Mathematics

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Sets

Sets Theory was develop by Cantor in the end of XIX century and it has influenced almost all areas of Mathematics and constitutes a pilar of Modern Mathematics. The term "set" was coined by Bernard Bolzano as the translation of the German "Menge", appearing in his work "The Paradoxes of the Infinite".

Definition

Sets A **set** is a collection of distinct and well-defined objects, of any kind. These objects are the **elements** of the set. Usually we use capital letters to designate a set and small letters for elements.

To say that x *is an element of set* X $x \in X$ meaning that "x belongs to X". To represent that "x is not in X" we write $x \notin X$ reading "x does not belong to X".

In what follows A and B are two arbitrary set.

Definition

A is a **subset** of B and we say that A **is contained** in B, writing $A \subseteq B$, if every element of A is also an element of B. Otherwise we write $A \nsubseteq B$, in which case at least one element from A is not an element of B.

B is a **superset** of A and we say that B **contains** A, writing $B \supseteq A$

$$A \subseteq B \leftrightarrow \forall a \in A, a \in B$$
$$A \nsubseteq B \leftrightarrow \exists a \in A, a \notin B$$
$$A \subseteq A, \forall A \quad (1)$$

$\forall$ For all, $\exists$ It exists

Definition

A and B are similar if they have the same elements and we write that $A = B$.

$$A = B \leftrightarrow \forall a \in A, a \in B \text{ and}$$
$$\forall a \in B, a \in A$$
$$A = B \leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Example

Let $A = \{1, 2, 3, 5\}$, $B = \{1, 3\}$ e $C = \{2, 4, 5\}$.

$$2 \in A, \text{ but } 2 \notin B.$$

$$B \subseteq A, \text{ but } B \subsetneq A.$$

$$B \nsubseteq C \text{ and } C \nsubseteq B.$$

Definition

Empty set An **empty set** is a set without any element represented by $\{\}$or $\varnothing$.

Some common sets are:

$\mathbb{N}$ - Set of natural numbers

$\mathbb{Z}$ -Set of integer numbers

$\mathbb{Q}$ - Set of racional numbers

$\mathbb{R}$ - Set of real numbers

$\mathbb{C}$ - Set of complex numbers.

Definition
We can represent a set using

Tabular Form Listing all the elements of a set, separated by commas and enclosed within curly brackets $\{\}$.

Descriptive Form State in words the elements of the set.

Set Builder Form Writing in symbolic form the common characteristics shared by all the elements of the set

Example

These are different representations of the same set

- Tabular Form - $A = \{1, 3, 5, 7, 9, \dots\}$.
- Descriptive Form - $A =$Set of positive odd integer.
- Set Builder Form - $A = \{x :\ x = 2n - 1,\ n \in \mathbf{N}\}$

Definition

Operations with sets Given sets A and B, we m

Union $A \cup B$ - set that consists of all elements belonging to either set A or set B (or both).

Intersection $A \cap B$ - set composed of all elements that belong to both A and B.

Setminus $A \setminus B$ - set composed by all elements in A that are not in B.

Complemet A^c - set of all elements in the universe U that are not in A. We admit that the admissible elements are restricted to some fixed class of objects U called the universal set (or universe). Also can be described as $U \setminus A$

Cartesian Product $A \times B$ - set consisting of all ordered pairs (a, b) for which $a \in A$ and $b \in B$.

Example

Given $A = \{1, 5\}$, $B = \{1, 4\}$ and $U = \{1, 2, 3, 4, 5\}$

$$\begin{aligned} A \cup B &= \{1, 4, 5\} \\ A \cap B &= \{1\} \\ A \setminus B &= \{5\} \\ A^c &= \{2, 3, 4\} \\ A \times B &= \{(1, 1), (1, 4), (5, 1), (5, 4)\} \\ B \times A &= \{(1, 1), (1, 5), (4, 1), (4, 5)\} \end{aligned}$$

1. Given $A = [-1, 4]$ and $B = \{0, 3, 5, 9\}$ find
   a) $B \setminus A$.
   b) $A \setminus B$.
   c) $B \cup A$.
   d) $B \cap A$.
2. Given $A = [-1, 4]$ and $B =]0, 4[\cup \{-1\}$ find
   a) $B \setminus A$.
   b) $A \setminus B$.
   c) $B \cup A$.
   d) $B \cap A$.

1. Given $A =]-\infty, 4]$ and $B =]-1, +\infty[$ find:
   a) $B \setminus A$.
   b) $A \setminus B$.
   c) $B \cup A$.
   d) $B \cap A$.
2. Given $A = \{-1, 0, 2\}$ and $B = \{-1, 1\}$ find:
   a) $B \setminus A$.
   b) $A \setminus B$.
   c) $B \cup A$.
   d) $B \cap A$.
   e) $B \times A$.

f) $B \times B$.

LOGIC Propositional calculus is a branch of logic. It is also called propositional logic, statement logic, sentential calculus, sentential logic, or sometimes zeroth-order logic. It deals with propositions (which can be true or false) and argument flow. Compound propositions are formed by connecting propositions by logical connectives. The propositions without logical connectives are called atomic propositions.

Unlike first-order logic, propositional logic does not deal with non-logical objects, predicates about them, or quantifiers. However, all the machinery of propositional logic is included in first-order logic and higher-order logics. In this sense, propositional logic is the foundation of first-order logic and higher-order logic.

Definition

Propositional logic is a mathematical model that allows us to reason about the truth or falsehood (T,F) of logical expressions.

Truth tables

Negation (NO)

p	$\sim p$
T	F
F	T

Conjunction (AND)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Disjunction (XOR)

p	q	$p \dot{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (IF THEN)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence (IIF)

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Properties	**Conjunction**
Comutativity	$(p \wedge q) \Leftrightarrow (q \wedge p)$
Associativity	$[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$
Idempotence	$(p \wedge p) \Leftrightarrow p$
Identity	$(p \wedge V) \Leftrightarrow p \Leftrightarrow (V \wedge p)$
Annihilator	$(p \wedge F) \Leftrightarrow F \Leftrightarrow (F \wedge p)$

Properties	Disjunction
Comutativity	$(p \vee q) \Leftrightarrow (q \vee p)$
Associativity	$[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$
Idempotence	$(p \vee p) \Leftrightarrow p$
Identity	$(p \vee F) \Leftrightarrow p \Leftrightarrow (F \vee p)$
Annihilator	$(p \vee V) \Leftrightarrow V \Leftrightarrow (V \vee p)$

Properties Disjunction - Conjunction

Distributivity of $\wedge$ over $\vee$

$$[p \wedge (q \vee r)] \;\Leftrightarrow\; [(p \wedge q) \vee (p \wedge r)]$$

$$[(q \vee r) \wedge p] \;\Leftrightarrow\; [(q \wedge p) \vee (r \wedge p)].$$

Distributivity of $\vee$ over $\wedge$,

$$[p \vee (q \wedge r)] \;\Leftrightarrow\; [(p \vee q) \wedge (p \vee r)]$$

$$[(q \wedge r) \vee p] \;\Leftrightarrow\; [(q \vee p) \wedge (r \vee p)].$$

Negation of $\vee$ **,** $\wedge$ **,** $\Rightarrow$**,** $\Leftrightarrow$,

$$\sim (p \vee q) \;\Leftrightarrow \sim p \wedge \sim q$$

$$\sim (p \wedge q) \;\Leftrightarrow \sim p \vee \sim q$$

$$\sim (p \Rightarrow q) \;\Leftrightarrow\; p \wedge \sim q$$

1. Prove that $(u \wedge v) \vee (u \wedge \sim v) \Leftrightarrow u$
2. Write the table of truth of $(u \wedge v) \vee (u \Rightarrow \sim v)$
3. Simplify $\sim ((u \wedge v) \vee (u \Rightarrow \sim v))$
4. Consider the following propositions

   $$\begin{aligned} p &: \text{ Paul studies Maths} \\ q &: \text{ Carla studies English} \\ r &: \text{ Rui studies Economics} \end{aligned}$$

   Knowing that

   $$((p \wedge q) \Rightarrow r) \Leftrightarrow (q \vee r) \Leftrightarrow \text{ FALSE}$$

   find the subjects that each student studies.
5. Write the table of truth of $(u \wedge v) \Rightarrow (u \vee \sim v)$

1- Considering $r \Leftrightarrow (p \vee \neg q) \Rightarrow (p \wedge q)$

1a- then

a) $r \Leftrightarrow q$

b) $r \Leftrightarrow p$

c) $r \Leftrightarrow p \wedge q$

1b- then

a) $\neg r \Leftrightarrow p \vee q$

b) $\neg r \Leftrightarrow p$

c) $\neg r \Leftrightarrow \neg q$

1c- For $p = F \wedge q = T$ find the logical value of r.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Find the logical value of

$$F \vee \neg T \rightarrow F \wedge T$$

and

$$(F \vee (\neg T \rightarrow F)) \wedge T$$

Representation of equivalence and implication

$$(p \Leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \vee (q \rightarrow p))$$

$$(p \rightarrow q) \Leftrightarrow (\neg p \wedge q)$$

INFERENCE RULE

In logic, a rule of inference, inference rule or transformation

rule is a logical form consisting of a function which takes premises, analyzes their syntax, and returns a conclusion (or conclusions).

$$P, Q \vdash S$$

For example, the rule of inference called modus ponens takes two premises, one in the form "If p then q" and another in the form "p", and returns the conclusion "q".

$$P \to Q, P \vdash Q$$

"If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."

1 $p \wedge s \Rightarrow e$

2 $p \wedge \neg s \Rightarrow \neg e$

3 $p \Rightarrow (s \wedge e) \vee (\neg s \wedge \neg e)$

Prove $1, 2 \vdash 3$

MODUS PONENS

Premise 1: If it's raining then it's cloudy.
Premise 2: It's raining.
Conclusion: It's cloudy.

Premise 1: $P \to Q$
Premise 2: P
Conclusion: Q

The same can be stated succinctly in the following way:

$$P \to Q, P \vdash Q$$

MODUS TOLLENS

If p then q; not q; therefore not p

$$((p \to q) \land \neg q) \vdash \neg p$$

Basic and Derived Argument Forms

Name	Sequent	Description
Modus Ponens	$((p \to q) \land p) \vdash q$	If p then q; p; therefore q
Modus Tollens	$((p \to q) \land \neg q) \vdash \neg p$	If p then q; not q; therefore not p
Hypothetical Syllogism	$((p \to q) \land (q \to r)) \vdash (p \to r)$	If p then q; if q then r; therefore, if p then r
Disjunctive Syllogism	$((p \lor q) \land \neg p) \vdash q$	Either p or q, or both; not p; therefore, q
Constructive Dilemma	$((p \to q) \land (r \to s) \land (p \lor r)) \vdash (q \lor s)$	If p then q; and if r then s; but p or r; therefore q or s
Destructive Dilemma	$((p \to q) \land (r \to s) \land (\neg q \lor \neg s)) \vdash (\neg p \lor \neg r)$	If p then q; and if r then s; but not q or not s; therefore not p or not r
Bidirectional Dilemma	$((p \to q) \land (r \to s) \land (p \lor \neg s)) \vdash (q \lor \neg r)$	If p then q; and if r then s; but p or not s; therefore q or not r
Simplification	$(p \land q) \vdash p$	p and q are true; therefore p is true
Conjunction	$p, q \vdash (p \land q)$	p and q are true separately; therefore they are true conjointly
Addition	$p \vdash (p \lor q)$	p is true; therefore the disjunction (p or q) is true
Composition	$((p \to q) \land (p \to r)) \vdash (p \to (q \land r))$	If p then q; and if p then r; therefore if p is true then q and r are true
De Morgan's Theorem (1)	$\neg(p \land q) \vdash (\neg p \lor \neg q)$	The negation of (p and q) is equiv. to (not p or not q)
De Morgan's Theorem (2)	$\neg(p \lor q) \vdash (\neg p \land \neg q)$	The negation of (p or q) is equiv. to (not p and not q)
Commutation (1)	$(p \lor q) \vdash (q \lor p)$	(p or q) is equiv. to (q or p)
Commutation (2)	$(p \land q) \vdash (q \land p)$	(p and q) is equiv. to (q and p)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(p is equiv. to q) is equiv. to (q is equiv. to p)
Association (1)	$(p \lor (q \lor r)) \vdash ((p \lor q) \lor r)$	p or (q or r) is equiv. to (p or q) or r
Association (2)	$(p \land (q \land r)) \vdash ((p \land q) \land r)$	p and (q and r) is equiv. to (p and q) and r
Distribution (1)	$(p \land (q \lor r)) \vdash ((p \land q) \lor (p \land r))$	p and (q or r) is equiv. to (p and q) or (p and r)
Distribution (2)	$(p \lor (q \land r)) \vdash ((p \lor q) \land (p \lor r))$	p or (q and r) is equiv. to (p or q) and (p or r)
Double Negation	$p \vdash \neg\neg p$	p is equivalent to the negation of not p
Transposition	$(p \to q) \vdash (\neg q \to \neg p)$	If p then q is equiv. to if not q then not p
Material Implication	$(p \to q) \vdash (\neg p \lor q)$	If p then q is equiv. to not p or q
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \to q) \land (q \to p))$	(p iff q) is equiv. to (if p is true then q is true) and (if q is true then p is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \land q) \lor (\neg p \land \neg q))$	(p iff q) is equiv. to either (p and q are true) or (both p and q are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \lor \neg q) \land (\neg p \lor q))$	(p iff q) is equiv to., both (p or not q is true) and (not p or q is true)
Exportation[10]	$((p \land q) \to r) \vdash (p \to (q \to r))$	from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$(p \to (q \to r)) \vdash ((p \land q) \to r)$	If p then (if q then r) is equivalent to if p and q then r
Tautology (1)	$p \vdash (p \lor p)$	p is true is equiv. to p is true or p is true
Tautology (2)	$p \vdash (p \land p)$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \lor \neg p)$	p or not p is true
Law of Non-Contradiction	$\vdash \neg(p \land \neg p)$	p and not p is false, is a true statement

$p(x)$ for $x \in A$

$$\forall x \in A, p(x)$$

$$\exists x \in A, p(x)$$

Example:

$$\forall x \in A, x^2 \geq 0$$

$$\forall x \in A, x^2 \geq 1$$

$$\exists x \in A, x^2 \geq 0$$

$$\exists x \in A, x^2 \geq 1$$

Negation

$$\neg(\forall x \in A, p(x)) \Leftrightarrow \exists x \in A, \neg p(x)$$

$$\neg(\exists x \in A, p(x)) \Leftrightarrow \forall x \in A, \neg p(x)$$