Efficiency of partially reduced-bias mean-of-order-p versus minimum-variance reduced-bias extreme value index estimation

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Abstract: A recent class of estimators of a positive extreme value index (EVI), related to a mean-of-order-p (MOP) class of EVI-estimators is enlarged and studied for finite samples through a Monte-Carlo simulation study. A comparison of this class and a representative class of minimum-variance reduced-bias (MVRB) EVI-estimators is performed. The class of MVRB EVI-estimators is related to a direct removal of the dominant component of the bias of the most popular estimator of a positive EVI, the Hill estimator, performed in such a way that the minimal asymptotic variance is kept at the same level.

Key Words: Heavy right-tails, Monte-Carlo simulations, Semi-parametric estimation, Statistics of extremes

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1 The estimators under study and scope of the paper

Let $X_1, \ldots, X_n$ be independent, identically distributed (i.i.d.) random variables (r.v.'s) with a common distribution function (d.f.) F. Let us denote the associated ascending order statistics (o.s.) by $X_{1:n} \leq \cdots \leq X_{n:n}$ and let us assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum, $X_{n:n}$, linearly normalized, i.e., $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate r.v. Then the limiting distribution is necessarily an *extreme value* (EV) distribution, with the functional form

$$EV_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0, & \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if} \quad \xi = 0. \end{cases} \tag{1}$$

The d.f. F is said to belong to the max-domain of attraction of EV_ξ, and we write $F \in \mathcal{D}_M(EV_\xi)$. The parameter ξ is the *extreme value index* (EVI), the primary parameter of extreme events. The EVI measures the heaviness of the right tail function

$\overline{F} := 1 - F$, and the heavier the tail, the larger the EVI is. In this paper we work with Pareto-type distributions, with a strict positive EVI, i.e. in $\mathcal{D}_M^+ := \mathcal{D}_M\,(EV_\xi)_{\xi>0}$. Essentially due to the fact that asymptotic properties of second-order parameters' estimators are known when $\rho < 0$, we often assume a right tail function,

$$\overline{F}(x) = 1 - F(x) = Cx^{-1/\xi}\big(1 + D_1 x^{\rho/\xi} + o\big(x^{\rho/\xi}\big)\big), \text{ as } x \to \infty,\ \xi > 0, \tag{2}$$

for $C > 0$, $D_1 \neq 0$, $\rho < 0$ (see [13]). Then, with the possible parameterization, $A(t) = \xi\beta t^\rho,\ \rho < 0$, and denoting by $U(t)$ the tail quantile function, $U(t) := F^{\leftarrow}(1 - 1/t)$, $t > 1$, with $F^{\leftarrow}(y) := \inf\{x : F(x) \geq y\}$, the generalized inverse function of F, we have

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{3}$$

a result more generally proved in [6] for $\rho \leq 0$ and $F \in \mathcal{D}_M^+$, where $|A|$ is necessarily a regularly varying function with an index of regular variation equal to ρ.

1.1 The class of EVI-estimators under play

For Pareto-type models, the most common EVI-estimators are the *Hill* (H) estimators, introduced in [14], which are the averages of the log-excesses, $\ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \leq i \leq k < n$, and can thus be written as

$$H(k) := \frac{1}{k}\sum_{i=1}^{k} \ln \frac{X_{n-i+1:n}}{X_{n-k:n}} = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k},\ 1 \leq k < n. \tag{4}$$

The H EVI-estimator is thus the logarithm of the geometric mean (or mean-of-order-0) of $\mathbb{U} := \{U_{ik} := X_{n-i+1:n}/X_{n-k:n},\ 1 \leq i \leq k < n\}$. More generally, the authors in [2] considered as basic statistics the *mean-of-order-p* (MOP) of $\mathbb{U}$, with $p \geq 0$, now written for any $p \in \mathbb{R}$:

$$A_p(k) = \begin{cases} \left(\frac{1}{k}\sum\limits_{i=1}^{k} U_{ik}^p\right)^{1/p}, & \text{if} \quad p \neq 0, \\ \left(\prod\limits_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if} \quad p = 0, \end{cases}$$

and an associated class of MOP EVI-estimators, that more generally than in [2], [3], [7] and [8], can be defined as

$$H_p(k) \equiv MOP_p(k) := \begin{cases} \frac{1 - A_p^{-p}(k)}{p} = \frac{1 - \left(\frac{1}{k}\sum\limits_{i=1}^{k} U_{ik}^p\right)^{-1}}{p}, & \text{if} \quad p < 1/\xi,\ p \neq 0, \\ \ln A_0(k) = H(k), & \text{if} \quad p = 0, \end{cases} \tag{5}$$

with $H_0(k) \equiv H(k)$, given in (4). In [11], for $p = 0$, and in [2], for $0 < p < 1/\xi$, was proved that if $F \in \mathcal{D}_M^+$ and k is intermediate, i.e. $k = k_n$, $1 \leq k < n$, is such that

$$k = k_n \to \infty \quad \text{and} \quad k_n = o(n), \text{ as } n \to \infty, \tag{6}$$

the estimators $H_p(k)$, in (5) are consistent for the estimation of ξ provided that $0 \le p < 1/\xi$. If we further assume the validity of the second-order condition in (3), with ρ possibly null, we can write for $0 \le p < 1/(2\xi)$ the asymptotic distributional representation,

$$H_p(k) \stackrel{d}{=} \xi + \frac{\sigma_{_{H_p}} Z_k^{(p)}}{\sqrt{k}} + b_{_{H_p}} A(n/k)(1 + o_p(1)),$$
$$b_{_{H_p}} = b_{_{H_p}}(\xi,\rho) = \frac{1-p\xi}{1-\rho-p\xi},\ \sigma^2_{_{H_p}} = \sigma^2_{_{H_p}}(\xi) = \frac{\xi^2(1-p\xi)^2}{1-2p\xi}, \quad (7)$$

where $Z_k^{(p)}$ is standard normal.

Remark 1. *We thus have an asymptotic normal behavior for $H_p(k)$, in* (5)*, if $0 \le p < 1/(2\xi)$ and k such that $\sqrt{k}A(n/k) \to \lambda$, finite. There is however a reasonably high asymptotic bias* (a *decreasing function of* p) *when $\lambda \ne 0$, i.e. when we slightly increase k up to values where the mean square error* (MSE) *of $H_p(k)$ is minimized.*

Remark 2. *Further note that for $p = -1$, in* (5)*, we get* $H_{-1}(k) := \left(\frac{1}{k}\sum_{i=1}^k X_{n-k:n}/X_{n-i+1:n}\right)^{-1} - 1$*, the so-called t-*Hill EVI*-estimator in* [15]*. Moreover, with the parameterization $p = 1-\beta$, we get the functional studied in* [1]*, for $\beta > 0$, or equivalently, $p < 1$.*

Working just for technical simplicity in the class of models in (2), the representation in (7), for $p = 0$, with $b_{_{H_0}} = 1/(1-\rho)$, led the authors in [4] to directly remove the dominant component of the bias of the H EVI-estimator, given by $\xi\beta(n/k)^\rho/(1-\rho)$, considering, for adequate second-order parameters' estimators, $(\hat\beta, \hat\rho)$, provided in [10], among others, the *corrected*-H (CH) *minimum-variance reduced-bias* (MVRB) EVI-estimator,

$$CH(k) \equiv CH_{\hat\beta,\hat\rho}(k) := H(k)\Big(1 - \frac{\hat\beta}{1-\hat\rho}\left(\frac{n}{k}\right)^{\hat\rho}\Big). \quad (8)$$

Similarly, and with values of p such that the asymptotic normality of the estimators in (5) was known to hold at the time, i.e. for $0 \le p < 1/(2\xi)$, as proved in [2], the authors in [3] noticed that there is an optimal value

$$p \equiv p_{_M} = \varphi_\rho/\xi, \quad \text{with} \quad \varphi_\rho = 1 - \rho/2 - \sqrt{\rho^2 - 4\rho + 2}\Big/2, \quad (9)$$

which maximises the asymptotic efficiency of the class of estimators in (5). Then, they considered an *optimal* MOP (OMOP) r.v., defined by $OMOP(k) := H_{p_M}(k)$, with $H_p(k)$ given in (5), deriving its asymptotic behaviour. Such a behaviour and some extra developments in [7], led the author in [8] to introduce a class of *partially* RBMOP (PRBMOP) EVI-estimators based on $H_p(k)$, in (5), given by

$$RB_p(k;\hat\beta,\hat\rho) \equiv PRBMOP_p(k) := H_p(k)\Big(1 - \frac{\hat\beta(1-\varphi(\hat\rho))}{1-\hat\rho-\varphi(\hat\rho)}\left(\frac{n}{k}\right)^{\hat\rho}\Big), \quad (10)$$

still dependent on a tuning parameter p.

We shall further estimate the optimal k-value for the H EVI-estimation, as given in [12], computing $\hat k_{0|H_0} = \left((1-\hat\rho)n^{-\hat\rho}/\left(\hat\beta\sqrt{-2\hat\rho}\right)\right)^{2/(1-2\hat\rho)}$, $H_{00} := H(\hat k_{0|H_0})$, and considering next the RB EVI-estimator

$$RB^*(k) := H_{\hat p_{_M}}(k), \qquad \hat p_{_M} = \varphi_{\hat\rho}/H_{00}, \quad (11)$$

with φ_ρ given in (9).

1.2 Scope of the paper

In Section 2, we state and prove a theorem related to the EVI-estimator in (5), that provides also the asymptotic behaviour of the class of EVI-estimators in (10), for any negative real p. In Section 3, and through the use of Monte Carlo simulation techniques, we derive finite sample distributional properties of the class of PRBMOP EVI-estimators, in (10), and the adaptive OMOP EVI-estimator in (11), comparatively to the class of MVRB EVI-estimators, in (8).

2 Asymptotic behaviour of the EVI-estimators for negative p

We next state and prove the main theoretical result in the article:

Theorem 1. *If $F \in \mathcal{D}_M^+$ and* (6) *holds, the estimators $H_p(k)$ in* (5) *are, for any real $p < 1/\xi$, consistent for the estimation of ξ. Moreover, for consistent estimators $(\hat\beta, \hat\rho)$ such that*

$$\hat\rho - \rho = o_p(1/\ln n), \quad \textit{as} \quad n \to \infty, \tag{12}$$

a property so far known to be achievable for models in (2)*, the $RB_p(k)$ estimators in* (10) *are also consistent for the* EVI*-estimation.*
Under the validity of the second-order condition in (3)*, with ρ possibly null, the asymptotic distributional representation in* (7) *follows for any real $p < 1/(2\xi)$. Moreover, also for any real $p < 1/(2\xi)$, and under condition* (12)*,*

$$RB_p(k;\hat\beta,\hat\rho) \overset{d}{=} \xi + \frac{\sigma_{H_p} Z_k^{RB_p}}{\sqrt{k}} + b_{RB_p} A(n/k)(1+o_p(1)),$$
$$b_{RB_p} = b_{RB_p}(\xi,\rho) = \frac{\rho(p\xi - \varphi_\rho)}{(1-p\xi-\rho)(1-\rho-\varphi_\rho)}, \tag{13}$$

with a dominant bias component $o_p(A(n/k))$ if and only if $p = p_M = \varphi_\rho/\xi$, with φ_ρ given in (9)*.*

Proof. Note that, with $U(\cdot)$ the tail quantile function, we can write the distributional identity $X = U(Y)$, with Y a standard Pareto r.v., i.e. a r.v. with d.f. $F_Y(y) = 1 - 1/y,\ y \geq 1$. For the o.s. associated with a strict Pareto sample $(Y_1, \ldots, Y_n)$, we have $Y_{n-i+1:n}/Y_{n-k:n} \overset{d}{=} Y_{k-i+1:k}$, $1 \leq i \leq k$. Moreover, $kY_{n-k:n}/n \xrightarrow{p} 1$, as $n \to \infty$, i.e. $Y_{n-k:n} \overset{p}{\sim} n/k$. Consequently, and provided that $k \to \infty$, with $k/n \to 0$, as $n \to \infty$, $U_{ik} \overset{p}{\sim} Y_{k-i+1:k}^\xi$, $1 \leq i \leq k$.

Further note that

$$\mathbb{E}(Y^a) = \frac{1}{1-a} \quad \text{if } a < 1, \quad \mathbb{V}ar(Y^a) = \frac{a^2}{(1-a)^2(1-2a)}, \quad \text{if} \quad a < 1/2. \tag{14}$$

Working next under the second-order framework in (3), and even more generally assuming that we can have $\rho = 0$, using the interpretation of the Box-Cox function as the logarithm when the power equals 0, we can write

$$T_p(k) := \frac{1}{k}\sum_{i=1}^k \left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^p = \frac{1}{k}\sum_{i=1}^k Y_i^{p\xi}\Big(1+A(n/k)\,(Y_i^\rho - 1)/\rho + o_p(A(n/k))\Big)^p$$
$$= \frac{1}{k}\sum_{i=1}^k Y_i^{p\xi} + pA(n/k)\,\frac{1}{k}\sum_{i=1}^k Y_i^{p\xi}(Y_i^\rho - 1)/\rho + o_p(A(n/k)).$$

On the basis of (14), a derivation similar to the one in [2], enables us to get the result in the theorem related to $H_p(k)$, in (5), for all $\xi > 0$ and $p < 1/(2\xi)$, even a negative real number, and for the EVI-estimator $H_p(k) = (1 - T_p^{-1}(k))/p$.

Noticing next that $RB_p(k;\beta,\rho) := H_p(k)\Big(1 - \frac{\beta(1-\varphi_\rho)}{1-\rho-\varphi_\rho}\left(\frac{n}{k}\right)^\rho\Big)$, we easily derive that the dominant component of the bias is given by

$$\frac{(1-p\xi)A(n/k)}{1-p\xi-\rho} - \frac{(1-\varphi_\rho)A(n/k)}{1-\rho-\varphi_\rho} = \frac{\rho(p\xi-\varphi_\rho)A(n/k)}{(1-p\xi-\rho)(1-\rho-\varphi_\rho)},$$

i.e. it is null only for $p = \varphi_\rho/\xi$. If we estimate consistently β and ρ through the estimators $\hat\beta$ and $\hat\rho$, and condition (12) holds, we can use Cramer's delta-method, and in the lines of [8], we get the result in the theorem not ony for $p \geq 0$, but for any negative real p. □

In Figure 1, to visualize the reduction in bias achieved by the PRBMOP EVI-estimation, a representation of $b_{H_p} = b_{H_p}(\xi,\rho)$ and $b_{RB_p} = b_{RB_p}(\xi,\rho)$, respectively given in (7) and (13), as functions of ξ, for $p = 0.1, 0.5, 1$ and $\rho = -1$, can be found.

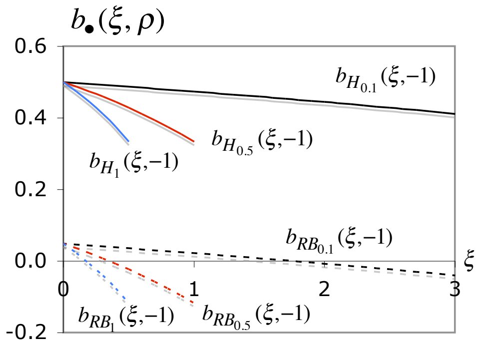

Figure 1: Values of $b_{H_p} = b_{H_p}(\xi,\rho)$ and $b_{RB_p} = b_{RB_p}(\xi,\rho)$, as functions of ξ, for $p = 0.1, 0.5, 1$ and $\rho = -1$

2.1 Quick and simple MVRB and PRBMOP EVI-estimators

Since the second-order reduced bias estimators in (8) and (10) depend on the estimation of the second order parameters β and ρ, and just as suggested in [10] for the MVRB EVI-estimator in (8), we could also have considered *quick and simple* estimators, a by-product of the estimators in (8) and (10), setting there $\hat\beta = 1$ and $\hat\rho = -1$. We then get

$$\overline{CH}(k) := H_{1,-1}(k) = H_0(k)\left(1 - \frac{a_{CH}k}{n}\right) \quad \text{and}$$
$$\overline{RB}_p(k) = RB_p(k;1,-1) = H_p(k)\left(1 - \frac{a_{RB}k}{n}\right) \quad (15)$$

with $a_{CH} = 1/2$, $a_{RB} = (1-\varphi(-1))/(2-\varphi(-1))$, $\varphi(-1) = (3-\sqrt{7})/2$, and where $H_p(k)$ stands for the MOP EVI-estimator in (5), with $H_0(k) \equiv H(k)$, the Hill EVI-estimator in (4).

If we consider the replacement of the estimators in (5) and (8) by the quick and simple estimators in (15), the dominant component of bias changes, but the asymptotic variance is still kept equal to $\sigma^2_{_{RB_p}}$, in (7). The dominant component of the asymptotic bias of $\overline{CH}(k)$ is then of a smaller order than $A(n/k)$ if and only if $(\beta, \rho) = (1, -1)$. For both $\overline{CH}(k)$ and $\overline{RB}_p(k)$ the dominant component of bias, of the order of k/n, is of a larger order than the one of the $H_p(k)$ EVI-estimators for models with $\rho < -1$.

Remark 3. *Note that the MOP EVI-estimator is a special case of the so-called quick and simple EVI-estimators, since it uses* $\widehat{\beta} = 0$ *in* (5)*.*

Remark 4. *The quick and simple estimators in* (15) *are adequate only if the guess* $\beta = 1$*,* $\rho = -1$ *captures properly the deviation of the underlying tail from a strict Pareto tail, but educated guesses may be much more precise than are the usually noisy estimates of higher order parameters. Note however that the second-order parameters' estimates proposed in this paper are quite reliable in the class of Hall-Welsh models.*

3 Monte Carlo simulations

We have performed extensive simulations associated with the Generalized Pareto (GP) model, related to $EV_\xi(\cdot)$, in (1), through the relationship $F(x) = 1 + \ln EV_\xi(x) = 1 - (1 + \xi x)^{-1/\xi}$, $x \geq 0$, $\xi > 0$, for which $\rho = -\xi$, and the Burr$_{\xi,\rho}$ model, with d.f. $F(x) = 1 - (1 + x^{-\rho/\xi})^{1/\rho}$, $x \geq 0$, $\xi > 0$, $\rho < 0$. In all Monte-Carlo simulation experiments we have considered multi-sample simulations of size 5000×20 and sample sizes $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000. For details on multi-sample simulation, we refer [9].

3.1 Mean values and MSE paths

For each value of n and for each of the aforementioned models, we have first simulated, as functions of k, the number of top o.s. involved in the estimation, and on the basis of first run of size 5000, the mean values (E) and root MSEs (RMSEs) of the EVI-estimators in (8) and (10), for values of $p = -1, -0.5, -0.25, -0.1$ and $p = \ell/(10\xi)$, $\ell = 1(1)9$, so that $p < 1/\xi$ and we have valid estimates. As an illustration, we present Figure 2 associated with Burr$_{1,-0.5}$ parents. We further notice that for all simulated parents with $\rho \in (-1, 0)$, RB^* is always in between the $PRBMOP$ for $\ell = 1$ (close to CH) and $\ell = 2$, both regarding mean values and RMSEs. In this same region of ρ-values, the best performance is always achieved in the region $5 \leq \ell \leq 9$, as can be seen in Figure 2.

3.1.1 Mean values and MSEs at optimal levels

We have computed the Hill EVI-estimator at the simulated value of $k_{0|H} := \arg\min_k RMSE\big(H(k)\big)$, the simulated optimal k in the sense of minimum RMSE. We have also computed RB_{p0}, i.e. the PRBMOP EVI-estimator $RB_p(k)$ computed at the simulated value of $k_{0|RB_p} := \arg\min_k RMSE\big(RB_p(k)\big)$. As an illustration of the bias reduction achieved with the PRBMOP EVI-estimators in (10) at optimal levels, see Table 1, related to the model $GP_{0.25}$. We there present, for $n = 100,\ 200,\ 500,\ 1000,\ 2000$ and 5000, the simulated mean values at optimal levels of H_{00}, CH_0 and RB_{p0} for $p = -1, -0.5, -0.25, -1$ and $p = \ell/(10\xi)$, considering the two regions, $\ell = 1, 2, 3, 4$, where we can guarantee consistency and asymptotic normality, and $\ell = 5, 6, 7, 8, 9$, where only consistency is assured by Theorem 1. We further consider RB^*_0. Information on 95% confidence intervals, computed on the basis

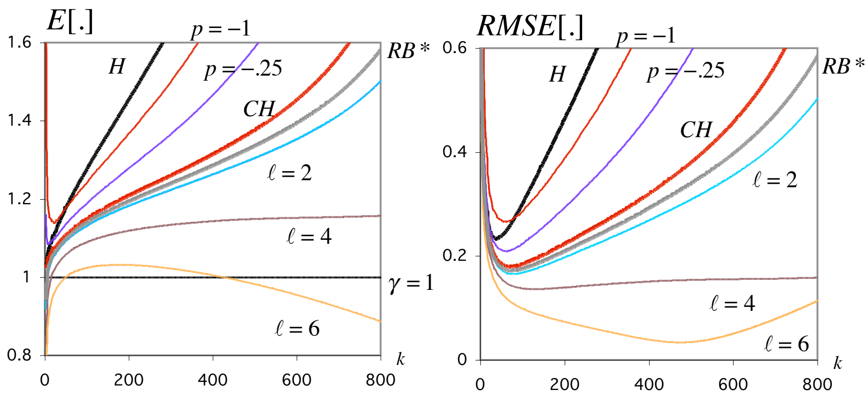

Figure 2: Mean values (left) and RMSEs (right) of $H(k)$, $CH(k)$, $RB^*(k)$ and $RB_p(k)$, $p = -1, -0.25$ and $p = \ell/10\xi$, $\ell = 2, 4, 6$, for a $\text{Burr}_{1,-0.5}$ underlying parent

of the 20 replicates with 5000 runs each, is also provided. For each region, and among the estimators considered, the one providing a smaller squared bias than the best one in the previous region is written in *italic*, and $\underline{\text{underlined}}$ whenever it turns out to be the best in a region, beating the best estimator in the previous region.

Table 1: Simulated mean values of H_{00}, CH_0, RB_{p0}, $p = -1, -0.5, -0.25$, RB_0^* and RB_{p0}, $p = \ell/(10\xi)$, $\ell = 2(1)9$, for $\text{GP}_{0.25}$ underlying parents, together with 95% confidence intervals

	GP_ξ parent, $\xi = 0.25$					
n	100	200	500	1000	2000	5000
H_{00}	0.419 ± 0.0024	0.390 ± 0.0028	0.365 ± 0.0018	0.347 ± 0.0016	0.335 ± 0.0012	0.320 ± 0.0011
CH_0	$\mathit{0.406} \pm 0.0030$	$\mathit{0.382} \pm 0.0017$	$\mathit{0.360} \pm 0.0017$	$\mathit{0.345} \pm 0.0018$	$\mathit{0.333} \pm 0.0013$	$\mathit{0.319} \pm 0.0009$
$p = -1$	0.452 ± 0.0027	0.416 ± 0.0026	0.381 ± 0.0016	0.361 ± 0.0019	0.345 ± 0.0013	0.327 ± 0.0008
$p = -0.5$	0.431 ± 0.0032	0.400 ± 0.0025	0.371 ± 0.0017	0.353 ± 0.0018	0.340 ± 0.0013	0.323 ± 0.0009
$p = -0.25$	0.419 ± 0.0031	0.391 ± 0.0017	0.366 ± 0.0018	0.350 ± 0.0019	0.337 ± 0.0014	0.321 ± 0.0009
RB_0^*	$\underline{\mathit{0.388}} \pm 0.0031$	$\underline{\mathit{0.370}} \pm 0.0023$	$\underline{\mathit{0.349}} \pm 0.0015$	$\underline{\mathit{0.336}} \pm 0.0015$	$\underline{\mathit{0.327}} \pm 0.0012$	$\underline{\mathit{0.314}} \pm 0.0008$
$\ell = 2$	0.361 ± 0.0027	0.351 ± 0.0020	0.338 ± 0.0016	0.328 ± 0.0015	0.321 ± 0.0012	0.310 ± 0.0007
$\ell = 3$	$\mathit{0.337} \pm 0.0022$	$\mathit{0.330} \pm 0.0024$	$\mathit{0.323} \pm 0.0016$	$\mathit{0.317} \pm 0.0013$	$\mathit{0.311} \pm 0.0011$	$\mathit{0.304} \pm 0.0008$
$\ell = 4$	$\underline{\mathit{0.295}} \pm 0.0002$	$\underline{\mathit{0.293}} \pm 0.0002$	$\underline{\mathit{0.293}} \pm 0.0001$	$\underline{\mathit{0.293}} \pm 0.0001$	$\underline{\mathit{0.293}} \pm 0.0001$	$\underline{\mathit{0.292}} \pm 0.0001$
$\ell = 5$	$\mathit{0.249} \pm 0.0002$	$\underline{\mathit{0.250}} \pm 0.0001$	$\underline{\mathit{0.250}} \pm 0.0001$	$\underline{\mathit{0.250}} \pm 0.0001$	$\underline{\mathit{0.250}} \pm 0.0001$	$\underline{\mathit{0.250}} \pm 0.0001$
$\ell = 6$	$\underline{\mathit{0.250}} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$
$\ell = 7$	$\mathit{0.249} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	0.250 ± 0.0001	0.250 ± 0.0001	0.250 ± 0.0001
$\ell = 8$	$\mathit{0.243} \pm 0.0002$	$\mathit{0.247} \pm 0.0001$	$\mathit{0.249} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$	$\mathit{0.250} \pm 0.0001$
$\ell = 9$	$\mathit{0.226} \pm 0.0002$	$\mathit{0.232} \pm 0.0001$	$\mathit{0.238} \pm 0.0001$	$\mathit{0.242} \pm 0.0001$	$\mathit{0.245} \pm 0.0001$	$\mathit{0.247} \pm 0.0001$

We next present in Table 2 the simulated values of the indicators,

$$REFF_{RB_p|H} := RMSE\,(H_{00})/RMSE\,(RB_{p0}), \tag{16}$$

again for a $GP_{0.25}$ model. Similar REFF-indicators have also been computed for the CH and RB^* EVI-estimators. In the first row of Table 2, we provide $RMSE_{00}$, the RMSE of H_{00}, so that we can easily recover the RMSE of all other estimators. The following rows provide the REFF-indicators of CH and RB_p, for the same values of p as in Table 1. We further present a similar REFF-indicator for RB^*. A similar mark (*italic* and/or $\underline{\text{underlined}}$) is used. Confidence intervals are not provided for REFF-indicators larger than 10, but are available from the authors.

Remark 5. *An indicator higher than one means a better performance than the* H *estimator, i.e. the higher these indicators are, the better the associated* EVI-*estimators perform, comparatively to* H_{00}.

Table 2: Simulated RMSE of H_{00} (first row) and $REFF$-indicators of CH, RB_p, $p = -1, -0.5, -0.25$, RB^* and RB_p, $p = \ell/(10\xi)$, $\ell = 2(1)9$, and for $\mathrm{GP}_{0.25}$ underlying parents, together with 95% confidence intervals

GP_ξ parent, $\xi = 0.25$						
n	100	200	500	1000	2000	5000
$RMSE_{00}$	0.237 ± 0.1756	0.196 ± 0.1684	0.155 ± 0.1592	0.131 ± 0.1525	0.112 ± 0.1460	0.092 ± 0.1377
CH_0	*1.148* ± 0.0049	*1.118* ± 0.0027	*1.088* ± 0.0025	*1.069* ± 0.0018	*1.057* ± 0.0018	*1.042* ± 0.0012
$p = -1$	0.911 ± 0.0058	0.926 ± 0.0035	0.939 ± 0.0032	0.941 ± 0.0032	0.947 ± 0.0030	0.948 ± 0.0019
$p = -0.5$	1.006 ± 0.0055	1.005 ± 0.0030	1.001 ± 0.0029	0.995 ± 0.0025	0.994 ± 0.0023	0.989 ± 0.0015
$p = -0.25$	1.064 ± 0.0050	1.051 ± 0.0027	1.037 ± 0.0026	1.026 ± 0.0023	1.021 ± 0.0020	1.012 ± 0.0012
RB^*	*1.231* ± 0.0034	*1.194* ± 0.0021	*1.157* ± 0.0020	*1.133* ± 0.0021	*1.115* ± 0.0016	*1.094* ± 0.0014
$\ell = 2$	*1.445* ± 0.0032	*1.350* ± 0.0023	*1.263* ± 0.0023	*1.213* ± 0.0024	*1.177* ± 0.0023	*1.138* ± 0.0022
$\ell = 3$	*1.717* ± 0.0038	*1.563* ± 0.0033	*1.420* ± 0.0036	*1.340* ± 0.0032	*1.279* ± 0.0032	*1.215* ± 0.0035
$\ell = 4$	*5.203* ± 0.0301	*4.460* ± 0.0218	*3.582* ± 0.0214	*3.037* ± 0.0131	*2.636* ± 0.0122	*2.172* ± 0.0080
$\ell = 5$	*34.900*	*40.247*	*53.187*	*65.706*	*76.072*	*96.346*
$\ell = 6$	*34.768*	*40.360*	*53.204*	*65.298*	*75.078*	*93.574*
$\ell = 7$	*30.255*	*36.245*	*47.763*	*57.696*	*65.384*	*77.688*
$\ell = 8$	18.053	*23.072*	*31.802*	*38.688*	*44.293*	*51.241*
$\ell = 9$	*8.791* ± 0.0708	*9.467* ± 0.0735	*10.932*	*12.456*	*14.347*	*17.536*

4 Concluding remarks

1. Note that for $5 \le \ell \le 9$, we have consistency of the estimators either in (5) or in (10), but no guarantee of asymptotic normality, and even of bias reduction comparatively to the MOP EVI-estimators. Despite of this comment the EVI-estimators in this region can be the ones that exhibit the best performance regarding both mean values and RMSEs.

2. The estimators for negative values of p can beat the Hill at optimal levels but never the CH EVI-estimators regarding both bias and RMSE. They are not indeed efficient, despite of the fact that the similar MOP EVI-estimator for $p = -1$ (see [1]) has revealed to be the most robust EVI-estimator, but with a low efficiency. This comment could thus lead to a discussion of robustness versus efficiency and to the need of an indicator that takes both concepts into account (see e.g. [5]).

3. For both mean values and RMSEs at optimal levels, and again if we restrict ourselves to the region of p-values where we can so far guarantee asymptotic normality, the best results were obtained at $p = 4/(10\xi)$ for most of the simulated models.

4. Regarding RMSE, the consistent and asymptotically normal PRBMOP EVI-estimators at optimal levels, can always beat the MVRB EVI estimators also at optimal levels for all $0 < p < 1/(2\xi)$. They can however be beaten by the only consistent PRBMOP EVI-estimators $\big(1/(2\xi) \le p < 1/\xi\big)$, at optimal levels, for most of the simulated parents.

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References

[1] Beran, J., Schell, D. and Stehlik, M. (2014) *The harmonic moment tail index estimator: asymptotic distribution and robustness*. Ann. Inst. Statist. Math., *66*, 193–220.

[2] Brilhante, M.F., Gomes, M.I. and Pestana, D. (2013) *A simple generalization of the Hill estimator.* Computational Statistics and Data Analysis, *57*:1, 518–535.

[3] Brilhante, M.F., Gomes, M.I. and Pestana, D. (2013) *The MOP EVI-Estimator Revisited.* In Pacheco, A., Santos, R., Oliveira, M.R., and Paulino, C.D. (eds.), *New Advances in Statistical Modeling and Application*. Studies in Theoretical and Applied Statistics, 163-175, Springer International Publishing, Switzerland.

[4] Caeiro, F., Gomes, M.I. and Pestana, D. (2005) *Direct reduction of bias of the classical Hill estimator.* Revstat, *3*:2, 113–136.

[5] Dell'Aquila, R. and Embrechts, P. (2006) *Extremes and robustness: a contradiction?* Fin. Mkts. Portfolio Mgmt., *20*, 103–118.

[6] Geluk, J. and de Haan, L. (1987) *Regular Variation, Extensions and Tauberian Theorems*. CWI Tract 40, Center for Mathematics and Computer Science, Amsterdam, Netherlands.

[7] Gomes, M.I., Brilhante, M.F. and Pestana, D. (2013) *New reduced-bias estimators of a positive extreme value index.* Comm. Statist. Simulation Comput., in press.

[8] Gomes, M.I., Brilhante, M.F., Caeiro, F. and Pestana, D. (2014) *A new partially reduced-bias mean-of-order p class of extreme value index estimators.* Notas e Comunicações CEAUL 04/2014.

[9] Gomes, M.I. and Oliveira, O. (2001) *The bootstrap methodology in Statistics of Extremes: choice of the optimal sample fraction.* Extremes, *4*:4, 331–358.

[10] Gomes, M.I. and Pestana, P. (2007) *A sturdy reduced-bias extreme quantile (VaR) estimator.* J. American Statistical Association, *102*:477, 280–292.

[11] de Haan, L. and Peng, L. (1998) *Comparison of extreme value index estimators.* Statistica Neerlandica, *52*, 60–70.

[12] Hall, P. (1982). *On some simple estimates of an exponent of regular variation*. J. Royal Statistical Society B, *44*, 37–42.

[13] Hall, P. and Welsh, A.W. (1985) *Adaptive estimates of parameters of regular variation.* Ann. Statist., *13*, 331–341.

[14] Hill, B.M. (1975) *A simple general approach to inference about the tail of a distribution.* Ann. Statist., *3*, 1163–1174.

[15] Stehlík M., Potocký R., Waldl H. and Fabián Z. (2010) *On the favourable estimation of fitting heavy tailed data*. Computational Statistics, *25*, 485–503.