Second-order Reduced-bias Tail Index and High Quantile Estimation

Caeiro, Frederico
FCT/UNL and CMA, Universidade Nova de Lisboa
Quinta da Torre, 2829-516 Caparica, Portugal
E-mail: fac@fct.unl.pt

Gomes, M. Ivette
Faculdade de Ciências de Lisboa (DEIO) and CEAUL, Universidade de Lisboa
Campo Grande, 1749-016, Lisboa, Portugal
E-mail: ivette.gomes@fc.ul.pt

1 Introduction and preliminaries

Heavy tailed-models are very useful in many fields, like Hydrology, Insurance and Finance, among others. In practice, it is often needed to estimate a high quantile, a value that is exceeded with a probability p, small. The semi-parametric estimation of this parameter depends not only on the estimation of the tail index $\gamma > 0$, the primary parameter in *Statistics of Extremes*, but also of a first order scale parameter or functional, here denoted C. A model F is said to be heavy-tailed if the tail function $\overline{F} := 1 - F \in RV_{-1/\gamma}$, $\gamma > 0$, where RV_α denotes the class of regularly varying functions with index of regular variation equal to α, i.e., non-negative measurable functions g such that, for all $x > 0$, $g(tx)/g(t) \to x^\alpha$, as $t \to \infty$. Let us denote $U(t) := F^{\leftarrow}(1 - 1/t) = \inf\{x : F(x) \geq 1 - 1/t\}$. Then, we may equivalently say that F is heavy-tailed if and only if $U \in RV_\gamma$, i.e.

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma, \qquad \text{for any } x > 0. \tag{1}$$

For small values of p, we want to estimate χ_{1-p}, a value such that $F(\chi_{1-p}) = 1-p$, a typical parameter in the most diversified areas of application. More specifically, we want to estimate

$$\chi_{1-p} = U(1/p), \qquad p = p_n \to 0, \quad np_n \to K \quad \text{as } n \to \infty, \quad K \in [0,1], \tag{2}$$

and we shall assume to be working in Hall-Welsh class of models (Hall and Welsh, 1985), where there exist $\gamma > 0$, $\rho < 0$, $C > 0$ and $\beta \neq 0$ such that

$$U(t) = Ct^\gamma(1 + \gamma\beta t^\rho/\rho + o(t^\rho)). \tag{3}$$

For some details in the paper we shall refer to a sub-class of Hall's class, such that

$$U(t) = Ct^\gamma(1 + \gamma\beta t^\rho/\rho + \beta' t^{2\rho} + o(t^{2\rho})), \tag{4}$$

i.e., relatively to Hall's class we merely make explicit a third order term $\beta' t^{2\rho}$, $\beta' \neq 0$. Such a class contains most of the heavy-tailed models important in applications, like the Fréchet, the Generalized Pareto and the Student's-t.

We are going to base inference on the largest k top order statistics (o.s.), and as usual in semi-parametric estimation of parameters of extreme events, we shall assume that k is an intermediate sequence of integers in $[1, n[$, i.e.,

$$k = k_n \to \infty, \qquad k/n \to 0, \qquad n \to \infty. \tag{5}$$

Since, from (2) and (3), $\chi_{1-p} = U(1/p) \sim Cp^{-\gamma}$, as $p \to 0$, an obvious estimator of χ_{1-p} is $\widehat{C}p^{-\hat\gamma}$, with $\widehat{C}$ and $\hat\gamma$ any consistent estimators of C and γ, respectively. Given a sample $(X_1, X_2, \ldots, X_n)$, let us denote $X_{i:n}$, $1 \le i \le n$, the set of associated ascending o.s. Denoting Y a standard Pareto model, i.e., a model such that $F_Y(y) = 1 - 1/y$, $y > 1$, the use of the universal uniform transformation enables us to write $X_{n-k:n} \stackrel{d}{=} U(Y_{n-k:n})$. Next, since $Y_{n-k:n} \stackrel{p}{\sim} (n/k)$ for intermediate k and whenever (3) holds, we get $X_{n-k:n} \stackrel{p}{\sim} CY^{\gamma}_{n-k:n} \stackrel{p}{\sim} C(n/k)^{\gamma}$, as $n \longrightarrow \infty$. Consequently, an obvious estimator of C, proposed by Hall and Welsh (1985), is

$$C_{\hat{\gamma}}(k) := X_{n-k:n} \left(\frac{k}{n}\right)^{\hat{\gamma}} \tag{6}$$

and

$$Q^{(p)}_{\hat{\gamma}}(k) = \widehat{C} p^{-\hat{\gamma}} = X_{n-k:n} \left(\frac{k}{np}\right)^{\hat{\gamma}} \tag{7}$$

is the obvious quantile-estimator at the level p (Weissman, 1978).

For heavy tails, the classical tail index estimator, usually the one which is plugged in (7), for a semi-parametric quantile estimation, is the Hill estimator $\hat{\gamma} = \hat{\gamma}(k) =: H(k)$ (Hill, 1975),

$$H(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik} = \frac{1}{k}\sum_{i=1}^{k} U_i, \tag{8}$$

where $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \le i \le k < n$, are the log-excesses, and

$$U_i := i\left(\ln X_{n-i+1:n} - \ln X_{n-i:n}\right), \quad 1 \le i \le k < n, \tag{9}$$

are the scaled log-spacings. We thus get the so-called classical quantile estimator, based on the Hill tail index estimator H, with the obvious notation, $Q^{(p)}_H(k)$.

In order to derive the asymptotic non-degenerate behaviour of semi-parametric estimators of extreme events' parameters, we need more than the first order condition in (1). A typical condition for heavy-tailed models, which holds for the models in (3), with

$$A(t) = \gamma\,\beta\, t^{\rho}, \qquad \gamma > 0,\ \beta \ne 0,\ \rho < 0, \tag{10}$$

is

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^{\gamma}}{A(t)} = x^{\gamma}\frac{x^{\rho}-1}{\rho} \qquad \text{iff} \qquad \lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^{\rho}-1}{\rho}, \tag{11}$$

for all $x > 0$, where A is a function of constant sign near infinity (positive or negative), and $\rho \le 0$ is the shape second order parameter.

Under the second order framework in (11) and for intermediate k, i.e., whenever (5) holds, we may guarantee the asymptotic normality of the Hill estimator $H(k)$, for an adequate k. Indeed, we may write (de Haan and Peng, 1998),

$$H(k) \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k + \frac{A(n/k)}{1-\rho}(1 + o_p(1)), \tag{12}$$

with $Z_k = \sqrt{k}\left(\sum_{i=1}^{k} E_i/k - 1\right)$, and $\{E_i\}$ i.i.d. standard exponential r.v.'s. Consequently, if we choose k such that $\sqrt{k}\,A(n/k) \to \lambda \ne 0$, finite, as $n \to \infty$, $\sqrt{k}(H(k) - \gamma)$ is asymptotically normal, with variance equal to γ^2 and a non-null bias given by $\lambda/(1-\rho)$. Most of the times, this type of estimates exhibits a strong bias for moderate k and sample paths with very short stability regions around the target value γ. This has recently led researchers to consider the possibility of dealing with the bias term in an appropriate way, building new estimators, $\hat{\gamma}_R(k)$ say, the so-called second order reduced-bias estimators. Then, for k intermediate, i.e., such that (5) holds, and under the second order framework in (11), we may write, with Z_k^R an asymptotically standard normal r.v.,

$$\hat{\gamma}_R(k) \stackrel{d}{=} \gamma + \frac{\gamma\sigma_R}{\sqrt{k}} Z_k^R + o_p(A(n/k)), \tag{13}$$

where $\sigma_R > 0$, being A again the function in (11). Consequently, the sequence of r.v.'s, $\sqrt{k}(\hat{\gamma}_R(k)-\gamma)$ is asymptotically normal with variance equal to $(\gamma\sigma_R)^2$ and a null mean value even when $\sqrt{k}\ A(n/k) \to \lambda \neq 0$, finite, as $n \to \infty$, possibly at expenses of an asymptotic variance $\gamma^2\sigma_R^2 > \gamma^2$. Gomes and Figueiredo (2006) suggest the use, in (7), of reduced-bias tail index estimators, like the ones in Gomes and Martins (2001, 2002) and Gomes *et al.* (2004), all with $\sigma_R > 1$ in (13), being then able to reduce also the dominant component of the classical quantile estimator's asymptotic bias.

More recently, Gomes *et al.* (2004), Caeiro *et al.* (2005) and Gomes *et al.* (2005) consider new classes of tail index estimators, for which (13) holds with $\sigma_R = 1$ at least for values k such that $\sqrt{k}\ A(n/k) \to \lambda$, finite, i.e., they are able to reduce bias keeping the same asymptotic variance of the classical estimator, provided that the second orders parameters are estimated at an adequate level, of a larger order than the level used to estimate the first order parameters. These classes depend on $(\hat{\beta}, \hat{\rho})$, an adequate consistent estimator of the vector (β, ρ) in (10). The influence of these tail index estimators in quantile estimation has been studied by Beirlant *et al.* (2006) and Gomes and Pestana (2007).

Also recently, new estimators of C have been proposed (Caeiro, 2006), where, instead of $X_{n-k:n}$ alone, a spacing $X_{n-[\theta k]:n} - X_{n-k:n}$, $0 < \theta < 1$, is considered. We shall here consider $\theta = 1/2$ and the replacement of $C_{\hat{\gamma}}(k)$ in (6) by

$$\widetilde{C}_{\hat{\gamma}_R}(k) := \frac{X_{n-[k/2]:n} - X_{n-k:n}}{2^{\hat{\gamma}_R} - 1} \left(\frac{k}{n}\right)^{\hat{\gamma}_R}, \tag{14}$$

where $\hat{\gamma}_R \equiv \hat{\gamma}_R(k)$ is a second order reduced-bias extreme value index estimator. Similarly to the way developed by Caeiro *et al.* (2005) for the extreme value index estimation, Caeiro (2006) has worked out the main dominant component of the asymptotic bias of $\widetilde{C}_{\hat{\gamma}_R}(k)$. With the parametrization $A(t) = \gamma\,\beta\,t^\rho$, already given in (10), such a component is given by $C \times \mathcal{B}(\gamma, \rho, \beta)$, where $\mathcal{B}(\gamma, \rho, \beta) = \gamma\,\beta\,(n/k)^\rho(2^{(\gamma+\rho)} - 1)/(\rho(2^\gamma - 1))$. It is thus sensible to consider the semi-parametric C-estimator,

$$\overline{C}_{\hat{\gamma}_R}(k) := \frac{X_{n-[k/2]:n} - X_{n-k:n}}{2^{\hat{\gamma}_R} - 1} \left(\frac{k}{n}\right)^{\hat{\gamma}_R} \times (1 - \mathcal{B}(\hat{\gamma}_R, \hat{\rho}, \hat{\beta})) \tag{15}$$

and the associated quantile estimator $\overline{Q}^{(p)}_{\hat{\gamma}_R}(k) \equiv \overline{Q}^{(p)}_{\hat{\gamma}_R,\hat{\rho},\hat{\beta}}(k)$, with

$$\overline{Q}^{(p)}_{\hat{\gamma}_R,\hat{\rho},\hat{\beta}}(k) := \frac{X_{n-[k/2]:n} - X_{n-k:n}}{2^{\hat{\gamma}_R} - 1} \left(\frac{k}{n\,p}\right)^{\hat{\gamma}_R} \times (1 - \mathcal{B}(\hat{\gamma}_R, \hat{\rho}, \hat{\beta})). \tag{16}$$

Moreover, we shall restrict our attention to the second order reduced-bias extreme value index estimator estimator introduced in Caeiro *et al.* (2005),

$$\overline{H}(k) \equiv \overline{H}_{\hat{\beta},\hat{\rho}}(k) := H(k)\left(1 - \tfrac{\hat{\beta}}{1-\hat{\rho}}\left(\tfrac{n}{k}\right)^{\hat{\rho}}\right) \tag{17}$$

for adequate consistent estimators $\hat{\beta}$ and $\hat{\rho}$ of the second order parameters β and ρ, respectively.

After a brief sketch on the estimation of the second order parameters, in Section 2, we provide, in Section 3, details on the reduced-bias estimators of γ and C, to be used for quantile estimation. Section 4 is devoted to the asymptotic behavior of quantile estimators and finally, in Section 5, we provide an illustration, for data from the field of finance.

2 Estimation of second order parameters

The reduced-bias tail index estimator in (17) requires the estimation of the second order parameters ρ and β in (10). Such an estimation will now be briefly discussed.

2.1 Estimation of the shape second order parameter ρ

We shall consider here particular members of the class of estimators of the second order parameter ρ proposed by Fraga Alves *et al.* (2003). Such a class of estimators may be parameterized by a *tuning* real parameter $\tau \in \mathbb{R}$ (Caeiro and Gomes, 2004). These ρ-estimators depend on the statistics

$$T_n^{(\tau)}(k) = \frac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}, \quad M_n^{(j)}(k) := \frac{1}{k}\sum_{i=1}^{k} \{\ln X_{n-i+1:n} - \ln X_{n-k:n}\}^j, \ j \geq 1,$$

with the notation $a^{b\tau} = b \ln a$ whenever $\tau = 0$. The statistics $T_n^{(\tau)}(k)$ converge towards $3(1-\rho)/(3-\rho)$, independently of τ, whenever the second order condition (11) holds and k is such that (5) holds and $\sqrt{k}\ A(n/k) \to \infty$, as $n \to \infty$. The ρ-estimators considered have the functional expression,

$$\hat{\rho}_n^{(\tau)}(k) := -\min\left(0, \left(3(T_n^{(\tau)}(k)) - 1\right)/\left(T_n^{(\tau)}(k) - 3\right)\right). \tag{18}$$

Remark 2.1. *Under adequate general conditions, and for an appropriate tuning parameter τ the ρ-estimators in* (18) *show highly stable sample paths as functions of k, for a wide range of large k-values.*

Remark 2.2. *The theoretical and simulated results in* Fraga Alves *et al.* (2003)*, together with the use of these estimators in different reduced-bias statistics, has led us to advise in practice the estimation of ρ through the estimator in* (18)*, computed at the value*

$$k_1 := \left[n^{0.995}\right], \tag{19}$$

not chosen in any optimal way, and the choice of the tuning parameter $\tau = 0$ for the region $\rho \in [-1, 0)$ and $\tau = 1$ for the region $\rho \in (-\infty, -1)$. Anyway, we again advise practitioners not to choose blindly the value of τ in (18)*. It is sensible to draw a few sample paths of $\hat{\rho}_n^{(\tau)}(k)$, as functions of k, electing the value of τ which provides higher stability for large k, by means of any stability criterion.*

2.2 Estimation of the scale second order parameter β

For the estimation of β we shall here consider the estimator in Gomes and Martins (2002):

$$\widehat{\beta}_{\widehat{\rho}}(k) := \left(\frac{k}{n}\right)^{\widehat{\rho}} \frac{\left(\frac{1}{k}\sum\limits_{i=1}^{k}\left(\frac{i}{k}\right)^{-\widehat{\rho}}\right) N_n^{(1)}(k) - N_n^{(1-\widehat{\rho})}(k)}{\left(\frac{1}{k}\sum\limits_{i=1}^{k}\left(\frac{i}{k}\right)^{-\widehat{\rho}}\right) N_n^{(1-\widehat{\rho})}(k) - N_n^{(1-2\widehat{\rho})}(k)}, \tag{20}$$

where $N_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k} (i/k)^{\alpha-1}\, U_i$, with U_i and $\hat{\rho} \equiv \hat{\rho}_n^{(\tau)}(k)$ defined in (9) and (18), respectively.

2.3 Asymptotic behaviour

In this paper, we intend to use the same level k_1 in (19) both for the estimation of ρ and β, through the estimators in (18) and (20), respectively, and we shall formalize, without proofs, the needed distributional properties of the estimators of (β, ρ), essentially for the class of models in (4).

Proposition 2.1 (Fraga Alves et al., 2003). *If the second order condition* (11) *holds, with* $\rho \leq 0$, k *is a sequence of intermediate integers, i.e.,* (5) *holds, and* $\lim_{n\to\infty} \sqrt{k}\ A(n/k) = \infty$, *then* $\hat{\rho}_n^{(\tau)}(k)$ *in* (18) *converges in probability towards* ρ, *as* $n \to \infty$. *Moreover, and now for models in* (4), $\hat{\rho}_n^{(\tau)}(k) - \rho = o_p(1/\ln(n/k)))$ *for values* k *such that* $\sqrt{k}A^2(n/k) \to \lambda_A$, *finite and non-null, and for values* k *such that* $\sqrt{k}A^2(n/k) \to \infty$ *for some* $\epsilon > 0$ *and* $k = O(n^{1-\epsilon})$.

Proposition 2.2 (Gomes and Martins, 2002). *If the second order condition* (11) *holds with* $A(t) = \gamma\ \beta\ t^\rho$, $\rho < 0$, *if* (5) *holds, and if* $\sqrt{k}A(n/k) \to \infty$, *then, with* $\hat{\beta}_\rho(k)$ *given in* (20), $\hat{\beta}_\rho(k)$ *is asymptotically normal and converges in probability towards* β, *as* $n \to \infty$.

Proposition 2.3 (Gomes, de Haan and Rodrigues, 2005). *Under the conditions in* Proposition 2.2, *with* $\hat{\rho}_n^{(\tau)}(k)$ *and* $\widehat{\beta}_{\widehat{\rho}}(k)$ *given in* (18) *and* (20), *respectively, and* $\hat{\rho} = \hat{\rho}_n^{(\tau)}(k)$ *for any* τ *and* k, *such that* $\hat{\rho} - \rho = o_p(1/\ln n)$, *as* $n \to \infty$, $\widehat{\beta}_{\widehat{\rho}}(k)$ *is consistent for the estimation of* β. *Moreover,* $\hat{\beta}_{\hat{\rho}}(k) - \beta \overset{p}{\sim} -\beta \ln(n/k)(\hat{\rho} - \rho) = o_p(1)$.

Remark 2.3. *We shall denote generically* $\hat{\rho}$ *any of the estimators in* (18), *computed at* k_1 *in* (19) *and* $\hat{\beta}$ *any estimator in* (20), *also computed at the value* k_1.

3 Reduced-bias estimation of γ and C

3.1 The asymptotic behaviour of the reduced-bias tail index estimators

We now state the following:

Proposition 3.1 (Caeiro *et al.*, 2005). *If* (11) *holds, if* $k = k_n$ *is a sequence of intermediate positive integers, i.e.,* (5) *holds, and if* $\sqrt{k}A(n/k) \to \lambda$, *finite and non necessarily null, as* $n \to \infty$, *then*

$$\sqrt{k}\left(\overline{H}_{\beta,\rho}(k) - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} Normal(0, \gamma^2).$$

This same limiting behaviour holds true if we replace $\overline{H}_{\beta,\rho}$ *by* $\overline{H}_{\hat{\beta},\hat{\rho}}$, *provided that* $\hat{\rho} - \rho = o_p(1/\ln n)$, *and we choose* $\hat{\beta} := \hat{\beta}_{\hat{\rho}}(k_1)$, *with* k_1 *and* $\hat{\beta}_{\hat{\rho}}(k)$ *given in* (19) *and* (20), *respectively. More specifically, and with* Z_k *an asymptotic standard normal r.v., we can then write*

$$\overline{H}_{\hat{\beta},\hat{\rho}}(k) \overset{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k + o_p(A(n/k)).$$

Remark 3.1. *Contrarily to what happens in Drees' class of functionals* (Drees, 1998), *where the minimal asymptotic variance of a reduced-bias tail index estimator is* $(\gamma(1-\rho)/\rho)^2$, *we have been here able to obtain a reduced-bias tail index estimator with an asymptotic variance* γ^2, *the asymptotic variance of Hill's estimator, the maximum likelihood estimator of* γ *for a strict Pareto model.*

3.2 The asymptotic behaviour of the C-estimator

We may state the following:

Proposition 3.2. *Let* F *be a model in Hall's class* (3). *If we consider the Hill estimator in* (8) *and plug it in* (6), *i.e., if we consider* $C_H(k)$, *the* C*-estimator proposed in* (6), *further assuming that* $\sqrt{k}A(n/k) \to \lambda$, *we have*

$$\frac{\sqrt{k}}{\ln n}\left(\frac{C_H(k) - C}{C}\right) \overset{d}{\longrightarrow} N\left(\frac{-\lambda}{(1-\rho)(1-2\rho)}, \frac{\gamma^2}{(1-2\rho)^2}\right).$$

We shall now consider the r.v.'s $\widetilde{C}_\gamma$ and $\overline{C}_\gamma$, with $\widetilde{C}_{\hat{\gamma}_R}$ and $\widetilde{C}_{\hat{\gamma}_R}$ given in (14) and (15), respectively:

Theorem 3.1. *Under the second order framework in* (11)*, for k values such that* (5) *holds and for models F in* (3)*,*

$$\widetilde{C}_\gamma(k) \stackrel{d}{=} C\left(1+\frac{\gamma\sigma_C}{\sqrt{k}}Z_k^C+\frac{2^{(\gamma+\rho)}-1}{2^\gamma-1}\,\frac{A(n/k)}{\rho}+o_p(A(n/k))\right) \tag{21}$$

and

$$\overline{C}_\gamma(k) \stackrel{d}{=} C\left(1+\frac{\gamma\sigma_C}{\sqrt{k}}Z_k^C+o_p(A(n/k))\right) \tag{22}$$

where $\sigma_C^2=1+\left(\frac{2^\gamma}{2^\gamma-1}\right)^2$ and Z_k^C is a sequence of asymptotically standard normal r.v.'s.

The following Corollary shows that for some intermediate k-values, only $\overline{C}_\gamma(k)$ has an asymptotic null mean value, keeping the same asymptotic variance as $\widetilde{C}_\gamma(k)$.

Corolary 3.1. *Under the conditions in Theorem 3.1 and for intermediate k such that $\sqrt{k}\,A(n/k)\to\lambda$,*

$$\sqrt{k}\left(\frac{\widetilde{C}_\gamma(k)-C}{C}\right)\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}N\left(\frac{\lambda(2^{(\gamma+\rho)}-1)}{\rho(2^\gamma-1)},\gamma^2\sigma_{C2}^2\right),\qquad \sqrt{k}\left(\frac{\overline{C}_\gamma(k)-C}{C}\right)\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}N\left(0,\gamma^2\sigma_C^2\right).$$

Theorem 3.2. *Under the conditions in Theorem 3.1, assume that $\sqrt{k}\,A(n/k)\to\lambda$ and $\hat{\gamma}_R\equiv\hat{\gamma}_R(k)$ is a second order reduced-bias extreme value index estimator, such that* (13) *holds. Then,*

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\widetilde{C}_{\hat{\gamma}_R}(k)-C}{C}\right)\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}N\left(0,\left(\frac{\gamma\sigma_R}{1-2\rho}\right)^2\right). \tag{23}$$

If we further consider $\hat{\rho}$ and $\hat{\beta}$ such that $\hat{\rho}-\rho=o_p(1/\ln n)$ and $\hat{\beta}-\beta=o_p(1)$, as $n\to\infty$,

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\overline{C}_{\hat{\gamma}_R}(k)-C}{C}\right)\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}N\left(0,\left(\frac{\gamma\sigma_R}{1-2\rho}\right)^2\right). \tag{24}$$

4 The asymptotic behaviour of reduced-bias quantile estimators

Details on semi-parametric estimation of extremely high quantiles for a general extreme value index $\gamma\in\mathbb{R}$ may be found in de Haan and Rootzn (1993) and more recently in Ferreira *et al.* (2003). Matthys and Beirlant (2003), Gomes and Figueiredo (2006), Mathys *et al.* (2004), Beirlant *et al.* (2006) and Gomes and Pestana (2007) deal with heavy tails and reduced-bias quantile estimation. Since we will work only with the asympotic unbiased extreme value estimator $\hat{\gamma}_R\equiv\overline{H}$ in (17), we shall next consider the high quantile estimator,

$$\overline{Q}_{\overline{H}}^{(p)}(k):=\frac{X_{n-[k/2]:n}-X_{n-k:n}}{2^{\overline{H}(k)}-1}\left(\frac{k}{n\,p}\right)^{\overline{H}(k)}\times(1-B_{1/2}(\overline{H}(k),\hat{\rho},\hat{\beta})). \tag{25}$$

We may state the following results:

Theorem 4.1. *Under the second order framework in* (11) *with $A(t)=\gamma\beta\,t^\rho$, for intermediate k, i.e., k such that* (5) *holds, whenever $\ln(np)/\sqrt{k}\to 0$, and $\sqrt{k}\,A(n/k)\to\lambda$, as $n\to\infty$,*

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{Q_{\overline{H}}^{(p)}(k)}{\chi_{1-p}}-1\right)\underset{n\to\infty}{\stackrel{d}{\longrightarrow}}N\left(\frac{\lambda}{1-\rho},\gamma^2\right). \tag{26}$$

Moreover, for $\hat{\rho}$ and $\hat{\beta}$ introduced in Remark 2.3, *such that $\hat{\rho} - \rho = o_p(1/\ln n)$, as $n \to \infty$,*

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{\overline{Q}_{\overline{H}}^{(p)}(k)}{\chi_{1-p}} - 1\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(0, \gamma^2\right). \tag{27}$$

Remark 4.1. *In equation (27) we have a mean value equal to 0, even if $\sqrt{k}A(n/k) \to \lambda \neq 0$, as $n \to \infty$.*

5 Application to Financial Data

We shall finally consider an illustration of the performance of the above mentioned estimators, reporting results associated to the Euro-UK Pound exchange rates from January 2, 2004 until December 29, 2006, which correspond to a sample of size $n = 771$. This data has been collected by the European System of Central Banks, and was obtained from http://www.bportugal.pt/.

The Value at Risk (VaR) is a common risk measure, defined as a large quantile of the log-returns, i.e., of $L_t = \ln(X_{t+1}/X_t)$, $1 \leq t \leq n-1$, assumed to be stationary and weakly dependent. Working with the $n^- = 384$ negative log-returns, we show in Figure 1 (left) the sample paths of the ρ-estimates associated to $\tau = 0$ and $\tau = 1$. They lead us to choose, on the basis of any stability criterion for large values of k, the estimate associated to $\tau = 0$. From the experience we have with this class of estimates, this means that $|\rho| \leq 1$ and the tuning parameter $\tau = 0$ is then advisable. We have got $\hat{\rho} = -0.61$. The use of $\hat{\beta}$ in (20), computed at the level k_1 in (19), i.e., at $k_1 = (n^-)^{0.995} = 372$, leads then us to the estimate $\hat{\beta} = 1.06$.

The sample paths of the classical Hill estimator in (8) and the second order reduced-bias tail index estimator $\overline{H}$ in (17) are presented also in Figure 1 (center). The associated Var-estimators in (7) and (25), respectively, for $p = 0.001$, are pictured in Figure 1 (right).

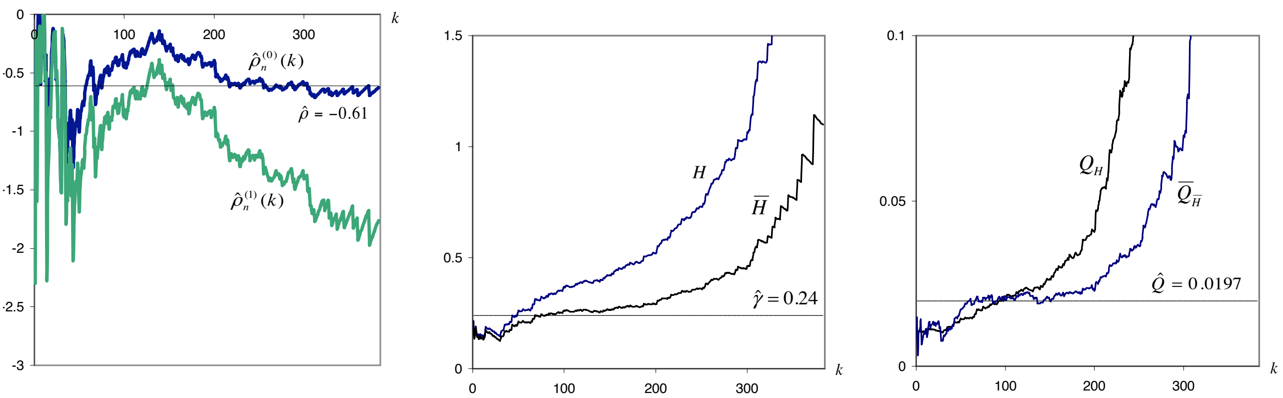

Figure 1: Estimates of the first-order parameter γ (*left*) and of the high quantile $\chi_{0.001}$ (*right*).

For the Hill estimator, as we know how to estimate the second order parameters ρ and β, we can estimate the optimal sample fraction and the extreme value index. We get $\hat{k}_0^H = 24$ and $H(24) = 0.16$. Since we do not have yet the possibility of adaptively estimate the optimal sample fraction associated to any second order reduced-bias estimator, the estimate pictured, $\hat{\gamma} = 0.24$, is the median of the $\overline{H}(k)$ estimates for k between k_0^H and $5 \times k_0^H$. A similar technique led us to the quantile estimate $\chi_{0.001} = 0.0197$, as pictured in Figure 1 (right).

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