Comparison of Asymptotically Unbiased Extreme Value Index estimators: a Monte Carlo Simulation Study

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Abstract: In this paper we are interested in the semi-parametric estimation of the extreme value index of a heavy-tailed model. We consider a class of consistent semi-parametric estimators, parameterized with two tuning parameters. Such parameters enables us to have an estimator with a null dominant component of asymptotic bias, and achieve a high efficiency comparatively to other classical estimators. After a brief review of the estimators under study, we provide a Monte Carlo simulation study of the estimators behaviour for finite sample sizes of some familiar models.

Key Words: Asymptotic properties, Extreme value index; Heavy tails; Monte Carlo simulation; Statistics of extremes.

AMS: 62G05, 62G20, 62G32.

1 Introduction

Let $X_1, X_2, \cdots, X_n$, be a set of independent, identically distributed (i.i.d.) random variables (r.v.'s) with a common distribution function (d.f.) F and $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the associated ascending order statistics (o.s.). Suppose F is a heavy tailed distribution. Then, the quantile function $U(t) := F^{\leftarrow}(1-1/t)$, $t \ge 1$, with $F^{\leftarrow}(y) := \inf\{x : F(x) \ge y\}$, is of regular variation with index $\gamma > 0$, i.e., for every $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\gamma}. \tag{1}$$

The parameter γ is the *extreme value index* (EVI), and measures the heaviness of the right tail function $\overline{F} := 1 - F$. The rate of convergence in the first order condition in (1), is ruled by the parameter $\rho \le 0$ in the relation

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \begin{cases} (x^{\rho}-1)/\rho, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{2}$$

which is assumed to hold for every $x > 0$, and where $|A(t)|$ must then be of regular variation with index ρ (see [20]). For a heavy tailed model, the classic semi-parametric EVI-estimator is the Hill estimator [21], the average of the log-excesses above a high threshold $X_{n-k:n}$, $\hat{\gamma}_n^H(k) := \frac{1}{k}\sum_{i=1}^{k}(\ln X_{n-i+1:n} - \ln X_{n-k:n})$, $k = 1, 2, \ldots, n-1$. Most classical semi-parametric EVI-estimators have a high variance for high thresholds, i.e., for small values of k and high bias for low thresholds, i.e., for large values of

k. Consequently, the mean squared error (MSE) has a very peaked pattern, making it difficult to determine the optimal k, where the MSE is minimal. The adequate accommodation of the bias of Hill's estimator has been extensively addressed in recent years by several authors. Recent overviews of statistics of univariate extremes can be found in [1, 11, 18, 23].

We shall consider the class of semi-parametric estimators of γ introduced in [3],

$$\hat{\gamma}_n^{(\delta,\alpha)}(k) := \frac{\Gamma(\alpha)}{M_n^{(\alpha-1)}(k)} \left(\frac{M_n^{(\delta\alpha)}(k)}{\Gamma(\delta\alpha+1)} \right)^{1/\delta}, \quad \delta > 0,\ \alpha \geq 1, \quad k = 1, 2, \ldots, n-1, \tag{3}$$

with tuning parameters (δ, α), which may be controlled at our ease, Γ denotes the complete Gamma function, $M_n^{(0)}(k) \equiv 1$, and

$$M_n^{(\omega)}(k) := \frac{1}{k} \sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\omega}, \quad \omega > 0$$

are consistent estimators of $\gamma^{\omega}\Gamma(\omega+1)$ whenever k is intermediate, i.e., whenever k is a sequence of integers between 1 and n such that $k \to \infty$ and $k/n \to 0$, as $n \to \infty$. The class of estimators in (3) generalizes the Hill estimator, which appears for $(\delta, \alpha) = (1, 1)$, and for $\delta = 2$ we obtain the class studied in [2, 9]. When $\alpha = 1$ we get the class of EVI-estimators studied in [16]. A class of location invariant estimators based on (3) with $\delta = 2$ can be found in [22].

Under the first order framework in (1) and for intermediate k, the statistics $\hat{\gamma}_n^{(\delta,\alpha)}(k)$ are consistent for γ, and under the second order condition in (2) and whenever $\sqrt{k}A(n/k) \to \lambda \neq 0$, finite, $\sqrt{k}(\hat{\gamma}_n^{(\delta,\alpha)}(k) - \gamma)$ is asymptotically normal, with asymptotic bias $\lambda b(\delta, \alpha)$ and asymptotic variance $\gamma^2\sigma^2_{\delta,\alpha}$, with

$$\sigma^2_{\delta,\alpha} := \frac{1}{\delta^2} \left\{ \frac{2\Gamma(2\delta\alpha)}{\delta\alpha\Gamma^2(\delta\alpha)} + \frac{\delta^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)} - \frac{2\Gamma((\delta+1)\alpha)}{\alpha\Gamma(\delta\alpha)\Gamma(\alpha)} - (\delta-1)^2 \right\}.$$

and $b(\delta, \alpha) := \left\{(1-\rho)^{-\delta\alpha} - \delta(1-\rho)^{1-\alpha} + (\delta-1)\right\}/(\delta\rho)$, if $\rho < 0$, being equal to 1 if $\rho = 0$ ([9]). Moreover, if $\rho < 0$ and $\delta > 1$ there is thus a value $\alpha_0^{(\delta)}(\rho)$ such that $b(\delta, \alpha_0^{(\delta)}(\rho)) = 0$, i.e., we have a EVI-estimator with a null dominant component of asymptotic bias (see [3]). For the particular but important cases to achieve a low asymptotic variance, $\delta = 1.5$ and $\delta = 2$, we have $\alpha_0^{(\delta)}(\rho)$ explicitly given by

$$\alpha_0^{(1.5)}(\rho) = -\frac{2\ln\left[(1-\rho)\left(\frac{1}{2} + \cos\left(\frac{\arctan(c(\rho))+4\pi}{3}\right)\right)\right]}{\ln(1-\rho)},$$

$$\alpha_0^{(2)}(\rho) = -\frac{\ln(1-\rho-\sqrt{(1-\rho)^2-1})}{\ln(1-\rho)},$$

with $c(\rho) = ((1-\rho)^3 - 2)/\sqrt{(1-\rho)^3-1}$. For simplicity and since it is important to reduce the bias, we shall assume $\rho < 0$ and consider the estimators

$$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(k), \qquad \hat{\gamma}_n^{(2,\hat{\alpha}_0)}(k), \qquad \text{with} \quad \hat{\alpha}_0 := \alpha_0^{(\delta)}(\hat{\rho}) \tag{4}$$

and $\hat{\rho}$ an adequate estimators of shape second-order parameter ρ introduced in (2). For comparison, we also consider the minimum-variance reduced-bias (MVRB) corrected-Hill (CH) estimator in [10], given by

$$\hat{\gamma}_n^{\mathrm{CH}}(k) := \hat{\gamma}_n^{\mathrm{H}}(k) \left(1 - \frac{\hat{\beta}(n/k)^{\hat{\rho}}}{1-\hat{\rho}}\right), \tag{5}$$

with $(\hat{\beta}, \hat{\rho})$ adequate estimators of the second-order parameters (β, ρ) such that $A(t) = \gamma \beta t^{\rho}$, $\rho < 0$. An asymptotic comparison between the estimator in (5) and others reduced bias EVI-estimators can be found in [5, 7].

The remainder of this paper is organized as follows: In the next section we present some estimators for the second order parameters β and ρ. In the last section we study the finite-sample properties of the estimators in (4) and (5), through Monte-Carlo simulation.

2 Estimation of the Second-Order Parameters

We have used particular members of the class of estimators of the second-order parameter ρ proposed in [14]. Such a class of estimators has been first parameterized by a tuning parameter $\tau \geq 0$, that can be straightforwardly considered as a real number [4], and is defined as

$$\hat{\rho}_\tau(k) := \min\left\{0\,,\, \frac{3(T_{n,k}^{(\tau)} - 1)}{T_{n,k}^{(\tau)} - 3}\right\}, \qquad T_{n,k}^{(\tau)} := \frac{\left(M_{n,k}^{(1)}\right)^{\tau} - \left(M_{n,k}^{(2)}/2\right)^{\tau/2}}{\left(M_{n,k}^{(2)}/2\right)^{\tau/2} - \left(M_{n,k}^{(3)}/6\right)^{\tau/3}}, \quad \tau \in \mathcal{R},$$

with the notation $a^{b\tau} = b \ln a$ if $\tau = 0$. Interesting alternative ρ-estimators have recently been introduced in [8, 12, 13, 15]. As suggested in previous articles, it is sensible not to choose the tuning parameter τ blindly. Here we use the same stability criterion used in [19]: Consider a sample with n positive values, compute $\{\hat{\rho}_\tau(k)\}_{k \in K}$, with $K = (\lfloor n^{0.995} \rfloor, \lfloor n^{0.999} \rfloor)$, compute their median, denoted η_τ, and compute $I_\tau := \sum_{k \in K} \left(\hat{\rho}_\tau(k) - \eta_\tau\right)^2$, $\tau = 0, 1$. Next choose $\tau^* = 0$ if $I_0 \leq I_1$; otherwise, choose $\tau^* = 1$ and compute $\hat{\rho} \equiv \hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$, with $k_1 = \lfloor n^{1-\epsilon} \rfloor$, $\epsilon = 0.001$.

For the estimation of the scale second-order parameter β we shall here consider the estimator in [17], given by

$$\hat{\beta}_{\hat{\rho}}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{D_{\hat{\rho},0}(k)\, D_{0,1}(k) - D_{\hat{\rho},1}(k)}{D_{\hat{\rho},0}(k)\, D_{\hat{\rho},1}(k) - D_{2\hat{\rho},1}(k)},$$

$$D_{\alpha_1,\alpha_2}(k) := \frac{1}{k} \sum_{i=1}^{k} \left(\frac{i}{k}\right)^{-\alpha_1} U_i^{\alpha_2}, \quad U_i := i\Big(\ln \frac{X_{n-i+1:n}}{X_{n-i:n}}\Big)$$

dependent on the estimator $\hat{\rho}$, suggested before. A reduced bias β-estimator can be found in [6].

3 Exact Distributional Properties of the EVI-estimators: a Monte Carlo Simulation Study

We are now interested in the comparative behaviour of the adaptive estimation of the EVI given in (4) and (5). We are going to consider here the Burr(γ,ρ) model, $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x > 0$, $\gamma > 0$, $\rho < 0$ with $\gamma = 0.5$ and $\rho = -1, -0.75, -0.5, -0.3$ ($\beta = 1$), the Fréchet(γ) model $F(x) = \exp(-x^{-1/\gamma})$, $x > 0$, $\gamma > 0$, with $\gamma = 0.5$ ($\rho = -1$, $\beta = 0.5$) and the Half-t_ν model, i.e., the absolute value of a Student's-t_ν model ($\gamma = 1/\nu$, $\rho = -2/\nu$, $\beta = (\nu + 1) d_\nu^2 / (\nu + 2)$, $d_\nu = (\mathcal{B}(\nu/2, 1/2)\nu/2)^{1/\nu}$). $\mathcal{B}$ denotes the complete Beta function.

On the basis of a Monte-Carlo simulation with 5000 runs, we present in table 1 the simulated values of the mean value (E) and the root mean squared error (RMSE) at the optimal level,

$$k_0 := \arg\min_k \{RMSE[\hat{\gamma}_n(k)]\}.$$

For each model, mean values with the smallest squared bias and RMSE are presented in **bold**.

Table 1: Simulated mean values / RMSE of the EVI-estimators at their simulated optimal level, $\hat{k}_0$.

n	100	200	500	1000	2000	5000
			Burr$(0.5,-1)$ **parent** ($\beta=1$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.5124 / 0.1040	0.5112 / 0.0810	**0.5079** / 0.0579	**0.5067** / 0.0433	0.5069 / 0.0292	0.5054 / 0.0179
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	0.5136 / 0.1044	0.5127 / 0.0815	0.5088 / 0.0582	0.5077 / 0.0436	0.5076 / 0.0297	0.5060 / 0.0186
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	**0.5038 / 0.0995**	**0.4893 / 0.0742**	0.4847 / **0.0554**	0.4825 / **0.0431**	**0.4981 / 0.0263**	**0.4996 / 0.0107**
			Burr$(0.5,-0.75)$ **parent** ($\beta=1$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	**0.5305 / 0.1107**	0.5315 / 0.0828	0.5281 / 0.0607	0.5248 / 0.0486	0.5199 / 0.0388	0.5160 / 0.0291
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	0.5310 / 0.1118	0.5324 / 0.0842	0.5289 / 0.0620	0.5255 / 0.0498	0.5210 / 0.0397	0.5164 / 0.0298
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.5468 / 0.1131	**0.5309 / 0.0753**	**0.5224 / 0.0497**	**0.5192 / 0.0363**	**0.5153 / 0.0276**	**0.5123 / 0.0205**
			Burr$(0.5,-0.5)$ **parent** ($\beta=1$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	**0.5557 / 0.1427**	**0.5617 / 0.1169**	**0.5555 / 0.0941**	**0.5494 / 0.0810**	0.5443 / **0.0687**	0.5367 / **0.0555**
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	0.5608 / 0.1448	0.5624 / 0.1191	0.5570 / 0.0958	0.5507 / 0.0824	**0.5439** / 0.0697	**0.5366** / 0.0561
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.6149 / 0.1771	0.5931 / 0.1376	0.5742 / 0.1062	0.5640 / 0.0900	0.5542 / 0.0757	0.5456 / 0.0618
			Burr$(0.5,-0.3)$ **parent** ($\beta=1$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.5686 / 0.2121	**0.5805 / 0.1892**	**0.5881 / 0.1649**	**0.5879 / 0.1461**	**0.5850 / 0.1293**	**0.5739 / 0.1101**
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	**0.5641 / 0.2113**	0.5812 / 0.1911	0.5953 / 0.1677	0.5896 / 0.1488	0.5869 / 0.1315	0.5775 / 0.1116
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.7356 / 0.3351	0.7029 / 0.2731	0.6624 / 0.2228	0.6522 / 0.1928	0.6266 / 0.1667	0.6045 / 0.1374
			Fréchet(0.5) **parent,** $\gamma=0.5$ ($\rho=-1, \beta=0.5$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.4969 / 0.1071	**0.5047** / 0.0792	0.5070 / 0.0577	0.5078 / 0.0452	0.5079 / 0.0341	0.5079 / 0.0234
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	**0.4984** / 0.1074	0.5061 / 0.0790	0.5075 / 0.0569	0.5082 / 0.0447	0.5078 / 0.0336	0.5076 / 0.0231
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.4852 / **0.0973**	0.4952 / **0.0712**	**0.4969 / 0.0511**	**0.4996 / 0.0393**	**0.4997 / 0.0289**	**0.4991 / 0.0187**
			Half-t_2 **parent** ($\gamma=0.5, \rho=-1, \beta=1.5$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.5226 / 0.1032	0.5231 / 0.0707	0.5164 / 0.0495	0.5135 / 0.0377	0.5100 / 0.0288	0.5063 / 0.0200
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	0.5237 / 0.1047	0.5246 / 0.0725	0.5179 / 0.0511	0.5147 / 0.0391	0.5111 / 0.0299	0.5070 / 0.0208
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	**0.5177 / 0.1006**	**0.5114 / 0.0615**	**0.5041 / 0.0379**	**0.5044 / 0.0255**	**0.5027 / 0.0181**	**0.5008 / 0.0116**
			Half-t_3 **parent** ($\gamma=0.3(3), \rho=-0.6(6), \beta=1.4165$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.3655 / **0.0864**	**0.3673 / 0.0694**	**0.3629 / 0.0542**	**0.3586 / 0.0448**	**0.3532 / 0.0364**	**0.3493 / 0.0279**
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	**0.3652** / 0.0880	**0.3673** / 0.0713	**0.3629** / 0.0557	0.3602 / 0.0459	0.3551 / 0.0372	0.3501 / 0.0284
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.3949 / 0.1045	0.3800 / 0.0768	0.3715 / 0.0571	0.3634 / 0.0462	0.3550 / 0.0369	0.3514 / 0.0283
			Half-t_4 **parent** ($\gamma=0.25, \rho=-0.5, \beta=1.3608$)			
$\hat{\gamma}_n^{(2,\hat{\alpha}_0)}(\hat{k}_0)$	0.2824 / **0.0794**	**0.2798 / 0.0666**	**0.2802 / 0.0559**	**0.2774 / 0.0468**	**0.2753 / 0.0404**	**0.2702 / 0.0323**
$\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}(\hat{k}_0)$	**0.2815** / 0.0804	0.2801 / 0.0678	0.2824 / 0.0571	0.2782 / 0.0478	0.2755 / 0.0411	0.2709 / 0.0327
$\hat{\gamma}_n^{CH}(\hat{k}_0)$	0.3286 / 0.1136	0.3130 / 0.0905	0.2977 / 0.0685	0.2878 / 0.0566	0.2808 / 0.0475	0.2771 / 0.0376

Some conclusions:

1. The simulated results suggest that the EVI-estimator with smallest squared bias has usually the smallest RMSE;

2. The simulated values for the EVI-estimators $\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}$ and $\hat{\gamma}_n^{(2,\hat{\alpha}_0)}$ are almost equal. But in almost every model and sample size, $\hat{\gamma}_n^{(2,\hat{\alpha}_0)}$ has a smaller squared bias and RMSE than $\hat{\gamma}_n^{(1.5,\hat{\alpha}_0)}$, at their respective simulated optimal level;

3. For models with $\rho \le -0.75$, $\hat{\gamma}_n^{CH}$ is usually the most efficient EVI-estimator at its optimal level. For models with $\rho > -0.75$, the best EVI-estimator is $\hat{\gamma}_n^{(2,\hat{\alpha}_0)}$.

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