Mathematics

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Sequences

Definition
(Sequence)
A sequence (infinite) is a function of $\mathbb{N}$ in $\mathbb{R}$

To simplify notation instead of $f(n)$ we use f_n and in general we adopt the letter u, v, w to designate sequences.

Unlike a set, the same elements can appear multiple times at different positions in a sequence, and order matters. The variable n is called an index. The position of an element in a sequence is its rank or index

Example

$$u_n = \frac{n+1}{n+2} \tag{1}$$

For u_n in 1, the element of rank 1 is $u_1 = \frac{1+1}{1+2} = 2/3$. The element of rank 5 is $6/7$.

A sequence can be defined by a list of its first elements, $v_n = \{1, 4, 9, 16, 25, \dots\}$ by the general term $v_n = n^2$ or by recursion. In a sequence defined by recursion a term depends on previous terms, like the Fibonacci numbers

Example

$$\begin{cases} w_1 = 0 \\ w_2 = 1 \\ w_{n+2} = w_n + w_{n+1} \end{cases}$$

For the Fibonacci sequence to find the element of rank 5 we have first to find the element of rank 3, $w_3 = w_2 + w_1 = 1$, of rank 4 $w_4 = w_3 + w_2 = 2$ and finally $w_5 = w_4 + w_3 = 2 + 1 = 3$.

Example

For the sequence defined by recursion:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + n + 1, \quad \forall n \in \mathbb{N} \end{cases}$$

the first 7 elements are $1, 3, 6, 10, 15, 21, 28$.

The sequence:

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_{n+2} = 2a_{n+1} + a_n, \quad \forall n \in \mathbb{N} \end{cases}$$

has as first elements $1, 1, 3, 7, 17, 41$.

Properties

There are several properties that are important to study a sequence.

Definition

(Increasing and decreasing)

A sequence u_n is said to be

- monotonically increasing if $u_{n+1} \geq u_n, \ \forall n \in \mathbb{N}.$
- strictly monotonically increasing if $u_{n+1} > u_n, \ \forall n \in \mathbb{N}.$

Definition

(Increasing and decreasing)

A sequence u_n is said to be

- monotonically decreasing if $u_{n+1} \leq u_n, \ \forall n \in \mathbb{N}$.
- strictly monotonically decreasing if $u_{n+1} < u_n, \ \forall n \in \mathbb{N}$.

Example

Lets study the monotonicity of the sequence

$$a_n = \frac{n+1}{2^n}.$$

By definition lets study the sign of

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)+1}{2^{n+1}} - \frac{n+1}{2^n} = \frac{n+2}{2^n \times 2} - \frac{n+1}{2^n} \\ &= \frac{n+2-(n+1)\times 2}{2^n \times 2} = \frac{n+2-2n-2}{2^n \times 2} \\ &= \frac{-n}{2^{n+1}}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

a_n is strictly monotonically decreasing.

Example

Now for

$$b_n = \frac{1}{7-2n}.$$

just looking at the first elements of this sequence,

$$b_1 = \frac{1}{5};\ \ b_2 = \frac{1}{3};\ \ b_3 = 1;\ \ b_4 = -1$$

we see that $b_1 < b_2 < b_3$ but $b_3 > b_4$ so we may conclude that b_n is not monotone.

Definition

(Bounded) A sequence u_n is said to be

- bounded from above if all the terms are less than some real number M, there is if,

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : u_n \leq M.$$

- bounded from below if all the terms are greater than some real number M, there is if,

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : u_n \geq M.$$

Definition

(Bounded) A sequence u_n is said to be

- bounded if it is both bounded from above and bounded from below,

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : |u_n| \leq M.$$

Arithmetic and Geometric Progressions

The sequence (a_n) with elements $1, 4, 7, 10, 13, \dots$ have a special feature. In fact we can easily note that for (a_n)

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 3, \quad \forall n \in \mathbb{N} \end{cases}$$

Sequences with these behavior are known as Arithmetic Progressions.

The sequence (b_n) with elements $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$, have a special feature. In fact for (b_n)

$$\begin{cases} b_1 = 1 \\ b_{n+1} = \frac{1}{2} b_n, \quad \forall n \in \mathbb{N} \end{cases}$$

Sequences with these behavior are known as Geometric Progressions.

Definition

(Progressions)

A sequence u_n is said to be

- an Arithmetic Progressions if the difference between the consecutive terms is constant.

$$\forall n \in \mathbb{N} : u_{n+1} = u_n + k = u_1 + nk, k \in \mathbb{R}$$

k is the common difference.

Definition

(Progressions)

A sequence u_n is said to be

- a Geometric Progressions if the quotient of any two successive members of the sequence is a constant

$$\forall n \in \mathbb{N} : u_{n+1} = r u_n = r^n u_1, r \in \mathbb{R} \setminus \{0\}$$

  $r \neq 0$ is the common ratio and u_1 is a scale factor

We may observe that an arithmetic progression is monotonically

- increasing if the common difference $k > 0$
- decreasing if $k < 0$.
- if $k = 0$ then the sequence is constant.

Regarding the monotonicity of a geometric progression with common ratio r and scale factor u_1 its is

- Increasing if $u_1 > 0$ and $r > 1$ or if $a_1 < 0$ and $0 < r < 1$;
- Decreasing if $a_1 > 0$ and $0 < r < 1$ or if $a_1 < 0$ and $r > 1$;

- Constant if $r = 1$;
- Not monotone if $r < 0$.

The sum S_n of the first n terms of an arithmetic progression (a_n), is given by

$$S_n = \frac{a_1 + a_n}{2} \times n.$$

The sum S_n of the first n terms of a geometric progression (a_n), is given by

$$S_n = a_1 \, \frac{1 - r^n}{1 - r}$$

where r is the common ratio and a_1 the scale factor.

Limits

Consider (a_n) the sequence $1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, \ldots, 1+\frac{1}{2^n}, \ldots$. This sequence is monotonically decreasing, with elements positive and approaching 1. In fact the distance between the elements of the sequence and 1, given by

$$|a_n - 1|$$

takes the values $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$. No matter how small we consider this distance, say ε, we know that we will find a rank p such that the distance of the elements of the sequence

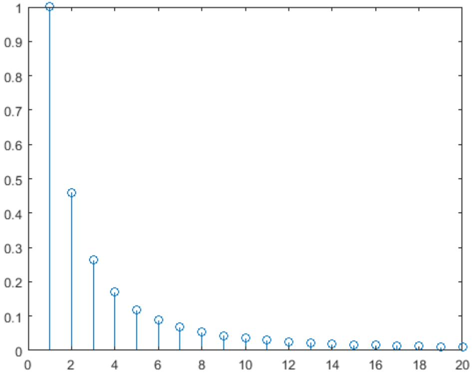

Figure 1: Plot $|a_n - 1|$.

For every real number $\varepsilon > 0$, there is a natural number p such that for every natural number $n > p$, we have $|a_n - 1| < \varepsilon$".

Definition

(Limit)

A sequence a_n is said to converge to the limit a and we write

$$\lim_{n \to +\infty} a_n = a \text{ or } a_n \to a \text{ if}$$

$$\forall \varepsilon > 0 \ \ \exists p \in \mathbb{N} \ \ \forall n \in \mathbb{N} : \quad n > p \Rightarrow |a_n - a| < \varepsilon.$$

Algebra of limits

We shall introduce some results regarding arithmetic operations on limits.

Theorem

If (a_n) and (b_n) are convergent sequences, then the sequence $(a_n + b_n)$ is convergent and

$$\lim (a_n + b_n) = \lim a_n + \lim b_n.$$

Theorem

If (a_n) and (b_n) are convergent sequences, then the sequence $(a_n \times b_n)$ is convergent and

$$\lim(a_n \times b_n) = \lim a_n \times \lim b_n.$$

Theorem

If (a_n) is a convergent sequence and p is a natural number, then the sequence $(a_n)^p$ is convergent and

$$\lim(a_n)^p = (\lim a_n)^p.$$

Theorem
If (a_n) and (b_n) are convergent sequences, then the sequence $(a_n - b_n)$ is convergent and

$$\lim\,(a_n - b_n) = \lim\; a_n - \lim\; b_n.$$

Theorem
If (a_n) and (b_n) are convergent sequences, $b_n \neq 0$, $\forall n \in \mathbb{N}$, and $\lim b_n \neq 0$ then the sequence, $\left(\dfrac{a_n}{b_n}\right)$ is convergent and

$$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}.$$

Theorem
If p is a natural number and (a_n) is a convergent sequence with non-negative elements, then the sequence $(\sqrt[p]{a_n})$ is convergent and

$$\lim \sqrt[p]{a_n} = \sqrt[p]{\lim a_n}.$$

Infinite limits

Theorem

A sequence (a_n) is said to tend to infinity (as n tends to infinity), or to have infinity as its limit, and we write $\lim a_n = +\infty$, if $\forall L > 0 \quad \exists p \in \mathbb{N} \quad \forall n \in \mathbb{N} :$ $n > p \Rightarrow a_n > L.$

Theorem

A sequence (a_n) is said to tend to minus infinity (as n tends to minus infinity), or to have $-\infty$ as its limit, and we write $\lim a_n = -\infty$, if $\forall L > 0 \;\; \exists p \in \mathbb{N} \;\; \forall n \in \mathbb{N} : \;\; n > p \Rightarrow a_n < -L.$

Question: What about $b_n = (-2)^n$?

Show that $\lim a_n = +\infty$ using the definition for

$$a_n = \begin{cases} n+1, & \text{se n é par} \\ n^2 - 10, & \text{se n é ímpar} \end{cases}$$

In $\overline{\mathbb{R}}$:

$$a \times \infty = \infty \qquad (a \neq 0)$$
$$\frac{a}{0} = \infty \qquad (a \neq 0)$$
$$\frac{a}{\infty} = 0 \qquad (a \neq \infty)$$
$$\frac{\infty}{a} = \infty \qquad (a \neq \infty)$$
$$\infty^p = \infty \qquad (p \in \mathbb{N})$$
$$\sqrt[p]{\infty} = \infty \qquad (p \in \mathbb{N})$$
$$\infty^k = 0 \qquad (k < 0)$$

Indeterminates

In calculus limits involving an algebraic combination of sequences are evaluated by replacing the sequences by their limits; if the expression obtained after this substitution cannot be evaluates because of lack of information it is said to take on an indeterminate form.

The most common indeterminate forms are:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 1^{\infty}, \infty - \infty, 0^0 \text{ and } \infty^0.$$

Special limits - ratio of polynomial in n

For $k, r \in \mathbb{N}$,

$$\lim \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0}{b_r n^r + b_{r-1} n^{r-1} + \cdots + b_0} = \begin{cases} \infty & \text{if } k > r \\ a_k/b_r & \text{if } k = r \\ 0 & \text{if } k < r \end{cases}$$

Example

$$\lim \left(\frac{n^2 - 3}{2n^2 + 1} \right) = 1/2$$

Exercices:

1. $\left(\frac{n^2 - 3}{2n^2 + 3n + 1} \right)$;
2. $\left(\frac{n^2 - 3}{n + 1} \right)$;
3. $\left(\frac{n^2 - 3}{4n^3 + n^2 + 1} \right)$;
4. $\left(\frac{4n^4 + n^3 + 2}{2n^4 + 6n + +1} \right)$;

Special limits - Generalization of ratio of polynomials

The previous result cam be generalized to powers of racional exponent, for example:

$$\lim \frac{\sqrt[3]{3n^3+3}}{\sqrt[2]{2n^2+3}} = \lim \frac{\sqrt[6]{(3n^3+3)^2}}{\sqrt[6]{(2n^2+3)^3}} = \lim \sqrt[6]{\frac{(3n^3+3)^2}{(2n^2+3)^3}} =$$

$$\sqrt[6]{\frac{3^2}{2^3}} = \frac{\sqrt[3]{3}}{\sqrt[2]{2}}$$

For $k, r \in \mathbb{Q}^+$,

$$\lim \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0}{b_r n^r + b_{r-1} n^{r-1} + \cdots + b_0} = \begin{cases} \infty & \text{if } k > r \\ a_k / b_r & \text{if } k = r \\ 0 & \text{if } k < r \end{cases}$$

Example

$$\lim \frac{\sqrt[2]{n^2 - 3}}{\sqrt[2]{4n^2 + n + 1}} = \frac{\sqrt[2]{1}}{\sqrt[2]{4}}$$

Exercices:

1. $\lim \dfrac{n^2 \sqrt[2]{n^2+1}}{\sqrt[2]{3n^6+n+1}}$;
2. $\lim \dfrac{2n \sqrt[2]{n-3}+n^2}{\sqrt[6]{4n^6+n^2+1}}$;
3. $\lim \dfrac{n \sqrt[4]{n^2-3}+n^2}{4n^3+1}$;
4. $\lim \dfrac{n^4+\sqrt[2]{n-3}+n^2}{\sqrt[5]{4n^{10}+n^2+1}}$;

Special limits - Exponential

Value a	Monotony a^n
$a > 1$	increasing
$a = 1$	constant
$0 < a < 1$	decreasing
$a = 0$	constant
$a < 0$	not monotone

Value of a	Limit of a^n
$a > 1$	$+\infty$
$a = 1$	1
$-1 < a < 1$	0
$a = -1$	does not exist
$a < -1$	∞

Example

$$\left(\frac{3}{4}\right)^n = 0$$

Exercise $\left(\dfrac{4n}{2n+1}\right)^n$

Special limits - Nepper

$$\lim \left(1 + \frac{k}{n}\right)^n = e^k$$

If $u_n \longrightarrow +\infty$

$$\lim \left(1 + \frac{k}{u_n}\right)^{u_n} = e^k$$

If $v_n \longrightarrow -\infty$

$$\lim \left(1 + \frac{k}{v_n}\right)^{v_n} = e^k$$

Example

$$\left(\frac{n+2}{n}\right)^{n+2} = \left(\frac{n+2}{n}\right)^{n} \left(\frac{n+2}{n}\right)^{2} =$$
$$= \left(1+\frac{2}{n}\right)^{n} \left(\frac{n+2}{n}\right)^{2} = e^2.1 = e^2$$

Exercise

1. $\left(\frac{n-3}{n}\right)^{n+1}$;
2. $\left(1-\frac{1}{n+1}\right)^{n}$;
3. $\left(1+\frac{2}{3n}\right)^{n}$;
4. $\left(\frac{2n-1}{3n+2}\right)^{n}$;
5. $\left(1-\frac{4}{n^2}\right)^{2n}$;
6. $\left(\frac{2n+3}{-3n+5}\right)^{4n}$;
7. $\left(1-\frac{2}{n^2}\right)^{n^3}$.

Special limits - Product of an infinitesimal by a bounded sequence

If $u_n \longrightarrow \infty$ and $v_n \longrightarrow 0$ then $\lim\left(u_n v_n\right) = 0.$

Example

To find $\lim\left((-1)^n \frac{1}{n^2+1}\right)$ we cannot apply the algebra of limits because $\lim(-1)^n$ does not exists but it is bounded since $-1 \leq (-1)^n \leq 1$. Since $\lim \frac{1}{n^2+1} \longrightarrow 0$ we may conclude that $\lim\left((-1)^n \frac{1}{n^2+1}\right) \longrightarrow 0.$

Exercise

1. $\lim \left(\frac{-1}{n+1}\right)^n$;
2. $\left(\sin(n)\dfrac{1}{n+1}\right)$;

Exercises

1. Consider the sequence $u_n = \dfrac{2n-1}{n+1}$.
   a) Find the terms of rank 5, 20 and n+1.
   b) Given the real numbers $\dfrac{29}{16}, \dfrac{40}{19}$ find it they are elements of u_n.
   c) Prove that:
      (i) (u_n) is monotonically increasing;
      (ii) $\forall n \in \mathbb{N},\ \frac{1}{2} \leq u_n < 2$;
      (iii) (u_n) is convergent.
   d) Find an upper and lower limit.

2. Given $u_n = \dfrac{\sqrt{2n}}{1+\sqrt{n}}$:

   a) Show that $\lim u_n = \sqrt{2}$

   b) Find the rank of the first element of the sequence that verifies

$$|u_n - \sqrt{2}| < 10^{-1}.$$

3. Show that the sequence $b_n = \dfrac{2^n}{(n+1)!}$ é is strictly increasing.

4. Consider

$$u_n = -2 \times 3^{n-5}.$$

   a) Show that u_n is a geometic progression .

b) Study its monotonicity.

c) Find $\sum_{k=2}^{8} u_k$.

5. In a aritmetic progression with common diffrence 5 we know that the element of rank 10 its three times the element of rank 8. Find the sum of the first 20 elements.

6. Find the limit of

   a) $\dfrac{4-n^2}{n^3-2}$

   b) $\dfrac{2}{n^3+5} \times \sqrt{n-3}$

c) $\dfrac{5^n + (-7)^{n+1}}{4^{n+2} - 3^n}$

d) $\left(\dfrac{n+5}{n+2}\right)^n$

e) $\left(\dfrac{n^3-2}{n^3}\right)^{n^2-3}$

7. Let (a_n) be the general term . Write a_{n+1}, a_{2n} and a_{n+p}, $p \in \mathbb{N}$, for the following cases:

a) $a_n = \dfrac{2^n}{n+1}$

b) $a_n = \dfrac{(n+1)!}{(3n-1)!}$

c) $a_n = \dfrac{(n-1)^2}{2n+1}$

d) $a_n = \sqrt[n]{\dfrac{(2n-1)!}{2^{n+1} + \log n}}$

e) $a_n = \dfrac{(n^2+1)!}{(n^2-1)!}$

8. Write the general term of the following sequences and check if they are bounded.

   a) The sequence formed by the simetrics of the perfect squares.

   b) The sequence of the powers of base (-2) and natural exponent.