Proposal for a PIIAL student project

Permutation reconstruction

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Introduction and background

The theory of permutation patterns and pattern avoidance has been an active field of research in the past decades. The fundamental notion of the theory is the pattern involvement relation. Writing a permutation $\pi \in S_n$ as a word $\pi_1\pi_2\ldots\pi_n$ where $\pi_i = \pi(i)$, a permutation $\tau \in S_\ell$ is called a *pattern* of π (or we say that π *involves* τ) if $\tau_1\tau_2\ldots\tau_\ell$ is order-isomorphic to some subword $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_\ell}$ of π.

For example, the permutation $\pi = 25134$ involves the following 4-patterns: 4123, 2134, 1423, and 2413. Note that the pattern 2413 occurs in π in two different ways; it is obtained by removing either the entry 3 or 4 from π. Similarly, the 3-patterns of π are 123 (2 occurrences), 132 (2 occurrences), 213 (2 occurrences), 231 (1 occurrence), and 312 (3 occurrences).

The notion of pattern involvement gives rise to the following reconstruction problem. Let n and k be positive integers. Is a permutation $\pi \in S_n$ uniquely determined by the collection of its $(n-k)$-patterns? By the word "collection" we usually mean a multiset, but we get a slightly different problem if we consider sets instead of multisets. The multiset of all $(n-k)$-patterns of π is referred to as the $(n-k)$-*deck* of π, and its elements are called the $(n-k)$-*cards* of π. Similarly, the set of all $(n-k)$-patterns of π is called the $(n-k)$-*set-deck* of π. (The parameter $n-k$ can be dropped from this nomenclature when it is clear from the context or irrelevant.)

The problem of reconstructing a permutation from its patterns has been investigated by several authors. It is known that for $n \geq 5$, every n-permutation is reconstructible from its $(n-1)$-set-deck (Ginsburg [1]) and from its $(n-1)$-deck (Smith [4], Raykova [3]). Raykova [3] and Smith [4] also considered $(n-k)$-decks for $k \geq 1$, and they proved the existence of and determined a few exact values and provided upper and lower bounds for the number N_k that is defined as the smallest number M such that all permutations of rank $n \geq M$ are reconstructible from their $(n-k)$-decks.

When a permutation is reconstructible, its deck contains enough information for uniquely determining the permutation. There may, nonetheless, be some redundancy in the deck; perhaps the permutation is uniquely determined by just a small number of cards. This raises the question how many cards are sufficient to guarantee reconstructibility. Given positive integers n and k, let us define $H_k(n)$ as the minimum number M such that every n-permutation is uniquely determined by any M of its $(n-k)$-cards. The problem of determining the numbers $H_k(n)$ was posed already by Ginsburg [1]. The case when $k = 1$ was recently settled in [2]: for $n \geq 5$, we have $H_1(n) = \lceil n/2 \rceil + 2$. Not much is known about these numbers for $k \geq 2$.

Problem statement

For $n \geq 6$, what is the smallest number M such that every n-permutation is uniquely determined by any M of its $(n-2)$-cards? In other words, what are the numbers $H_2(n)$?

There are also other possible questions, problems, or tasks surrounding this theme, such as the following:

- How about $H_k(n)$ for $k > 2$?
- Describe an efficient method for deciding whether a given collection of $(n-k)$-permutations is a (partial) deck of some n-permutation.
- Improve known upper or lower bounds for the numbers N_k.

Prerequisites

Basic knowledge of discrete mathematics. Mathematical maturity. Desirable: programming skills.

References

[1] J. Ginsburg, Determining a permutation from its set of reductions, *Ars Combin.* **82** (2007) 55–67.

[2] M. J. Gouveia, E. Lehtonen, Permutation reconstruction from a few large patterns, manuscript in preparation.

[3] M. Raykova, Permutation reconstruction from minors, *Electron. J. Combin.* **13** (2006) #R66.
http://doi.org/10.37236/1092

[4] R. Smith, Permutation reconstruction, *Electron. J. Combin.* **13** (2006) #N11.
http://doi.org/10.37236/1149