{k+n+1}c_{k+n+2}\dots c_{s+n}|}\right)^{1/n} \le \limsup_{n\to+\infty} \left(\frac{|c_{k+1}c_{k+2}\dots c_s|}{C^{s-k}}\right)^{1/n} = 1. \tag{31}$$

On the other hand, for every $k \ge m$ and every $n \in \mathbb{N}$, we obtain $C \le |c_{k+1}c_{k+2}\dots c_{k+n}|^{1/n} < +\infty$. From the latter inequalities and (30)–(31) we get the second group of inequalities in (28). The monotonicity of the sequence $\{\rho_k^+(c)\}_{k=m}^{+\infty}$ implies that the limit $\rho_+(c)$ exists and belongs to $[C, +\infty]$.

The "negative" case is considered analogously. Part (a) is proved. Parts (b) and (c) are proved using the same idea. Let us prove (c).

Since $\rho_+(c) > 1$, by part (a) there exists a number $M \ge m$ such that $\rho_M^+(c) > 1$. Therefore, there exists $q > 1$ and a sequence $n_j \to +\infty$ as $j \to +\infty$ such that

$$|c_M c_{M+1}\dots c_{M+n_j}|^{1/n_j} > q > 1, \quad j \in \mathbb{N}. \tag{32}$$

Then, taking into account (29) and (32), we get

$$\|\varphi\|_{l^p} \ge |\varphi_{M+n_j+1}| > |dc_{m+1}c_{m+2}\dots c_{M-1}|q^{n_j} \to +\infty \quad \text{as} \quad j \to +\infty$$

whenever $d \in \mathbb{C}\setminus\{0\}$. Proposition is proved. ■

## 4.2. Non-invertibility conditions

Let the both coefficients degenerate. If $b_n \neq 0$, we put $c_n := a_n/b_n$ for $n \in \mathbb{Z}$.

**Proposition 12.** *Suppose $a, b \in l^\infty$, and there exist $k, m \in \mathbb{Z}$ such that*

$$a_n \neq 0 \quad for \quad n \in \mathbb{Z} \setminus \{k\}, \qquad b_n \neq 0 \quad for \quad n \in \mathbb{Z} \setminus \{m\}, \qquad a_k = b_m = 0. \tag{33}$$

*Then the operator $D$ is not invertible on $l^p$.*

***Proof.*** Assume that the operator $D$ is invertible and consider the following three subcases.

1. If $m < k$ then the function $\varphi : \mathbb{Z} \to \mathbb{C}$ given by

$$\varphi_n = \begin{cases} 0, & n \leq m, \\ 1, & n = m+1, \\ c_{m+1}c_{m+2}\dots c_{n-1}, & m+2 \leq n \leq k, \\ 0, & n > k+1 \end{cases}$$

satisfies the equality $D\varphi = 0$. Since $\varphi$ has finite support, $\varphi \in l^p$. Hence, $\varphi \in \operatorname{Ker} D \setminus \{0\}$.

2. If $m = k$, then the non-homogeneous equation $D\varphi = \{\delta_{m,n}\}_{n\in\mathbb{Z}}$, where $\delta_{m,n}$ is the Kronecker delta, is unsolvable on $l^p$ because, in view of (33),

$$0 = a_m\varphi_m - b_m\varphi_{m+1} \neq \delta_{m,m} = 1. \tag{34}$$

3. If $m > k$, then the non-homogeneous equation $D\varphi = \{\delta_{m,n}\}_{n\in\mathbb{Z}}$ is unsolvable on $l^p$ again, because from the system

$$a_n\varphi_n - b_n\varphi_{n+1} = 0 \quad (n = k, k+1, \dots, m-1)$$

it follows, due to $a_k = 0$, that $\varphi_{k+1} = \varphi_{k+2} = \dots = \varphi_m = 0$, and hence, we get (34) again.

Thus, in each subcase we get a contradiction. ∎

Now we consider the case when the second coefficient vanishes only at one point.

**Proposition 13.** *Suppose $a \in \mathcal{G}l^\infty$, $b \in l^\infty$, and there exists $m \in \mathbb{Z}$ such that $b_m = 0$ and $b_n \neq 0$ for all $n \neq m$. If $\rho_+(a/b) < 1$, then the operator $D$ is not invertible on $l^p$.*

***Proof.*** Taking into account that $b_m = 0$, $a_m \neq 0$ and $a_n b_n \neq 0$ for all $n \neq m$, we deduce that the function $\varphi : \mathbb{Z} \to \mathbb{C}$ given by

$$\varphi_n = \begin{cases} 0, & n \leq m, \\ 1, & n = m+1, \\ c_{m+1}c_{m+2}\dots c_{n-1}, & n > m+1 \end{cases} \tag{35}$$

satisfies the equation $D\varphi = 0$. Since $\rho_+(a/b) < 1$, from Proposition 11(a) we get $\rho_m^+(a/b) < 1$. Then, by definition, there exist numbers $q \in (0,1)$ and $N \in \mathbb{N}$ such that

$$|c_{m+1}c_{m+2}\dots c_{m+n}|^{1/n} < q < 1 \quad \text{for all} \quad n \geq N.$$

Thus, $|\varphi_n| < q^{n-m}$ for $|n| \geq N$. Therefore, the function $\varphi$ given by (35) belongs to $l^p$. Consequently, $\varphi \in \operatorname{Ker} D \setminus \{0\}$. This means that $D$ is not invertible. ∎

Finally, we consider the case when the both coefficients do not vanish on $\mathbb{Z}$.

**Proposition 14.** *Suppose $a, b \in \mathcal{G}l^\infty$. If $\rho_+(a/b) < 1$ and $\rho_-(a/b) < 1$, then the operator $D$ is not invertible on $l^p$.*

***Proof.*** If $\rho_-(c) < 1$ and $\rho_+(c) < 1$, then there exist numbers $q \in (0,1)$ and $N \in \mathbb{N}$ such that

$$|c_{-1}c_{-2}\dots c_{-n}|^{-1/n} < q < 1, \quad |c_0 c_1 \dots c_{n-1}|^{1/n} < q < 1,$$

for all $n \geq N$, respectively. Therefore, the function $\varphi : \mathbb{Z} \to \mathbb{C}$ given by

$$\varphi_n = \begin{cases} c_n^{-1}c_{n+1}^{-1}\cdots c_{-1}^{-1}, & n < 0, \\ 1, & n = 0, \\ c_0 c_1 \dots c_{n-1}, & n > 0 \end{cases} \tag{36}$$

belongs to $l^p$, in view of the estimate $|\varphi_n| < q^{|n|}$ for $|n| \geq N$. It is easily seen that $\varphi \in \operatorname{Ker} D \backslash \{0\}$. Thus, $D$ is not invertible. ∎

**Proposition 15.** *Suppose $a \in \mathcal{G}l^\infty$, $b \in l^\infty$, and $b_n \neq 0$ for all $n \in \mathbb{Z}$. If $r(b/a) > 1$ and $\rho_+(a/b) > 1$, then the operator $D$ is not invertible on $l^p$.*

***Proof.*** Assume that the operator $D$ is invertible on $l^p$. Since $r(b/a) > 1$, there exist numbers $q > 1$ and $M \in \mathbb{N}$ such that for every $m \geq M$ there exists $k_m \in \mathbb{Z}$ for which

$$\left| c_{k_m-1}^{-1} c_{k_m-2}^{-1} \cdots c_{k_m-m}^{-1} \right|^{1/m} > q > 1. \tag{37}$$

Consider the non-homogeneous equation $D\varphi = \{\delta_{k_m,n}\}_{n\in\mathbb{Z}}$. It is equivalent to the system

$$c_n\varphi_n - \varphi_{n+1} = \delta_{k_m,n}/b_n \quad (n \in \mathbb{Z}).$$

This equation can have only the solution $\varphi^{(m)} : \mathbb{Z} \to \mathbb{C}$ given by

$$\varphi_n^{(m)} = \begin{cases} dc_n^{-1}c_{n+1}^{-1}\cdots c_{k_m-1}^{-1}, & n < k_m, \\ d, & n = k_m, \\ dc_{k_m} - 1/b_{k_m}, & n = k_m + 1, \\ (dc_{k_m} - 1/b_{k_m})c_{k_m+1}c_{k_m+2}\cdots c_{n-1}, & n > k_m + 1, \end{cases}$$

where $d \in \mathbb{C}$. It follows from Proposition 11(c) that $\varphi^{(m)}$ would belong to $l^p$ only if $d = 1/a_{k_m}$.
On the other hand, from (37) it follows that

$$\|\varphi^{(m)}\|_{l^p} \geq |\varphi_{k_m-m}^{(m)}| = \left| a_{k_m}^{-1} c_{k_m-1}^{-1} c_{k_m-2}^{-1} \cdots c_{k_m-m}^{-1} \right| > Cq^m \to +\infty \quad \text{as} \quad m \to +\infty,$$

where $C > 0$ is the lower bound of $|a_n|^{-1}$ for $n \in \mathbb{N}$. But this contradicts the invertibility of $D$ because for every $m \geq M$,

$$\|\varphi^{(m)}\|_{l^p} \leq \|D^{-1}\|_{\mathfrak{B}(l^p)} \|\{\delta_{k_m,n}\}_{n\in\mathbb{Z}}\|_{l^p} = \|D^{-1}\|_{\mathfrak{B}(l^p)} < +\infty. \quad ∎$$

**Proposition 16.** *Suppose $a, b \in \mathcal{G}l^\infty$. If $r(a/b) > 1$ and $\rho_-(a/b) > 1$, then the operator $D$ is not invertible on $l^p$.*

This statement is proved by analogy with Proposition 15 making use of Proposition 11(b).

## 4.3. Invertibility criterion

Now we are in a position to prove the main result of this section.

**Theorem 17.** *Suppose $a, b \in l^\infty$. The operator $D := aI - bV$ is invertible on $l^p$ if and only if one of the following two alternative conditions holds:*

$$\text{(i)} \quad \inf_{n\in\mathbb{Z}} |a_n| > 0 \quad \textit{and} \quad r(b/a) < 1, \qquad \text{(ii)} \quad \inf_{n\in\mathbb{Z}} |b_n| > 0 \quad \textit{and} \quad r(a/b) < 1. \tag{38}$$

*If $D$ is invertible, then its inverse is given by*

$$D^{-1} = \sum_{n=0}^{\infty}\Big((b/a)V\Big)^n a^{-1}I \quad \textit{in case (i)}; \qquad D^{-1} = -V^{-1}\sum_{n=0}^{\infty}\Big((a/b)V^{-1}\Big)^n b^{-1}I \quad \textit{in case (ii)}. \tag{39}$$

***Proof.*** *Sufficiency.* Let $a \in \mathcal{G}l^\infty$ and $r(b/a) < 1$. Then $D = a\big(I - (b/a)V\big)$, where $b/a \in l^\infty$. Since the operator $aI$ is invertible on $l^p$ and since the operator $I - (b/a)V$ also is invertible on $l^p$ in view of the inequality $r(b/a) < 1$, we infer that $D$ is invertible on $l^p$ too, and its inverse is given by the first formula in (39). Sufficiency of (ii) and (39) in case (ii) are obtained analogously.

*Necessity.* Assume $D$ is invertible on $l^p$ and (38) does not hold. Then one of the following four conditions is satisfied.

1. Let $\inf_{n\in\mathbb{Z}} |a_n| = 0$, $\inf_{n\in\mathbb{Z}} |b_n| = 0$. Then for every $\varepsilon > 0$ there exist $\widetilde{a}, \widetilde{b} \in l^\infty$ and $k, m \in \mathbb{Z}$ such that

$$\widetilde{a}_n \neq 0 \quad \text{for} \quad n \in \mathbb{Z}\setminus\{k\}, \qquad \widetilde{b}_n \neq 0 \quad \text{for} \quad n \in \mathbb{Z}\setminus\{m\}, \qquad \widetilde{a}_k = \widetilde{b}_m = 0,$$

and $\|a - \widetilde{a}\|_{l^\infty} < \varepsilon/2$, $\|b - \widetilde{b}\|_{l^\infty} < \varepsilon/2$. If $\varepsilon$ is sufficiently small, then the operator $\widetilde{D} := \widetilde{a}I - \widetilde{b}V$ is invertible together with $D$ because $\|D - \widetilde{D}\|_{\mathfrak{B}(l^p)} < \varepsilon$. But, on the other hand, $\widetilde{D}$ is not invertible, due to Proposition 12. So, we arrive at a contradiction.

2. Let $\inf_{n\in\mathbb{Z}} |a_n| > 0$, $\inf_{n\in\mathbb{Z}} |b_n| = 0$, $r(b/a) \geq 1$. Then either $b_m = 0$ for some $m \in \mathbb{Z}$, or $b_n \neq 0$ for all $n \in \mathbb{Z}$ but $\inf_{n\in\mathbb{Z}} |b_n| = 0$. Consider these two subcases separately.

(a) If $b_m = 0$ for some $m \in \mathbb{Z}$, then for $\varepsilon > 0$ we define the functions $b_\varepsilon$, $\widetilde{b}_\varepsilon : \mathbb{Z} \to \mathbb{C}$ by

$$(b_\varepsilon)_n := \begin{cases} b_n, & |b_n| \geq \varepsilon, \quad n \neq m, \\ \varepsilon, & |b_n| < \varepsilon, \quad n \neq m, \\ 0, & n = m, \end{cases} \qquad \widetilde{b}_\varepsilon := (1+\varepsilon)b_\varepsilon. \tag{40}$$

Then, by Proposition 11(a), the quantities $\rho_+(a/b_\varepsilon)$ and $\rho_+(a/\widetilde{b}_\varepsilon)$ are well defined. Moreover,

$$r(\widetilde{b}_\varepsilon/a) = (1+\varepsilon)r(b_\varepsilon/a) > r(b_\varepsilon/a) \geq r(b/a) \geq 1, \quad \rho_+(a/\widetilde{b}_\varepsilon) = (1+\varepsilon)^{-1}\rho_+(a/b_\varepsilon). \tag{41}$$

If $\rho_+(a/b_\varepsilon) \leq 1$, then we put $\widetilde{b} := \widetilde{b}_\varepsilon$ and choose $\varepsilon > 0$ so small that the operator $\widetilde{D} := aI - \widetilde{b}V$ is invertible together with $D$, and $\rho_+(a/\widetilde{b}) < 1$ by (41). But, on the other hand, $\widetilde{b}_m = 0$ and $\widetilde{b}_n \neq 0$ for all $n \neq m$. Hence, by Proposition 13, the operator $\widetilde{D}$ is not invertible.

If $\rho_+(a/b_\varepsilon) > 1$, then consider the function $\widetilde{b} \in \mathcal{G}l^\infty$ given by

$$\widetilde{b}_n := \begin{cases} (1+\varepsilon)(b_\varepsilon)_n, & n \neq m, \\ (1+\varepsilon)\varepsilon, & n = m. \end{cases}$$

By (41), we get $r(\widetilde{b}/a) > 1$. Further, we can choose $\varepsilon > 0$ so small that $\rho_+(a/\widetilde{b}) = \rho_+(a/\widetilde{b}_\varepsilon) > 1$ and the operator $\widetilde{D} := aI - \widetilde{b}V$ is invertible together with $D$. On the other hand, by Proposition 15, the operator $\widetilde{D}$ is not invertible.

(b) If $b_n \neq 0$ for all $n \in \mathbb{Z}$ but $\inf_{n\in\mathbb{Z}} |b_n| = 0$, then, by Proposition 11(a), $\rho_+(a/b)$ is well defined.

If $\rho_+(a/b) > 1$, then we can choose $\varepsilon > 0$ so small that for $\widetilde{b} := (1+\varepsilon)b$ we have

$$r(\widetilde{b}/a) = (1+\varepsilon)r(b/a) > r(b/a) \geq 1, \quad \rho_+(a/\widetilde{b}) = (1+\varepsilon)^{-1}\rho_+(a/b) > 1, \tag{42}$$

and the operator $\widetilde{D} := aI - \widetilde{b}V$ is invertible together with $D$. On the other hand, in view of Proposition 15, the operator $\widetilde{D}$ is not invertible.

If $\rho_+(a/b) \leq 1$, then for a given $\varepsilon > 0$ there exists $m \in \mathbb{Z}$ such that $|b_m| < \varepsilon$. We consider the function $\widetilde{b} := \widetilde{b}_\varepsilon$, where $\widetilde{b}_\varepsilon$ is given by (40). Then, taking into account (41), we have $\rho_+(a/\widetilde{b}) < 1$ and $\widetilde{b}_m = 0$. Clearly, we can choose $\varepsilon > 0$ so small that the operator $\widetilde{D} := aI - \widetilde{b}V$ is invertible together with $D$. On the other hand, the operator $\widetilde{D}$ is not invertible, due to Proposition 13.

Thus, case 2 is completely considered.

3. Let $\inf_{n\in\mathbb{Z}} |a_n| = 0$, $\inf_{n\in\mathbb{Z}} |b_n| > 0$, $r(a/b) \geq 1$. Then for the operator $bI - aV^{-1}$, all the conditions of the previous case are satisfied. Hence, by case 2, the operator $bI - aV^{-1}$ is not invertible. On the other hand, the operator $bI - aV^{-1}$ is invertible on $l^p$ simultaneously with $D$.

4. Let $\inf_{n\in\mathbb{Z}} |a_n| > 0$, $\inf_{n\in\mathbb{Z}} |b_n| > 0$, $r(b/a) \geq 1$, $r(a/b) \geq 1$. In this case the characteristics $\rho_\pm(a/b)$ are well defined, due to Proposition 11(a), and one of the following three conditions is fulfilled.

(a) If $\rho_+(a/b) \leq 1$ and $\rho_-(a/b) \leq 1$, then setting

$$\widetilde{b}_n = \begin{cases} (1+\varepsilon)b_n, & n \geq 0, \\ (1-\varepsilon)b_n, & n < 0, \end{cases}$$

where $\varepsilon \in (0,1)$, we get

$$\rho_+(a/\widetilde{b}) = (1+\varepsilon)^{-1}\rho_+(a/b) < \rho_+(a/b) \leq 1, \quad \rho_-(a/\widetilde{b}) = (1-\varepsilon)\rho_-(a/b) < \rho_-(a/b) \leq 1. \tag{43}$$

Since $\|b - \widetilde{b}\|_{l^\infty} \leq \varepsilon \|b\|_{l^\infty}$, we can choose $\varepsilon$ so small that the operator $\widetilde{D} := aI - \widetilde{b}V$ is invertible together with $D$. But, by Proposition 14 and (43), the operator $\widetilde{D}$ is not invertible.

(b) If $\rho_+(a/b) > 1$, then setting $\widetilde{b} := (1+\varepsilon)b$ and choosing a sufficiently small $\varepsilon > 0$, we get (42), which contradicts the invertibility of the operator $\widetilde{D} := aI - \widetilde{b}V$, by Proposition 15.

(c) If $\rho_-(a/b) > 1$, then put $\widetilde{a} := (1+\varepsilon)a$ where $\varepsilon > 0$. Therefore, for a sufficiently small $\varepsilon > 0$,

$$r(\widetilde{a}/b) = (1+\varepsilon)r(a/b) > r(a/b) \geq 1, \quad \rho_-(\widetilde{a}/b) = (1+\varepsilon)^{-1}\rho_-(a/b) > 1. \tag{44}$$

We can choose $\varepsilon$ so small that the operator $\widetilde{D} := \widetilde{a}I - bV$ is invertible together with $D$. On the other hand, from (44) and Proposition 16 we deduce that the operator $\widetilde{D}$ is not invertible. This completes the proof in case 4.

Thus, in all the cases 1–4 we get contradictions, that completes the proof. ■

## 5. Invertibility criteria for binomial functional operators

In this section we get invertibility criteria on $L^p$, $1 \leq p \leq \infty$, for functional operators of the form

$$A = aI - bU_\alpha \tag{45}$$

where $a, b, \log \alpha' \in L^\infty$. By Lemma 7, with the operator (45) we associate the operator-valued function $\mathcal{A} \in L^\infty(\gamma, \mathfrak{B}(l^p))$ given by

$$\mathcal{A} : t \mapsto \Big( a[\alpha_i(t)]\delta_{i,j} - b[\alpha_i(t)]\delta_{i,j-1} \Big)_{i,j\in\mathbb{Z}},$$

where $\delta_{i,j}$ is the Kronecker delta. Considering vectors in $l^p$ as complex-valued functions on $\mathbb{Z}$, we can rewrite the operators $\mathcal{A}(t) \in \mathfrak{B}(l^p)$, defined for almost all $t \in \gamma$, as difference operators of the form

$$A_t = a_t I - b_t V \in \mathfrak{B}(l^p), \qquad a_t : n \mapsto a[\alpha_n(t)], \quad b_t : n \mapsto b[\alpha_n(t)] \quad (n \in \mathbb{Z}).$$

Here $a_t, b_t$ belong to $l^\infty$ and the isometric shift operator $V$ is given by $(Vf)_n = f_{n+1}$ for $n \in \mathbb{Z}$.
From Theorems 9 and 17 we directly obtain the following criterion.

**Theorem 18.** *The operator* (45) *is invertible on the space $L^p$ if and only if for almost every $t \in \gamma$ one of the following two alternative conditions holds:*

(i) $\inf_{n\in\mathbb{Z}} |a[\alpha_n(t)]| > 0$ *and* $r(b_t/a_t) < 1$, (ii) $\inf_{n\in\mathbb{Z}} |b[\alpha_n(t)]| > 0$ *and* $r(a_t/b_t) < 1$,

*and the operator-valued function $t \mapsto (\mathcal{A}(t))^{-1}$, where $(\mathcal{A}(t))^{-1}$ is the matrix of the difference operator*

$$(A_t)^{-1} := \begin{cases} \sum_{n=0}^{\infty} \Big( (b_t/a_t)V \Big)^n (a_t)^{-1} I & \text{in case (i)}, \\ -V^{-1} \sum_{n=0}^{\infty} \Big( (a_t/b_t)V^{-1} \Big)^n (b_t)^{-1} I & \text{in case (ii)}, \end{cases} \tag{46}$$

*belongs to $L^\infty(\gamma, \mathfrak{B}(l^p))$.*

Making use of Theorem 18 and Corollary 4, we get the following more pleasant criterion.

**Theorem 19.** *The operator* (45) *is invertible on the space $L^p$ if and only if there exists partitioning of $\mathbb{I}$ into two measurable $\alpha$-invariant subsets $\mathbb{I}_a$ and $\mathbb{I}_b$ such that*

$$\text{(i)} \quad a \in \mathcal{G}L^\infty(\mathbb{I}_a), \quad r\Big( ((b/a)U_\alpha)|_{L^p(\mathbb{I}_a)} \Big) < 1;$$

and

$$\text{(ii)} \quad b \in \mathcal{G}L^\infty(\mathbb{I}_b), \quad r\Big( ((a/b)U_\alpha^{-1})|_{L^p(\mathbb{I}_b)} \Big) < 1.$$

***Proof.*** *Sufficiency.* Let $\operatorname{mes} \mathbb{I}_a > 0$ and $\operatorname{mes} \mathbb{I}_b > 0$. Taking into account that the measurable sets $\mathbb{I}_a$ and $\mathbb{I}_b$ are $\alpha$-invariant, we get

$$A = \operatorname{diag}\{A_1, A_2\} \in \mathfrak{B}(L^p(\mathbb{I}_a) \dot{+} L^p(\mathbb{I}_b)) \quad \text{where} \quad A_1 := A|_{L^p(\mathbb{I}_a)}, \quad A_2 := A|_{L^p(\mathbb{I}_b)}. \tag{47}$$

By (i) and (ii), the operator restrictions $(I - (b/a)U_\alpha)|_{L^p(\mathbb{I}_a)}$ and $(I - (a/b)U_\alpha^{-1})|_{L^p(\mathbb{I}_b)}$ are invertible. Hence,

$$A_1^{-1} = \sum_{n=0}^{\infty} \Big( (b/a)U_\alpha \Big)^n a^{-1} \chi_a I, \quad A_2^{-1} = -U_\alpha^{-1} \sum_{n=0}^{\infty} \Big( (a/b)U_\alpha^{-1} \Big)^n b^{-1} \chi_b I \tag{48}$$

where $\chi_a$ and $\chi_b$ are the characteristic functions of $\mathbb{I}_a$ and $\mathbb{I}_b$, respectively. Finally, (47) and (48) imply that the operator $A = aI - bU_\alpha$ is invertible too, and

$$A^{-1} = \sum_{n=0}^{\infty} \Big((b/a)U_\alpha\Big)^n a^{-1}\chi_a I - U_\alpha^{-1} \sum_{n=0}^{\infty} \Big((a/b)U_\alpha^{-1}\Big)^n b^{-1}\chi_b I \tag{49}$$

Clearly, (49) remains valid if $\operatorname{mes} \mathbb{I}_a = 0$ or $\operatorname{mes} \mathbb{I}_b = 0$. Sufficiency is proved.

*Necessity.* If $A$ is invertible, then for almost every $t \in \gamma$ one of the conditions (i)–(ii) of Theorem 18 is fulfilled. Let $\gamma_a$ denote the set of $t \in \gamma$ for which condition (i) of Theorem 18 holds, and $\gamma_b := \gamma \setminus \gamma_a$.

Setting

$$\mathbb{I}_a := \bigcup_{n\in\mathbb{Z}} \alpha_n(\gamma_a), \quad \mathbb{I}_b := \bigcup_{n\in\mathbb{Z}} \alpha_n(\gamma_b),$$

we infer from (1) that $\mathbb{I} = \mathbb{I}_a \cup \mathbb{I}_b$, $\mathbb{I}_a \cap \mathbb{I}_b = \emptyset$, and $\alpha(\mathbb{I}_a) = \mathbb{I}_a$, $\alpha(\mathbb{I}_b) = \mathbb{I}_b$. Clearly, in view of (46), the matrix of the operator $(A_t)^{-1}$ has the form

$$(\mathcal{A}(t))^{-1} = (c_{j-i}[\alpha_i(t)])_{i,j\in\mathbb{Z}}, \quad t \in \gamma,$$

where for almost all $t \in \gamma$,

$$c_n(t) := \begin{cases} \dfrac{b(t)}{a(t)} \dfrac{b[\alpha(t)]}{a[\alpha(t)]} \cdots \dfrac{b[\alpha_{n-1}(t)]}{a[\alpha_{n-1}(t)]} \dfrac{\chi_a(t)}{a[\alpha_n(t)]}, & n \in \{0,1,2,\dots\}, \\ -\dfrac{a[\alpha_{-1}(t)]}{b[\alpha_{-1}(t)]} \dfrac{a[\alpha_{-2}(t)]}{b[\alpha_{-2}(t)]} \cdots \dfrac{a[\alpha_{n+1}(t)]}{b[\alpha_{n+1}(t)]} \dfrac{\chi_b(t)}{b[\alpha_n(t)]}, & n \in \{-1,-2,\dots\}. \end{cases} \tag{50}$$

By Lemma 8, the formulas (50), defining the entries of $(\mathcal{A}(\cdot))^{-1}$, are valid for almost all $t \in \mathbb{I}$.

Since, by Theorem 18, the matrix function $\mathcal{A}^{-1} : t \mapsto (\mathcal{A}(t))^{-1}$ is measurable on $\gamma$, so is its $(0,0)$-entry

$$c_0 = (\mathcal{A}^{-1}e_0, e_0) = \chi_a/a,$$

which implies the measurability of the sets $\gamma_b = \{t \in \gamma : c_0(t) = 0\}$ and $\gamma_a = \gamma \setminus \gamma_b$ Consequently, the both sets $\mathbb{I}_a$ and $\mathbb{I}_b$ are measurable too. Moreover, since $\mathcal{A}^{-1} \in L^\infty(\gamma, \mathfrak{B}(l^p))$, we conclude that $c_0 \circ \alpha_n = \chi_a/(a \circ \alpha_n) \in L^\infty(\gamma)$ for all $n \in \mathbb{Z}$. Hence, $a \in \mathcal{G}L^\infty(\mathbb{I}_a)$. Analogously, $b \in \mathcal{G}L^\infty(\mathbb{I}_b)$.

Let $\operatorname{mes} \gamma_a > 0$ and $\operatorname{mes} \gamma_b > 0$. Since the sets $\mathbb{I}_a$ and $\mathbb{I}_b$ are $\alpha$-invariant and since $A$ is invertible on $L^p$, it follows from (47) that the restrictions $A_1$ and $A_2$ are invertible on $L^p(\mathbb{I}_a)$ and $L^p(\mathbb{I}_b)$, respectively. As $A_1^{-1} = (\sigma^{-1}\mathcal{A}^{-1}\sigma)|_{L^p(\mathbb{I}_a)}$ and $A_2^{-1} = (\sigma^{-1}\mathcal{A}^{-1}\sigma)|_{L^p(\mathbb{I}_b)}$, where $\sigma$ is given by (22), we derive from (50) that, by analogy with (46), the operators $A_1^{-1}$ and $A_2^{-1}$ have the form (48). On the other hand, by Corollary 4, $A_1^{-1} \in \mathcal{W}_p(\mathbb{I}_a)$ and $A_2^{-1} \in \mathcal{W}_p(\mathbb{I}_b)$. Thus,

$$A_1^{-1} = \sum_{n=0}^{\infty} c_n U_\alpha^n \in \mathcal{W}_p(\mathbb{I}_a), \quad A_2^{-1} = -\sum_{n=1}^{\infty} c_{-n} U_\alpha^{-n} \in \mathcal{W}_p(\mathbb{I}_b),$$

where $c_n$ are given by (50) for $t \in \mathbb{I}$. Hence,

$$\sum_{n=0}^{\infty} \|c_n\|_{L^\infty(\mathbb{I}_a)} < \infty, \quad \sum_{n=1}^{\infty} \|c_{-n}\|_{L^\infty(\mathbb{I}_b)} < \infty. \tag{51}$$

Due to the Beurling-Gelfand formula for the spectral radius,

$$r_1 := r\Big(((b/a)U_\alpha)|_{L^p(\mathbb{I}_a)}\Big) = \lim_{n\to+\infty} \|c_n(a\circ\alpha_n)\|^{1/n}_{L^\infty(\mathbb{I}_a)} = \lim_{n\to+\infty} \|c_n\|^{1/n}_{L^\infty(\mathbb{I}_a)},$$
$$r_2 := r\Big(((a/b)U_\alpha^{-1})|_{L^p(\mathbb{I}_b)}\Big) = \lim_{n\to+\infty} \|c_{-n}(b\circ\alpha_{-n})\|^{1/n}_{L^\infty(\mathbb{I}_b)} = \lim_{n\to+\infty} \|c_{-n}\|^{1/n}_{L^\infty(\mathbb{I}_b)}. \quad (52)$$

By the Cauchy test for the convergence of positive series, from (51) and (52) we get $r_1 \le 1$ and $r_2 \le 1$. Since the invertibility of $A$ is stable under small perturbations of coefficients, it is easily seen that actually $r_1 < 1$ and $r_2 < 1$.

The cases $\operatorname{mes}\gamma_a = 0$ and $\operatorname{mes}\gamma_b = 0$ are considered analogously. ■

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