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FREDERICO CAEIRO AND M. IVETTE GOMES

BIAS REDUCTION IN THE ESTIMATION OF PARAMETERS OF RARE EVENTS

In this paper we consider a class of consistent semi-parametric estimators of a positive tail index γ, parametrized in two *tuning* or *control* parameters α and θ. Such control parameters enable us to have access, for any available sample, to an estimator of γ with a null dominant component of asymptotic bias, and with a reasonably flat *Mean Squared Error* pattern, as a function of k, the number of top order statistics considered. Those control parameters depend on a second order parameter ρ, which needs to be adequately estimated so that we may achieve a high efficiency relatively to the classical Hill estimator. We then obviously need to have access to a larger number of top order statistics than the number needed for optimal estimation through the Hill estimator.

THE CLASS OF SEMI-PARAMETRIC ESTIMATORS

In this paper we deal with semi-parametric estimators of the *tail index* γ, with a null dominant component of asymptotic bias. Such a kind of estimators has usually revealed nice smooth sample paths, as functions of the number k of top order statistics considered, quite close to the true value of the unknown parameter for a wide region of k values, and with a "bath-tube" Mean Squared Error, $MSE(k)$, in that same region, making thus the exact determination of the optimal k less relevant, from a practical point of view (Peng, 1998; Feuerverger and Hall, 1999; Beirlant *et al.*, 1999; Gomes *et al.*, 2000a; Gomes and Martins, 2001; Caeiro and Gomes, 2001).

We shall consider here heavy tails, i.e. $\gamma > 0$ in the *Extreme Value* distribution function (d.f.)

$$EV_\gamma(x) := \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \quad 1+\gamma x > 0,$$

to which $X_{n:n} := \max(X_1, \cdots, X_n)$ is attracted, after suitable linear normalization. We then say that the underlying model F is in the max-domain of attraction of EV_γ, and denote this by $F \in D_M(EV_\gamma)$. As usual, $X_{i:n}$, $1 \le i \le n$, denotes the i-th ascending order statistic associated to the random sample $(X_1, \cdots, X_n)$, from the unknown distribution function F.

With $U(t) := F^{\leftarrow}(1-1/t)$, $t \ge 1$, $F^{\leftarrow}$ denoting the generalized inverse function of F, we have (Gnedenko, 1943; de Haan, 1970) the first order condition,

$$F \in D_M(EV_\gamma), \gamma > 0 \quad \text{iff} \quad 1-F \in RV_{-1/\gamma} \quad \text{iff} \quad U \in RV_\gamma,$$

where RV_β stands for the class of *regularly varying* functions at infinity with *index of regular variation* β, i.e., positive measurable functions g such that $\lim\limits_{t\to\infty} g(tx)/g(t) = x^\beta$, for all $x > 0$.

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The class of semi-parametric estimators of γ to be considered here is

$$\gamma_n^{(\theta,\alpha)}(k) := \frac{\Gamma(\alpha)}{M_n^{(\alpha-1)}(k)} \left(\frac{M_n^{(\theta\alpha)}(k)}{\Gamma(\theta\alpha+1)}\right)^{1/\theta}, \quad \alpha \geq 1, \quad \theta > 0,$$

parametrized in the *tuning* parameters α and θ, which may be controlled at our ease, where $M_n^{(0)} \equiv 1$, and

$$M_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha}, \quad \alpha > 0,$$

is a consistent estimator of $\Gamma(\alpha+1)\gamma^{\alpha}$, whenever k is *intermediate*, i.e.,

$$k = k_n \to \infty, \text{ and } k = o(n), \text{ as } n \to \infty.$$

The estimator $M_n^{(1)}(k)$, also denoted $\gamma_n^H(k)$, is the classical Hill's estimator (Hill, 1975).

The class of estimators herewith introduced generalizes the Hill estimator, which appears for $(\theta,\alpha)=(1,1)$, and for $\theta = 2$ we obtain a class, studied in Caeiro and Gomes (2001), which generalizes the estimator $\gamma_n^{(1)}(k) := \sqrt{M_n^{(2)}(k)/2}$, studied in Gomes *et al.* (2000a). In section 2 of this paper we shall derive the asymptotic distributional properties of the estimators in this class, under a general second order framework, and will see that for $\theta > 1$ is always possible to find a control parameter α which makes null the dominant component of asymptotic bias of our tail index estimator, and which depends on the second order parameter ρ, which, on its turn should first be properly estimated on the basis of our sample. In section 3 we shall work under a third order framework, and shall make a few comments on a class of estimators of the second order parameter ρ, recently introduced by Fraga Alves et al. (2001). Finally, in section 4, we shall incorporate these ρ-estimators in our class of tail index estimators, removing the dominant component of asymptotic bias.

ASYMPTOTIC PROPERTIES OF THE ESTIMATORS

Under the first order framework and for intermediate k, the statistics $\gamma_n^{(\theta,\alpha)}(k)$ are consistent for γ, and under some extra mild conditions on the second order behaviour of the model F underlying the data they are asymptotically normal, with an asymptotic bias, perhaps non-null, whenever $\lim_{n\to\infty} \sqrt{k}A(n/k) = \lambda \neq 0$, finite, where $A(t)$ (and concomitantly ρ) measures the rate of convergence, in the first order condition, of $\ln U(tx) - \ln U(t)$ towards $\gamma \ln x$. Indeed, apart from the first order condition we often assume that there exists a function $A(t)$ of constant sign near infinity, such that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^{\rho}-1}{\rho},$$

for every $x > 0$, where ρ (≤ 0) is a *second order parameter.* The limit function in the above mentioned expression must be of the stated form, and $|A(t)| \in RV_{\rho}$ (Geluk and de Haan, 1987).

We first state, without proof, which may be seen in Gomes and Martins (2001), and had already been suggested in Dekkers *et al.* (1989), the main result used to derive the asymptotic properties of $\gamma_n^{(\theta,\alpha)}(k)$.

Proposition 1. *Under the validity of the first order condition and for k intermediate, the statistic $M_n^{(\alpha)}(k)$ converges in probability towards $\Gamma(\alpha+1)\ \gamma^\alpha$. If we further assume the general second order framework, the following asymptotic distributional representation holds for $M_n^{(\alpha)}(k)$,*

$$\gamma^\alpha \Gamma(\alpha+1) + \frac{\gamma^\alpha \sigma_\alpha}{\sqrt{k}}\ Z_n^{(\alpha)} + \gamma^{\alpha-1}\Gamma(\alpha+1)\frac{1-(1-\rho)^\alpha}{\rho(1-\rho)^\alpha} A(n/k)(1+o_p(1)),$$

where $Z_n^{(\alpha)}$ is an asymptotically standard normal r.v., and

$$\sigma_\alpha := \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}.$$

More than this: the r.v.'s $(Z_n^{(\alpha)}, Z_n^{(\beta)})$ have a covariance structure given by

$$\sigma_{\alpha,\beta} := \frac{\Gamma(\alpha+\beta+1) - \Gamma(\alpha+1)\Gamma(\beta+1)}{\sigma_\alpha \sigma_\beta}.$$

By continuity, the bias term $\frac{1-(1-\rho)^\alpha}{\rho(1-\rho)^\alpha}$ must be replaced by α if $\rho = 0$.

We next present the main result in this section.

Theorem 1. *Under the conditions and notations of Proposition 1, the asymptotic distributional representation*

$$\gamma + \frac{\gamma\sqrt{v_{\theta,\alpha}}}{\sqrt{k}}\ P_n^{(\theta,\alpha)} + b_{\theta,\alpha}(\rho)\ A(n/k) + o_p(A(n/k)) + o_p(1/\sqrt{k})$$

holds true for the statistic $\gamma_n^{(\theta,\alpha)}(k)$, where $P_n^{(\theta,\alpha)}$ is asymptotically standard normal,

$$v_{\theta,\alpha} = \frac{1}{\theta^2}\left\{\frac{2\Gamma(2\theta\alpha)}{\theta\alpha\Gamma^2(\theta\alpha)} + \frac{\theta^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)} - \frac{2\Gamma((\theta+1)\alpha)}{\alpha\Gamma(\theta\alpha)\Gamma(\alpha)} - (\theta-1)^2\right\},$$

and

$$b_{\theta,\alpha}(\rho) = \frac{1}{\theta\rho}\left\{(1-\rho)^{-\theta\alpha} - \theta(1-\rho)^{-\alpha+1} + (\theta-1)\right\} \quad \text{if} \quad \rho < 0,$$

being equal to 1 if $\rho = 0$.

Then, since $k = k_n \to \infty$, $\gamma_n^{(\theta,\alpha)}(k)$ is consistent for the estimation of γ, and if $\sqrt{k}A(n/k) \to \lambda$, finite, $\sqrt{k}\left(\gamma_n^{(\theta,\alpha)}(k) - \gamma\right)$ is asymptotically normal, with asymptotic variance $\gamma^2\ v_{\theta,\alpha}$ and asymptotic bias $\lambda\ b_{\theta,\alpha}(\rho)$.

Moreover, for every $\rho \in \Re^-$ and $\theta > 1$ there is a value $\alpha_0^{(\theta)} \equiv \alpha_0^{(\theta)}(\rho)$ such that $b_{\theta,\alpha_0^{(\theta)}}(\rho) = 0$, i.e. $\gamma_n^{(\theta,\alpha_0^{(\theta)})}(k)$ has a null asymptotic bias, even when $\sqrt{k}A(n/k) \to \lambda \neq 0$, as $n \to \infty$.

Proof of Theorem 1. We merely need to use Proposition 1, together with the fact that we have $(1+x)^\beta = 1 + \beta\ x + o(x)$ and $\frac{1}{1+x} = 1 - x + o(x)$, as $x \to 0$. Then, since

$$\left(\frac{M_n^{(\theta\alpha)}(k)}{\Gamma(\theta\alpha+1)}\right)^{1/\theta}$$

is equally distributed to

$$\left\{\gamma^{\theta\alpha}\left(1 + \frac{\sigma_{\theta\alpha}\ Z_n^{(\theta\alpha)}}{\sqrt{k}\ \Gamma(\theta\alpha+1)} + \frac{1-(1-\rho)^{\theta\alpha}}{\gamma\rho(1-\rho)^{\theta\alpha}} A(n/k)(1+o_p(1))\right)\right\}^{1/\theta}$$

which is on its turn equally distributed to

$$\gamma^{\alpha}\left\{1+\left(\frac{\sigma_{\theta\alpha}\, Z_n^{(\theta\alpha)}}{\theta\sqrt{k}\,\Gamma(\theta\alpha+1)}+\frac{1-(1-\rho)^{\theta\alpha}}{\theta\gamma\rho(1-\rho)^{\theta\alpha}}A(n/k)\right)(1+o_p(1))\right\},$$

and $\frac{\Gamma(\alpha)}{M_n^{(\alpha-1)}(k)}$ has the same distribution as

$$\gamma^{1-\alpha}\left\{1-\frac{\sigma_{\alpha-1}\, Z_n^{(\alpha-1)}}{\sqrt{k}\,\Gamma(\alpha)}-\frac{1-(1-\rho)^{\alpha-1}}{\gamma\rho(1-\rho)^{\alpha-1}}A(n/k)(1+o_p(1))\right\},$$

it follows that $\gamma_n^{(\theta,\alpha)}(k)$ is equally distributed to

$$\gamma\left\{1+\frac{1}{\sqrt{k}}\left(\frac{\sigma_{\theta\alpha}Z_n^{(\theta\alpha)}}{\theta\Gamma(\theta\alpha+1)}-\frac{\sigma_{\alpha-1}Z_n^{(\alpha-1)}}{\Gamma(\alpha)}\right)\right\}$$
$$+\left(\frac{1-(1-\rho)^{\theta\alpha}}{\theta\rho(1-\rho)^{\theta\alpha}}-\frac{1-(1-\rho)^{\alpha-1}}{\rho(1-\rho)^{\alpha-1}}\right)A(n/k)(1+o_p(1))+o_p(1/\sqrt{k}).$$

Then, from the covariance structure between $Z_n^{(\alpha)}$ and $Z_n^{(\beta)}$, we easily get that the variance of

$$\left(\frac{\sigma_{\theta\alpha}Z_n^{(\theta\alpha)}}{\theta\Gamma(\theta\alpha+1)}-\frac{\sigma_{\alpha-1}Z_n^{(\alpha-1)}}{\Gamma(\alpha)}\right)$$

is equal to $v_{\theta\alpha}$, and consequently $P_n^{(\theta,\alpha)}$ is given by

$$P_n^{(\theta,\alpha)}=\frac{1}{\sqrt{v_{\theta\alpha}}}\left(\frac{\sigma_{\theta\alpha}}{\theta\Gamma(\theta\alpha+1)}Z_n^{(\theta\alpha)}-\frac{\sigma_{\alpha-1}}{\Gamma(\alpha)}Z_n^{(\alpha-1)}\right).$$

The bias term and the first part of the theorem follows then straightforwardly.

The second part of the theorem comes from the fact that

$$\lim_{\alpha\to 1} b_{\theta,\alpha}(\rho)=\frac{1-(1-\rho)^{\theta}}{\theta\rho(1-\rho)^{\theta}}\geq 0,\quad \forall\rho<0,\ \forall\theta>0,$$

$$\lim_{\alpha\to+\infty} b_{\theta,\alpha}(\rho)=\frac{\theta-1}{\rho\theta}<0,\quad \forall\rho<0,\ \theta>1,$$

together with the fact that $b_{\theta,\alpha}(\rho)$ is a continuous function of α. If $\theta>1$, there is thus a value $\alpha_0^{(\theta)}\in[1,+\infty)$ such that $b_{\theta,\alpha_0^{(\theta)}}(\rho)=0$. □

For values of α close to α_0 we thus get a dominant component of bias close to 0, and consequently we expect to have then, at the optimal level, a high asymptotic efficiency relatively to $\gamma_n^{(1)}(k)$ at its optimal level, and *a fortiori* relatively to the Hill estimator, also at its optimal level, as may be easily derived in the lines of Gomes and Martins (2001) and Caeiro and Gomes (2001).

Remark 1. For the particular but important cases, $\theta=2$ and $\theta=3$, we have

$$\alpha_0^{(2)}\equiv\alpha_0^{(2)}(\rho)=-\frac{\ln\left[1-\rho-\sqrt{(1-\rho)^2-1}\right]}{\ln(1-\rho)},$$

and

$$\alpha_0^{(3)}\equiv\alpha_0^{(3)}(\rho)=-\frac{\ln\left\{2\sqrt{1-\rho}\,\cos\left(\frac{\arctan\left(-\sqrt{(1-\rho)^3-1}\right)+4\pi}{3}\right)\right\}}{\ln(1-\rho)},$$

respectively.

A THIRD ORDER FRAMEWORK AND ESTIMATION OF THE SECOND ORDER PARAMETER

We shall assume now that a third order condition holds, and more than that, we assume that such a condition may be written as

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx)-\ln U(t)-\gamma \ln x}{A(t)} - \frac{x^\rho - 1}{\rho}}{B(t)} = \frac{x^{2\rho}-1}{2\rho},$$

where $|B|$ must then be of regular variation, also with index of regular variation $\rho < 0$. Although not so general as the conditions assumed in other papers, like the ones of Gomes and de Haan (1999), Gomes *et al.* (2000b) and Fraga Alves *et al.* (2001), such a condition holds for most of the well-known heavy tail models, like the *Fréchet*, the *Generalized Pareto*, the *Burr*, the *Student* models, and a large variety of models in Hall's class of distributions (Hall, 1982; Hall and Welsh, 1985), for which we have, as $x \to \infty$, a tail function of the type

$$1 - F(x) = Cx^{-1/\gamma}\left(1 + D_1\, x^{\rho/\gamma} + D_2\, x^{2\rho/\gamma}(1 + o(1))\right).$$

Let E denote an exponential r.v., with d.f. $F_E(x) = 1 - \exp(-x)$, $x > 0$, and, with the same notation as in Gomes *et al.* (2000b) and Fraga Alves *et al.* (2001), let us put

$$\mu_\alpha^{(1)} := E[E^\alpha] = \Gamma(\alpha+1), \quad \overline{\mu}_\alpha^{(1)} = 1$$

$$\overline{\sigma}_\alpha^{(1)} := \frac{\sqrt{Var[E^\alpha]}}{\mu_\alpha^{(1)}} = \frac{\sqrt{\Gamma(2\alpha+1)-\Gamma^2(\alpha+1)}}{\alpha\Gamma(\alpha)} = \sqrt{\frac{2\Gamma(2\alpha)}{\alpha\Gamma^2(\alpha)} - 1},$$

$$\overline{\mu}_\alpha^{(2)}(\rho) := \frac{1}{\mu_\alpha^{(1)}}\, E\left[E^{\alpha-1}\left(\frac{e^{\rho E}-1}{\rho}\right)\right] = \frac{1-(1-\rho)^\alpha}{\alpha\,\rho\,(1-\rho)^\alpha},$$

$$\overline{\sigma}_\alpha^{(2)}(\rho) := \frac{\sqrt{Var\left[E^{\alpha-1}\left(\frac{e^{\rho E}-1}{\rho}\right)\right]}}{\mu_\alpha^{(1)}} = \frac{\sqrt{\mu_{2\alpha}^{(3)}(\rho) - \left(\mu_\alpha^{(2)}(\rho)\right)^2}}{\mu_\alpha^{(1)}},$$

with

$$\overline{\mu}_\alpha^{(3)}(\rho) := \frac{1}{\mu_\alpha^{(1)}}\, E\left[E^{\alpha-2}\left(\frac{e^{\rho E}-1}{\rho}\right)^2\right] = \frac{2\left\{\mu_{\alpha-1}^{(2)}(2\rho) - \mu_{\alpha-1}^{(2)}(\rho)\right\}}{\rho\,\mu_\alpha^{(1)}},$$

which is equal to $\frac{1}{\rho^2}\ln\frac{(1-\rho)^2}{1-2\rho}$ if $\alpha = 1$ and equal to $\frac{(1-2\rho)^{1-\alpha}-2(1-\rho)^{1-\alpha}+1}{\rho^2\,\alpha\,(\alpha-1)}$ if $\alpha \neq 1$.

In the lines of Theorem 1 we may further say that:

Theorem 2. *If, with the same notation as before, we further assume the general third order framework, if we choose the tuning parameters θ and α such that*

$$\alpha\overline{\mu}_{\alpha\theta}^{(2)}(\rho) - (\alpha-1)\,\overline{\mu}_{\alpha-1}^{(2)}(\rho) = 0,$$

and if we also assume that $\sqrt{k}A(n/k) \to \infty$ as $n \to \infty$, the asymptotic distributional representation,

$$\gamma + \frac{\gamma\sqrt{v_{\theta,\alpha}}P_n^{(\theta,\alpha)}}{\sqrt{k}} + \frac{A^2(n/k)\ \left(\alpha(\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(\rho) - \alpha^2(\theta-1)(\overline{\mu}_{\alpha\theta}^{(2)}(\rho))^2\right)}{2\,\gamma}$$
$$+\frac{A^2(n/k)}{2\,\gamma}\left(-(\alpha-1)(\alpha-2)\,\overline{\mu}_{\alpha-1}^{(3)}(\rho) + 2(\alpha-1)^2\left(\overline{\mu}_{\alpha-1}^{(2)}(\rho)\right)^2\right)$$
$$+A(n/k)\ B(n/k)\left(\alpha\overline{\mu}_{\alpha\theta}^{(2)}(2\rho) - (\alpha-1)\,\overline{\mu}_{\alpha-1}^{(2)}(2\rho)\right)\ \ (1+o_p(1))$$

holds for $\gamma_n^{(\theta,\alpha)}(k)$. *Consequently* $\sqrt{k}\left\{\gamma_n^{(\theta,\alpha)}(k)-\gamma\right\}$ *is asymptotically normal with null mean value, not only when* $\sqrt{k}A(n/k)\to 0$, *but also whenever* $\sqrt{k}A(n/k)\to\lambda$, *finite or infinite, provided that* $\sqrt{k}A^2(n/k)\to 0$, *as* $n\to\infty$. *If* $\sqrt{k}A^2(n/k)\to\lambda_A$, *finite (and then* $\sqrt{k}A(n/k)\,B(n/k)\to\lambda_B$, *also finite), there is a non-null asymptotic bias given by*

$$\frac{\lambda_A}{2\,\gamma}\ \left(\alpha(\alpha\theta-1)\overline{\mu}^{(3)}_{\alpha\theta}(\rho)-\alpha^2(\theta-1)(\overline{\mu}^{(2)}_{\alpha\theta}(\rho))^2-(\alpha-1)(\alpha-2)\ \overline{\mu}^{(3)}_{\alpha-1}(\rho)\right)$$
$$+\frac{\lambda_A}{\gamma}(\alpha-1)^2\left(\overline{\mu}^{(2)}_{\alpha-1}(\rho)\right)^2+\lambda_B\left(\alpha\overline{\mu}^{(2)}_{\alpha\theta}(2\rho)-(\alpha-1)\ \overline{\mu}^{(2)}_{\alpha-1}(2\rho)\right).$$

Proof of Theorem 2. Under the third order condition, assuming that k is intermediate, and using the same arguments as in Dekkers *et al.* (1989), in lemma 2 of Draisma *et al.* (1999) and more recently in Gomes *et al.* (2000b) and Fraga Alves *et al.* (2001), we may write the distributional representation

$$\begin{aligned}
\left(\frac{M_n^{(\alpha\theta)}(k)}{\mu^{(1)}_{\alpha\theta}}\right)^{1/\theta} &= \gamma^{\alpha}\left(1+\frac{\overline{\sigma}^{(1)}_{\alpha\theta}}{\theta}\,\frac{P_n^{(\alpha\theta)}}{\sqrt{k}}\right)\\
&\quad+\alpha\,\gamma^{\alpha-1}\left(\overline{\mu}^{(2)}_{\alpha\theta}(\rho)\ A(n/k)+\overline{\sigma}^{(2)}_{\alpha\theta}(\rho)\ \frac{A(n/k)}{\sqrt{k}}\ \overline{P}_n^{(\alpha\theta)}\right)\\
&\quad+\frac{\alpha\,\gamma^{\alpha-2}A^2(n/k)}{2}\left((\alpha\theta-1)\overline{\mu}^{(3)}_{\alpha\theta}(\rho)+\alpha(1-\theta)(\overline{\mu}^{(2)}_{\alpha\theta}(\rho))^2\right)\\
&\quad+\alpha\,\gamma^{\alpha-1}\ \overline{\mu}^{(2)}_{\alpha\theta}(2\rho)\ A(n/k)\ B(n/k)\ (1+o_p(1)),
\end{aligned}$$

where $P_n^{(\alpha\theta)}$ and $\overline{P}_n^{(\alpha\theta)}$ are asymptotically standard Normal r.v.'s.

Also

$$\begin{aligned}
\left(\frac{M_n^{(\alpha-1)}(k)}{\mu^{(1)}_{\alpha-1}}\right)^{-1} &= \gamma^{1-\alpha}\left(1-\overline{\sigma}^{(1)}_{\alpha-1}\frac{P_n^{(\alpha-1)}}{\sqrt{k}}\right)\\
&\quad-\frac{\alpha-1}{\gamma^{\alpha}}\left(\overline{\mu}^{(2)}_{\alpha-1}(\rho)+\overline{\sigma}^{(2)}_{\alpha-1}(\rho)\ \frac{\overline{P}_n^{(\alpha-1)}}{\sqrt{k}}\right)A(n/k)\\
&\quad-\frac{\alpha-1}{2\gamma^{\alpha+1}}\ A^2(n/k)\left((\alpha-2)\ \overline{\mu}^{(3)}_{\alpha-1}(\rho)-2(\alpha-1)\ (\overline{\mu}^{(2)}_{\alpha-1}(\rho))^2\right)\\
&\quad-\frac{\alpha-1}{\gamma^{\alpha}}\ \overline{\mu}^{(2)}_{\alpha-1}(2\rho)\ A(n/k)\ B(n/k)\ (1+o_p(1)).
\end{aligned}$$

Then

$$\begin{aligned}
\gamma_n^{(\theta,\alpha)}(k) &= \gamma+\frac{\gamma}{\sqrt{k}}\left(\frac{\overline{\sigma}^{(1)}_{\alpha\theta}\ P_n^{(\alpha\theta)}}{\theta}-\overline{\sigma}^{(1)}_{\alpha-1}P_n^{(\alpha-1)}\right)\\
&\quad+A(n/k)\ \left(\alpha\overline{\mu}^{(2)}_{\alpha\theta}(\rho)-(\alpha-1)\ \overline{\mu}^{(2)}_{\alpha-1}(\rho)\right)\\
&\quad+\frac{A(n/k)}{\sqrt{k}}\left(\alpha\overline{\sigma}^{(2)}_{\alpha\theta}(\rho)\ \overline{P}_n^{(\alpha\theta)}-(\alpha-1)\ \overline{\sigma}^{(2)}_{\alpha-1}(\rho)\ \overline{P}_n^{(\alpha-1)}\right)\\
&\quad+\frac{A^2(n/k)}{2\,\gamma}\ \left(\alpha(\alpha\theta-1)\overline{\mu}^{(3)}_{\alpha\theta}(\rho)-\alpha^2(\theta-1)(\overline{\mu}^{(2)}_{\alpha\theta}(\rho))^2-(\alpha-1)(\alpha-2)\ \overline{\mu}^{(3)}_{\alpha-1}(\rho)\right)\\
&\quad+\frac{A^2(n/k)}{\gamma}(\alpha-1)^2\left(\overline{\mu}^{(2)}_{\alpha-1}(\rho)\right)^2(1+o_p(1))\\
&\quad+A(n/k)\ B(n/k)\left(\alpha\overline{\mu}^{(2)}_{\alpha\theta}(2\rho)-(\alpha-1)\ \overline{\mu}^{(2)}_{\alpha-1}(2\rho)\right)\quad(1+o_p(1)),
\end{aligned}$$

If we choose α and θ such that

$$\alpha\overline{\mu}^{(2)}_{\alpha\theta}(\rho) - (\alpha-1)\,\overline{\mu}^{(2)}_{\alpha-1}(\rho) = 0,$$

the rate of convergence is still of the order of $\frac{1}{\sqrt{k}}$, and the asymptotic bias is null not only when $\sqrt{k}\ A(n/k) \to 0$ but also when $\sqrt{k}\ A(n/k) \to \lambda \neq 0$. Indeed the minimum asymptotic mean squared error is going to be attained whenever $\sqrt{k}\ A(n/k) \to \infty$ and $\sqrt{k}\ A^2(n/k) \to \lambda_A \neq 0$, finite. Then, also $\sqrt{k}\ A(n/k)\ B(n/k) \to \lambda_B \neq 0$, finite. □

We shall consider here particular members of the class of estimators of the second order parameter ρ proposed by Fraga Alves *et al.* (2001). Under adequate general conditions, they are semi-parametric asymptotically normal estimators of ρ, which show highly stable sample paths as functions of k, the number of top order statistics used, for a wide range of large k-values. Such a class of estimators is parametrized in a tuning parameter τ and depends on the statistics

$$T_n^{(\tau)}(k) := \frac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}$$

for $\tau > 0$, with

$$T_n^{(0)}(k) := \frac{\ln\left(M_n^{(1)}(k)\right) - \frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right)}{\frac{1}{2}\ln\left(M_n^{(2)}(k)/2\right) - \frac{1}{3}\ln\left(M_n^{(3)}(k)/6\right)},$$

obtained by continuity arguments. These statistics converge towards $3(1-\rho)/(3-\rho)$, independently of τ, whenever the second order condition holds and k is such that $k = o(n)$ and $\sqrt{k}\ A(n/k) \to \infty$, as $n \to \infty$. The ρ-estimators may thus be defined as

$$\widehat{\rho}_n^{(\tau)}(k) := \min\left(0,\ \frac{3(T_n^{(\tau)}(k)-1)}{T_n^{(\tau)}(k)-3}\right).$$

We shall formalize, without proofs, the main distributional results of these ρ-estimators. Proofs may be found in Fraga Alves *et al.* (2001).

Proposition 2. *If the first order condition holds, k is a sequence of intermediate integers, and*

$$\lim_{n\to\infty} \sqrt{k}A(n/k) = \infty,$$

then $\widehat{\rho}_n^{(\tau)}(k)$ is consistent for the estimation of ρ, i.e., converges in probability towards ρ, as $n \to \infty$.

Proposition 3. *If the third order condition holds, together with $\lim_{n\to\infty} \sqrt{k}A(n/k) = \infty$, and also the validity of the following conditions:*

$$\lim_{n\to\infty} \sqrt{k}\ A^2(n/k) = \lambda_1,$$

and

$$\lim_{n\to\infty} \sqrt{k}\ A(n/k)\ B(n/k) = \lambda_2,$$

both finite, we may guaranteee asymptotic normality of the estimators $\widehat{\rho}_n^{(\tau)}(k)$, and

$$\widehat{\rho}_n^{(\tau)}(k) - \rho = O_p\left(\frac{1}{\sqrt{k}A(n/k)}\right).$$

Remark 2. The theoretical and simulated results in Fraga Alves *et al.* (2001), together with the use of these estimators in the Generalized Jackknife statistics of Gomes et al. (2000a) lead us to advise in practice the consideration of the level

$$k_1 = \min(n-1, [2n/\ln\ln n])$$

(not chosen in an optimal way), and of the tuning parameters $\tau = 0$ for the region $\rho \in [-1, 0)$ and $\tau = 1$ for the region $\rho \in (-\infty, -1)$. Anyway, we advise practitioners not to choose blindly the value of τ. It is sensible to draw a few sample paths of $\widehat{\rho}_n^{(\tau)}(k)$, as functions of k, electing the value of τ which provides higher stability for large k, by means of any stability criterion.

Distributional properties of the "asymptotically unbiased" estimators of the tail index

We shall now restrict ourselves to the case $\theta = 2$, merely for sake of simplicity of the expressions, and shall consider the estimators

$$\gamma_n^{(\widehat{\alpha})}(k) := \frac{\Gamma(\widehat{\alpha})}{M_n^{(\widehat{\alpha}-1)}(k)} \left(\frac{M_n^{(2\widehat{\alpha})}(k)}{\Gamma(2\widehat{\alpha}+1)}\right)^{1/2},$$

with

$$\widehat{\alpha} = -\frac{\ln\left[1-\widehat{\rho}-\sqrt{(1-\widehat{\rho})^2-1}\right]}{\ln(1-\widehat{\rho})}, \quad \widehat{\rho} = \widehat{\rho}_n^{(\tau)}(k_1),$$

any of the ρ-estimators given before, computed at the level k_1.

Theorem 3. *If the second order condition holds, if $k = k_n$ is a sequence of intermediate positive integers, i.e., and if $\sqrt{k}A(n/k) \underset{n\to\infty}{\longrightarrow} \lambda$, finite, non necessarily null, then $\sqrt{k}\left(\gamma_n^{(\widehat{\alpha})}(k) - \gamma\right)$ are asymptotically normal with null mean value. Indeed, with Z_k denoting an asymptotically standard normal r.v., we have, for the tail index estimators under study, the distributional representation*

$$\gamma_n^{(\widehat{\alpha})}(k) = \gamma + \frac{\gamma\,\varphi(\rho)}{\sqrt{k}}\,Z_k + o_p(A(n/k)).$$

Proof of Theorem 3. The proof for the r.v. $\gamma_n^{(\alpha_\rho)}(k)$, $\alpha_\rho = \alpha_0^{(2)}(\rho)$ follows the same lines of the proof of Theorem 1, and we get straightforwardly

$$\varphi^2(\rho) = \frac{1}{4}\left\{\frac{\Gamma(4\alpha_\rho)}{\alpha_\rho\Gamma^2(2\alpha_\rho)} + \frac{4\Gamma(2\alpha_\rho-1)}{\Gamma^2(\alpha_\rho)} - \frac{2\Gamma((3)\alpha_\rho)}{\alpha_\rho\Gamma(\alpha_\rho)\Gamma(\alpha_\rho)} - 1\right\},$$

with

$$\alpha_\rho = \alpha_0^{(2)}(\rho) = -\frac{\ln\left[1-\rho-\sqrt{(1-\rho)^2-1}\right]}{\ln(1-\rho)}.$$

The distributional representation in the theorem is due to the fact that we may write

$$\gamma_n^{(\widehat{\alpha})}(k) = \gamma_n^{(\alpha_\rho)}(k) + (\widehat{\rho}-\rho)\xi(k)(1+o_p(1)),$$

with $\xi(k) = O_p\left(\frac{1}{\sqrt{k}}\right)$. □

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DEPARTAMENTO DE MATEMÁTICA, F.C.T., UNIVERSIDADE NOVA DE LISBOA, QUINTA DA TORRE 2825-114 CAPARICA, PORTUGAL
E-mail: fac@mail.fct.unl.pt

C.E.A.U.L., D.E.I.O. (F.C.U.L.) (UNIVERSITY OF LISBON), EDIFÍCIO C2, PISO 2, CAMPO GRANDE, 1749-016 LISBOA, PORTUGAL
E-mail: ivette.gomes@fc.ul.pt

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